Publ. Math. Debrecen
76/3 (2010), 317–328

Rate of convergence for certain optimal stopping problems

By ANDRIY KLESOV (Kiev)

Dedicated to the 100th anniversary of the birthday of Béla Gyires

Abstract. We prove that the rate of convergence of a solution of the optimal stopping problem for a Lévy process on an interval $[0,T]$ to that on the interval $[0,\infty)$ is exponential as $T\to\infty$.

1. Introduction

The first paper to deal with a stopping time of a Lévy process in the context we consider below is MORDECKI [3] where an explicit expression is found for an optimal stopping time for a reward functions of either $(X_t-K)^+$ or $(K-X_t)^+$. MORDECKI [3] found that the optimal stopping time is of a threshold type.

A new approach appeared in [5] where the Appel polynomials are applied for optimal stopping problems of the discussed type. An analogue of MORDECKI's [3] result had been obtained in [5] for the discrete Markov chains and for the reward functions $g(x)=(x^n)^+$, $n\in\mathcal{N}$. It is proved in [5] that the rate of convergence of the solution of the optimal stopping problem on a finite interval converges to that on the infinite interval $[0,\infty]$. We shall concentrate on a generalization of this result for a broad class of Lévy processes.

A generalization of the result of [5] for general Lévy-type processes and the reward function $g(x)=(x^n)^+$, $n\in\mathcal{N}$, can be found in [2]. The most general

2000 *Mathematics Subject Classification:* 60G40, 91B70, 60H30.
Key words and phrases: optimal stopping time, Lévy processes, rate of convergence, appel polynomials.
Supported in part by a DFG grant.

result up to now is obtained in [4]. An explicit form of the optimal stopping moment for the optimal stopping problem for homogeneous Lévy processes and the reward function $g(x) = (x^\eta)^+$, $\eta > 0$, are found in [5]. The optimal stopping moment is constructed in [5] by using the Appel polynomials. However the rate of convergence is nor discussed in [5], at all.

In the current paper we find the rate of convergence of a solution of the optimal stopping problem for a Lévy process on an interval $[0, T]$ to that on the interval $[0, \infty)$ as $T \to \infty$. It turns out that the rate of convergence is exponential.

2. Lévy–Itô decomposition

For convenience, we recall the well known Lévy–Itô decomposition for Lévy processes.

Theorem 2.1 (Lévy–Itô decomposition)**.** *Let X_t be a Lévy process. Then there exists a triplet of stochastic processes $X_t^{(1)}$, $X_t^{(2)}$, and $X_t^{(3)}$ such that*

$$X_t = X_t^{(1)} + X_t^{(2)} + X_t^{(3)}, \tag{2.1}$$

where $X_t^{(1)}$ is a Brownian motion with drift, $X_t^{(2)}$ a compound Poisson process, $X_t^{(3)}$ a square integrable pure jump martingale.

The compound Poisson process $X_t^{(2)}$ in 2.1 is usually constructed from a simple Poisson process. We will assume that the intensity $\lambda(t)$ of the simple Poisson process is such that

$$\sum_{m=1}^{\infty} \frac{\lambda(2^m)}{2^m} < \infty. \tag{2.2}$$

Another useful assumption we use below for the process $X_t^{(3)}$ of 2.1 is that

$$\int_1^\infty \frac{\mathbf{E}\left|X_t^{(3)}\right|^\eta}{e^{\eta q t}}\, dt < \infty. \tag{2.3}$$

for some $\eta > 0$ and $q > 1$.

3. Main result

Theorem 3.1 (Main result)**.** *Fix $\eta > 0$ and $q > 1$. Let $(X_t, t \geq 0)$ be a Lévy process such that*

$$\mathbf{E}(X_t^+)^\eta < \infty$$

and that X_t admits the Lévy–Itô decomposition (2.1) *without drift. Let $X_0 = x$.*

Assume that the square integrable pure jump process $X_t^{(3)}$ in representation (2.1) *satisfies condition* (2.3)*.*

We further assume that the compound Poisson process $X_t^{(2)}$ in representation (2.1) *is such that*

$$X_t^{(2)} = \sum_{k \leq N_t} \xi_k \tag{3.1}$$

where the random variables ξ_k, $k \geq 1$, are nonnegative, independent, identically distributed, and such that

$$\mathbf{E}\xi_k^{\eta \vee 1} < \infty \quad \text{for some } \eta > 0.$$

The symbol N_t in representation (3.1) *stands for a simple Poisson process with intensity $\lambda(t)$ such that the process N_t and the sequence $\{\xi_k\}$ are independent. Moreover we assume that the intensity λ satisfies condition* (2.2)*.*

Let $T > 0$ and let $\mathcal{M}$ and $\mathcal{M}_T$ denote the sets of all stopping times $\tau \in [0, \infty]$ and $\tau \in [0, T]$, respectively. Let $g(x)$ denote the function $(x^+)^\eta$ and let

$$V(x) = \sup_{\tau \in \mathcal{M}} \mathbf{E}(e^{-q\tau} g(X_\tau) \mathcal{I}_{\{\tau < \infty\}}), V(x, T) = \sup_{\tau \in \mathcal{M}_T} \mathbf{E}(e^{-q\tau} g(X_\tau)).$$

Then there exist a number $T_0 > 0$, an universal constant $c > 0$, and, for a given real number x, there exists a constant $C(x)$ such that

$$0 \leq V(x) - V(x, T) \leq C(x) e^{-cT} \tag{3.2}$$

for all $T > T_0$.

Remark 1. Theorem 3.1 is a generalization of Theorem 3 of the paper [5].

4. Auxiliary results

The proof of Theorem 3.1 is based on several auxiliary results.

Lemma 1. *Let $(X_t, t \geq 0)$ be a process such that it can be decomposed into a sum $X_t = P_t + Q_t + R_t$. Let η, q, $g(x)$, $V(x)$, and $V(x, T)$ be defined as in Theorem 3.1.*

Then the conclusion of Theorem 3.1 holds if

$$\mathbf{E}\left(\sup_{s\ge t}\frac{|P_s|}{s^\theta}\right)^\eta<\infty,\quad \mathbf{E}\left(\sup_{s\ge t}\frac{|Q_s|}{s^\theta}\right)^\eta<\infty,\quad \mathbf{E}\left(\sup_{s\ge t}\frac{|R_s|}{s^\theta}\right)^\eta<\infty.$$

for some $\theta>0$ *and all* $t>0$.

Proof. Note that $V(x)\ge V(x,T)$, since $\mathcal{M}_T\subset\mathcal{M}$. Now let τ^* be a positive root of the Appel polynomial constructed from the random variable $M_{\tau,q}=\sup_{0\le t<\tau}X_t$, where τ is a random variable such that $P\{\tau>t\}=e^{-tq}$. Then

$$\begin{aligned}V(x,T)&=\sup_{\tau\in\mathcal{M}}\mathbf{E}(e^{-q\tau}g(X_\tau))\ge\mathbf{E}\big(g(X_{\min(\tau^*,T)})e^{-q\min(\tau^*,T)}\big)\\&\ge\mathbf{E}\big(g(X_{\min(\tau^*,T)})e^{-q\tau^*}\mathcal{I}_{\{\tau^*\le T\}}\big),\end{aligned}$$

since $\tau^*\wedge T\in\mathcal{M}_T$.

Thus

$$V(x)-V(x,T)\le\mathbf{E}\big(g(X_{\tau^*})e^{-q\tau^*}\mathcal{I}_{\{T<\tau^*<\infty\}}\big).$$

Since the function $e^{-qs}s^{\theta\eta}$ is decreasing on the semiaxis being far enough of the origin,

$$\begin{aligned}V(x)-V(x,T)&\le\mathbf{E}\big(g(X_{\tau^*})e^{-q\tau^*}\mathcal{I}_{\{T<\tau^*<\infty\}}\big)\le\mathbf{E}\left[\sup_{s\ge T}\frac{(P_s+Q_s+R_s)^\eta}{e^{qs}}\right]\\&\le 2^{2\eta}\left(\mathbf{E}\left[\sup_{s\ge T}\frac{|P_s|^\eta}{e^{qs}}\right]+\mathbf{E}\left[\sup_{s\ge T}\frac{|Q_s|^\eta}{e^{qs}}\right]+\mathbf{E}\left[\sup_{s\ge T}\frac{|R_s|^\eta}{e^{qs}}\right]\right)\\&=2^{2\eta}\left(\mathbf{E}\left[\sup_{s\ge T}\frac{|P_s|^\eta}{s^{\theta\eta}}\cdot\frac{s^{\theta\eta}}{e^{qs}}\right]+\mathbf{E}\left[\sup_{s\ge T}\frac{|Q_s|^\eta}{s^{\theta\eta}}\cdot\frac{s^{\theta\eta}}{e^{qs}}\right]+\mathbf{E}\left[\sup_{s\ge T}\frac{|R_s|^\eta}{s^{\theta\eta}}\cdot\frac{s^{\theta\eta}}{e^{qs}}\right]\right)\\&\le 2^{2q}\cdot c'\cdot\frac{T^{\theta\eta}}{e^{qT}}\left(\mathbf{E}\left[\sup_{s\ge T}\frac{|P_s|}{s^\theta}\right]^\eta+\mathbf{E}\left[\sup_{s\ge T}\frac{|Q_s|}{s^\theta}\right]^\eta+\mathbf{E}\left[\sup_{s\ge T}\frac{|R_s|}{s^\theta}\right]^\eta\right).\end{aligned}$$

Thus (3.2) holds for an arbitrary $c<q$, sufficiently large T_0, and appropriate $C(x)$. □

Lemma 2 (Wiener process). *Let* $\eta>0$ *and let* $X_t^{(1)}=W_t$, $t>0$, *be a Wiener process. If* $\theta>\frac{1}{2}$ *and* $T>0$, *then*

$$\mathbf{E}\left(\sup_{t\ge T}\frac{|W_t|}{t^\theta}\right)^\eta<\infty.$$

PROOF. It is known that if X is a nonnegative random variable, then

$$\mathbf{E}X^\eta = \eta \int_0^\infty x^{\eta-1}\mathbf{P}(X \ge x)\,dx.$$

Without loss of generality assume that $T = 1$. Using the latter formula we get

$$\begin{aligned}
\mathbf{E}\Big(\sup_{s\ge 1}\frac{|W_s|}{s^\theta}\Big)^\eta &= \eta\int_0^\infty x^{\eta-1}\mathbf{P}\Big(\sup_{s\ge 1}\frac{|W_s|}{s^\theta}\ge x\Big)\,dx \\
&\le \eta\int_0^\infty x^{\eta-1}\sum_{m=0}^\infty \mathbf{P}\Big(\sup_{2^m\le s\le 2^{m+1}}\frac{|W_s|}{s^\theta}\ge x\Big)\,dx \\
&\le \eta\sum_{m=0}^\infty\int_0^\infty x^{\eta-1}\mathbf{P}\Big(\sup_{2^m\le s\le 2^{m+1}}|W_s|\ge x2^{m\theta}\Big)\,dx.
\end{aligned}$$

For any $y > 0$,

$$\mathbf{P}\Big(\sup_{2^m\le s\le 2^{m+1}}|W_s|\ge y\Big)\le \mathbf{P}\Big(\sup_{s\le 2^{m+1}}|W_s|\ge y\Big) = 2\mathbf{P}(|W_{2^{m+1}}|\ge y).$$

Thus

$$\begin{aligned}
\mathbf{E}\left(\sup_{s\ge 1}\frac{|W_s|}{s^\theta}\right)^\eta &\le 2\eta\sum_{m=0}^\infty\int_0^\infty x^{\eta-1}\mathbf{P}(|W_{2^{m+1}}|\ge x2^{m\theta})\,dx \\
&= 2\eta\sum_{m=0}^\infty\int_0^\infty\Big(\frac{y}{2^{m\theta}}\Big)^{\eta-1}\mathbf{P}(|W_{2^{m+1}}|\ge y)\frac{dy}{2^{m\theta}} \\
&= 2\eta\sum_{m=0}^\infty\frac{1}{2^{m\theta\eta}}\int_0^\infty y^{\eta-1}\mathbf{P}(|W_{2^{m+1}}|\ge y)\,dy \\
&= 2\eta\sum_{m=0}^\infty\frac{1}{2^{m\theta\eta}}\mathbf{E}|W_{2^{m+1}}|^\eta. \qquad (4.1)
\end{aligned}$$

Since W_t is a Gaussian random variable with zero mean and variance t, we have

$$\mathbf{E}|W_t|^\eta = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^\infty |x|^\eta e^{-x^2/2t}\,dx = \frac{t^{\eta/2}}{\sqrt{2\pi t}}\int_{-\infty}^\infty |x|^\eta e^{-x^2/2}\,dx = \kappa t^{\eta/2},$$

where

$$\kappa = \sqrt{\frac{2}{\pi}}\int_0^\infty x^\eta e^{-x^2/2}\,dx = 2^{\eta-\frac12}\pi^{-\frac12}\Gamma(\eta)$$

and where Γ is the gamma function. Thus

$$\sum_{m=0}^\infty\frac{1}{2^{m\theta\eta}}\mathbf{E}|W_{2^{m+1}}|^\eta = \kappa\sum_{m=0}^\infty\frac{2^{(m+1)/2}}{2^{m\theta\eta}} = \kappa 2^{\eta/2}\sum_{m=0}^\infty 2^{m\eta\left(\frac12-\theta\right)} < \infty,$$

since $\theta > \frac12$. This completes the proof. □

Remark 2. The method used in the proof of Lemma 2 fits the case of the Wiener process with a drift $O(t^\theta)$, as well.

Lemma 3 (Simple Poisson process)**.** *Let $\eta > 0$ and $q > 0$ and let $\Pi(t)$ be the simple Poisson process with intensity $\lambda(t)$. If*

$$\int_1^\infty e^{-\eta q t} \max\{\lambda(t), \lambda^\eta(t)\} dt < \infty,$$

then for every $T > 0$

$$\mathbf{E}\left(\sup_{t \geq T} \frac{\Pi(t)}{e^{qt}}\right)^\eta < \infty.$$

Lemma 3 implies the corresponding result for the difference of two Poisson process, that, in turn, allows one to consider the processes with both positive and negative jumps.

Corollary 1. *Let $\eta > 0$ and $p > 1/2$. Let $\Pi^1(t)$ and $\Pi^2(t)$ be two Poisson processes with intensities $\lambda_1(t) \to \infty$ and $\lambda_2(t) \to \infty$, respectively. If*

$$\int_1^\infty e^{-\eta q t} \lambda_i^\eta(t) dt < \infty, \quad i = 1, 2,$$

then for every $T > 0$

$$\mathbf{E}\left(\sup_{t \geq T} \left| \frac{\Pi^1(t) - \Pi^2(t)}{e^{qt}} \right| \right)^\eta < \infty.$$

In order to prove Lemma 3 we need both upper and lower bounds for moments of the Poisson distribution. The exact values of such moments can easily be evaluated for integer η, however this is not the case for non-integer η and thus we need to use the following estimates.

Lemma 4 (upper bound)**.** *Let $\Pi \in Po(\lambda)$, $\lambda > 0$ and $\eta > 0$. Then there exists a constant $c > 0$, that does not depend on λ, such that*

$$\mathbf{E}(\Pi^\eta) \leq c\lambda^\eta$$

if $\lambda \geq 1$, and

$$\mathbf{E}(\Pi^\eta) \leq c\lambda$$

if $0 < \lambda < 1$.

Lemma 5 (lower bound)**.** *Let $\Pi \in Po(\lambda)$, $\lambda > 0$ and $\eta > 0$.Then there exists a constant $c > 0$, that does not depend on λ, such that*

$$\mathbf{E}(\Pi^\eta) \geq c\lambda^\eta$$

if $\lambda \geq 1$, and

$$\mathbf{E}(\Pi^\eta) \geq c\lambda$$

if $0 < \lambda < 1$.

Although the constants in Lemmas 4 and 5 are denoted by the same symbol c, they are different, in fact.

First we show that Lemma 3 follows from Lemmas 4 and 5 and then prove Lemmas 4 and 5 themselves.

Proof of Lemma 3. Without loss of generality assume that $T = 1$. We have

$$\begin{aligned}\mathbf{E}\left(\sup_{t\geq 1}\frac{\Pi(t)}{e^{qt}}\right)^{\eta} &\leq \sum_{k=1}^{\infty}\mathbf{E}\left(\sup_{k\leq t<k+1}\frac{\Pi(t)}{e^{qt}}\right)^{\eta}\\ &\leq \sum_{k=1}^{\infty}e^{-qk\eta}\mathbf{E}(\Pi^{\eta}(k+1)) \leq e^{q}\sum_{k=1}^{\infty}e^{-qk\eta}\mathbf{E}(\Pi^{\eta}(k)).\end{aligned}$$

Lemma 4 implies that

$$\mathbf{E}\left(\sup_{t\geq 1}\frac{\Pi(t)}{e^{qt}}\right)^{\eta} \leq ce^{q}\sum_{k=1}^{\infty}e^{-qk\eta}\max\left\{\lambda(k),\lambda^{\eta}(k)\right\}.$$

Since

$$\begin{aligned}\int_{1}^{\infty}\frac{\max\left\{\lambda(t),\lambda^{\eta}(t)\right\}}{e^{qt\eta}}\,dt &= \sum_{k=1}^{\infty}\int_{k}^{k+1}\frac{\max\left\{\lambda(t),\lambda^{\eta}(t)\right\}}{e^{qt\eta}}\,dt\\ &\geq e^{-q}\sum_{k=1}^{\infty}\frac{\max\left\{\lambda(k),\lambda^{\eta}(k)\right\}}{e^{qk\eta}}.\end{aligned}$$

Lemma 3 is proved. □

Proof of Lemma 4. Set $p_k = e^{-\lambda}\frac{\lambda^k}{k!}$, $k = 0, 1, \ldots$. First let us consider the case $0 < \lambda < 1$:

$$\mathbf{E}(\Pi^{\eta}) = \sum_{k=1}^{\infty}k^{\eta}p_k \leq \sum_{k=1}^{\infty}k^{[\eta]+1}p_k = \mathbf{E}(\Pi^{[\eta]}+1). \tag{4.2}$$

Let f denote the moment generating function of the Poisson distribution with parameter λ and let $f^{(i)}$ denote its derivative of order i. Then $f(t) = e^{\lambda(t-1)} = e^{-\lambda}\cdot e^{\lambda t}$. Thus $f^{(i)}(t) = \lambda^i\cdot f(t)$, whence $f^{(i)}(1) = \lambda^i$, $i \geq 1$. Since the moment of any order j is a linear combination of derivatives $f'(1), f''(1), \ldots, f^{(j)}(1)$, the expectation $\mathbf{E}(\Pi^{[\eta]+1})$ is a linear combination of $\lambda, \lambda^2, \ldots, \lambda^{[\eta]+1}$. Using the triangular inequality, we get $\mathbf{E}(\Pi^{[\eta]+1}) \leq c\lambda$ for some constant $c > 0$ if $\lambda < 1$, that, taking into account (4.2) proves the second part of Lemma 4.

Now let $\lambda \geq 1$. As before, set $m = [\lambda]$. If $0 < \eta < 1$, then

$$\begin{aligned}\mathbf{E}(\Pi^\eta) &= \sum_{k=0}^{\infty} k^\eta p_k = \sum_{1\leq k\leq m} k^\eta p_k + \sum_{k>m} k^\eta p_k \leq m^\lambda + \sum_{1\leq k\leq m} p_k + \sum_{k>m} k^\eta p_k \\ &= m^\lambda + \sum_{1\leq k\leq m} p_k + \lambda \sum_{k>m} \frac{p_{k-1}}{k^{1-\eta}} \leq m^\lambda \sum_{1\leq k\leq m} p_k + \frac{\lambda}{m^{1-\eta}} \sum_{k>m} p_{k-1} \\ &= m^\lambda \left(\sum_{1\leq k\leq m} p_k + \sum_{k>m-1} p_k \right) = m^\lambda (1 + P(\Pi = m)) \leq 2m^\eta. \end{aligned} \tag{4.3}$$

Since $m \leq \lambda$, the first part of Lemma 4 is proved for all $0 < \eta < 1$. If $\eta \geq 1$, then

$$\mathbf{E}(\Pi^\eta) = \sum_{k=1}^{\infty} k^\eta p_k = \lambda \sum_{k=1}^{\infty} k^{\eta-1} p_{k-1} = \lambda \sum_{k=0}^{\infty} (k+1)^{\eta-1} p_k \leq 2^{\eta-1} \lambda \sum_{k=0}^{\infty} k^{\eta-1} p_k.$$

Continuing these estimations, we obtain

$$\mathbf{E}(\Pi^\eta) \leq d\lambda^{[\eta]} \sum_{k=0}^{\infty} k^{\eta-[\eta]} p_{k-1}, \quad d = 2^{(\eta-1)+(\eta-2)+\cdots+(\eta-[\eta])}.$$

If $\eta \in \mathcal{N}$, then this inequality coincides with the statement of the first part of Lemma 4. If $\eta \notin \mathcal{N}$, we use lemma 4 for the case of $0 < \eta < 1$ and get

$$\mathbf{E}(\Pi^\eta) \leq d\lambda^{[\eta]} \cdot 2\lambda^{\eta-[\eta]},$$

which completes the proof of Lemma 4. Thus Lemma 4 is proved. □

Proof of Lemma 5. Set $p_k = e^{-\lambda} \frac{\lambda^k}{k!}$, $k = 0, 1, \ldots$. First consider the case of $0 < \lambda < 1$:

$$\mathbf{E}(\Pi^\eta) = \sum_{k=0}^{\infty} k^\eta p_k > p_1 = e^{-\lambda} \lambda \geq \frac{\lambda}{e},$$

that proves the second part of Lemma 5. Now let $\lambda \geq 1$. Set $m = [\lambda]$. Starting from the case $0 < \eta < 1$:

$$\begin{aligned}\mathbf{E}(\Pi^\eta) &= \sum_{k=0}^{\infty} k^\eta p_k = \sum_{1\leq k\leq m} k^\eta p_k + \sum_{k>m} k^\eta p_k = \lambda \sum_{1\leq k\leq m} \frac{p_{k-1}}{k^{1-\eta}} + \sum_{k>m} k^\eta p_k \\ &\geq \frac{\lambda}{m^{1-\eta}} \sum_{1\leq k\leq m} p_{k-1} + m^\eta \sum_{k>m} p_k = \frac{\lambda}{m^{1-\eta}} \sum_{0\leq k\leq m-1} p_k + m^\eta \sum_{k>m} p_k\end{aligned}$$

$$\geq m^{\eta}\left(\sum_{0\leq k\leq m-1} p_k + \sum_{k>m} p_k\right) = m^{\eta}(1 - P(\Pi = m))$$
$$\geq 2^{-\eta}\lambda^{\eta}(1 - \mathbf{P}(\Pi = m)). \tag{4.4}$$

We show that there exists a constant $c > 0$ such that

$$1 - \mathbf{P}(\Pi = m) \geq c \tag{4.5}$$

if $\lambda \geq 1$.

Using Stirling's formula:

$$\mathbf{P}(\Pi = m) = e^{-\lambda}\frac{\lambda^m}{m!} = e^{-\lambda}\frac{\lambda^m}{\sqrt{2\pi m}\cdot m^m \cdot e^{-m+\theta_m}},$$

where $0 < \theta_m < \frac{1}{12m}$. Since

$$\left(\frac{\lambda}{m}\right)^m \leq \left(\frac{m+1}{m}\right)^m = (1 + \frac{1}{m})^m \leq e, \quad e^{-\lambda+m+\theta_m} \leq 1,$$

we have

$$\mathbf{P}(\Pi = m) \leq \frac{e}{\sqrt{2\pi m}} \leq \frac{e}{\sqrt{4\pi}} < 1, \quad m \geq 2$$

If $1 \leq \lambda \leq 2$, then

$$\mathbf{P}(\Pi = m) = \mathbf{P}(\Pi = 1) = e^{-\lambda}\lambda < 1.$$

This implies (4.5). Inequality (4.5) proves Lemma 5 for $0 < \eta < 1$. In order to complete the proof of Lemma 5, consider the case of $\eta \geq 1$:

$$\mathbf{E}(\Pi^{\eta}) = \sum_{k=1}^{\infty} k^{\eta} p_k = \lambda \sum_{k=1}^{\infty} k^{\eta-1}\frac{\lambda^{k-1}}{(k-1)!}e^{-\lambda}$$
$$= \lambda \sum_{k=0}^{\infty}(k+1)^{\eta-1}\frac{\lambda^k}{k!}e^{\lambda} \geq \lambda \sum_{k=0}^{\infty} k^{\eta-1}\frac{\lambda^k}{k!}e^{-\lambda} = \lambda\mathbf{E}(\Pi^{\eta-1}).$$

Continuing with these estimates, we obtain

$$\mathbf{E}(\Pi^{\eta}) \geq \lambda^{[\eta]}\sum_{k=0}^{\infty} k^{\eta-[\eta]}\frac{\lambda^k}{k!}e^{\lambda}.$$

This inequality coincides with the second part of Lemma 5 if $\eta \in \mathcal{N}$. For $\eta \notin \mathcal{N}$, we use Lemma 5 for $\eta < 1$:

$$E(\Pi^{\eta}) \geq \lambda^{[\eta]} \cdot c\lambda^{\eta-[\eta]}.$$

Note that constant c is the same as in the case of $0 < \eta < 1$, that is, it does not depend on η. Thus, Lemma 5 is proved. □

Lemma 6 (compound Poisson process)**.** *Let $X_t^{(2)}$ be a compound Poisson process represented in the form of* (3.1) *where N_t is a simple Poisson process whose intensity satisfies* (2.2)*. We also assume that the random variables ξ_k, $k \ge 1$, are independent, identically distributed, and such that*

$$\mathbf{E}\xi_k^{\eta \vee 1} < \infty$$

for some $\eta > 0$. We further assume that the process N_t and the sequence ξ_k, $k \ge 1$, are independent. Then

$$\mathbf{E}\left(\sup_{s \ge t} \frac{\left|X_s^{(2)}\right|}{s}\right)^{\eta} < \infty.$$

for all $t > 0$.

Proof. We provide the proof for the case of $\eta = 1$. Other cases are proved similarly. Put $\mu = \mathbf{E}\xi_1$. Then

$$\begin{aligned}
\mathbf{E}\left(\sup_{t \ge 1} \frac{1}{t} \sum_{k \le N_t} \xi_k\right) &= \int_0^\infty \mathbf{P}\left(\sup_{t \ge 1} \frac{1}{t} \sum_{k \le N_t} \xi_k \ge x\right) dx \\
&\le \int_0^\infty \sum_{m=0}^\infty \mathbf{P}\left(\sup_{2^m \le t \le 2^{m+1}} \frac{1}{t} \sum_{k \le N_t} c \ge x\right) dx \\
&\le \int_0^\infty \sum_{m=0}^\infty \mathbf{P}\left(\frac{1}{2^m} \sum_{k \le N_{2^{m+1}}} \xi_k \ge x\right) dx \\
&= \int_0^\infty \sum_{m=0}^\infty \sum_{l=0}^\infty \mathbf{P}(N_{2^{m+1}} = l) \mathbf{P}\left(\frac{1}{2^m} \sum_{k \le l} \xi_k \ge x\right) dx \\
&= \sum_{m=0}^\infty \sum_{l=0}^\infty \mathbf{P}(N_{2^{m+1}} = l) \mathbf{P}\left(\frac{1}{2^m} \sum_{k \le l} \xi_k \ge x\right) dx \\
&= \sum_{m=0}^\infty \sum_{l=0}^\infty \mathbf{P}(N_{2^{m+1}} = l) \mathbf{E}\left[\frac{1}{2^m} S_l\right] \\
&= \mu \sum_{m=0}^\infty \sum_{l=0}^\infty \mathbf{P}(N_{2^{m+1}} = l) \frac{l}{2^m} \\
&= \mu \sum_{m=0}^\infty \frac{1}{2^m} \sum_{l=0}^\infty l \mathbf{P}(N_{2^{m+1}} = l) = \mu \sum_{m=0}^\infty \frac{1}{2^m} \mathbf{E} N_{2^{m+1}} \\
&= \mu \sum_{m=0}^\infty \frac{\lambda(2^{m+1})}{2^m} \mathbf{E} N_{2^{m+1}} = 2\mu \sum_{m=1}^\infty \frac{\lambda(2^m)}{2^m} < \infty. \qquad \square
\end{aligned}$$

Lemma 7 (martingale)**.** *Let Y_t be a stochastic process such that $|Y_t|$ is a right continuous submartingale. Let $q > 0$, $\eta > 1$, $T > 0$. If* (2.3) *holds, then*

$$\mathbf{E}\left(\sup_{t\geq T}\frac{|Y_t|}{e^{qt}}\right)^\eta < \infty$$

for all $T > 0$.

Remark 3. Our assumption that $|Y_t|$ is a right continuous submartingale is weaker than the assumption that Y_t is a right continuous submartingale and, moreover, that Y_t is a martingale.

The following two properties are well known for submartingales. Namely, if Y_t is submartingale and $\mathbf{E}|Y_t|^\eta < \infty$ for some $\eta > 1$, then

$$\mathbf{E}|Y_t|^\eta \quad \text{is nondecreasing in } t. \tag{4.6}$$

Lemma 8 ([1], p. 140, Theorem 6.2.16)**.** *Let Y_t, $t \geq 0$, be a right continuous submartingale. Let A be a certain subset of real numbers and let $Y^*(\omega) = \sup_{t\in A} Y_t(\omega)$. If $p > 1$, then $Y^* \in \mathbf{L}_p$ if and only if*

$$\sup_{t\in A}\|Y_t\|_{\mathbf{L}_p} < \infty.$$

In particular, if $\frac{1}{r} = 1 - \frac{1}{p}$, then

$$\|Y^*\|_{\mathbf{L}_p} \leq r\sup_{t\in A}\|Y_t\|_{\mathbf{L}_p}.$$

In fact, we only need the following particular case of Lemma 8 corresponding to the case of $A = [k, k+1]$ and for $\left|X_t^{(3)}\right|$ instead of Y_t:

$$\mathbf{E}\left(\sup_{k\leq t\leq k+1}\left|X_t^{(3)}\right|\right)^\eta \leq \left(1-\frac{1}{\eta}\right)^{-\eta}\mathbf{E}\left|X_{k+1}^{(3)}\right|^\eta. \tag{4.7}$$

Proof of Lemma 7. Without loss of generality we assume that $T = 1$. It follows from (4.7) that

$$\begin{aligned}
\mathbf{E}\left(\sup_{t\geq 1}\frac{|Y_t|}{e^{qt}}\right)^\eta &\leq \sum_{k=1}^\infty \mathbf{E}\left(\sup_{k\leq t\leq k+1}\frac{|Y_t|}{e^{qt}}\right)^\eta \leq \sum_{k=1}^\infty e^{-qk\eta}\mathbf{E}\left(\sup_{k\leq t\leq k+1}|Y_t|\right)^\eta \\
&\leq \left(1-\frac{1}{\eta}\right)^{-\eta}\sum_{k=1}^\infty e^{-qk\eta}\mathbf{E}\,|Y_{k+1}|^\eta \\
&\leq \left(1-\frac{1}{\eta}\right)^{-\eta} e^{2q\eta}\sum_{k=1}^\infty e^{-q(k+1)\eta}\mathbf{E}\,|Y_k|^\eta
\end{aligned}$$

$$\leq \left(1 - \frac{1}{\eta}\right)^{-\eta} e^{2q\eta} \sum_{k=1}^{\infty} \int_k^{k+1} \frac{\mathbf{E}\,|Y_t|^\eta}{e^{qt\eta}}\,dt$$
$$= \left(1 - \frac{1}{\eta}\right)^{-\eta} e^{2q\eta} \int_1^{\infty} \frac{\mathbf{E}\,|Y_t|^\eta}{e^{qt\eta}}\,dt < \infty. \qquad \square$$

Remark 4. Lemma 7 can also be proved for the case of $\eta = 1$. However the condition for this case is as follows

$$\int_1^\infty \frac{\mathbf{E}\,|Y_t| \ln^+ |Y_t|}{e^{qt}}\,dt < \infty$$

where $\ln^+ z = \ln(1+z)$ for $z \geq 0$. The idea of the proof remains the same, but another Doob's inequality applies.

5. Proof of Theorem 3.1

First we write down the Lévy–Itô decomposition (2.1). Then we put $P_s = X_s^{(1)}$, $Q_s = X_s^{(2)}$, and $R_s = X_s^{(3)}$. The assumptions of Lemma 1 hold for P_s, Q_s, and R_s by Lemmas 2, 6, and 7, respectively. Therefore Theorem 3.1 follows from Lemma 1.

References

[1] R. J. Elliott and P. E. Kopp, Mathematics of Financial Markets, 2nd edition, *Springer, Berlin*, 2005.

[2] A. E. Kyprianou and B. A. Surya, On the Novikov-Shiryaev optimal stopping problems in continuous time, *Electron. Comm. Probab.* **10** (2005), 146–151.

[3] E. Mordecki, Optimal stopping and perpetual options for Lévy processes, *Finance Stoch.* **6**, no. 4 (2002), 473–493.

[4] A. Novikov and A. Shiryaev, On a solution of the optimal stopping problem for processes with independent increments, *Stochastics* **79**, no. 3–4 (2007), 393–406.

[5] A. A. Novikov and A. N. Shiryaev, On an effective solution of the optimal stopping problem for random walks, *Theory Probab. Appl.* **49**, no. 2 (2004), 344–354.

ANDRIY KLESOV
NATIONAL TECHNICAL UNIVERSITY OF UKRAINE "KPI"
PEREMOGY AVENUE, 56
03037 KIEV
UKRAINE

E-mail: andrey.klesov@gmail.com

(Received June 22, 2009; revised March 29, 2010)