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How short might be the longest run in a dynamical coin tossing sequence

By PÁL RÉVÉSZ

Dedicated to the 100th anniversary of the birthday of Béla Gyires

Abstract. Let $X_1, X_2, \dots$ denote i.i.d. random bits each taking the values 1 and 0 with respective probabilities 1/2 and 1/2. A well-known theorem of Erdős and Rényi [2] describes the limit distribution of the length of the longest contiguous run of ones in $X_1, X_2, \dots, X_n$ as $n \to \infty$. Benjamini et al. ([1] Theorem 1.4) demonstrated the existence of unusual times, provided that the bits undergo equilibrium dynamics in time. In fact they prove that the dynamics produces much longer runs than the original model. In the present paper we study the length of the shortest run in the presence of the dynamics.

1. Introduction

Let $X_1, X_2, \dots$ be a sequence of i.i.d.r.v.'s with distribution

$$\mathbf{P}\{X_1 = 1\} = \mathbf{P}\{X_1 = 0\} = \frac{1}{2}.$$

Further let

$$S_n = X_1 + X_2 + \dots + X_n, \qquad I(n,a) = \max_{0 \le k \le n-a} (S_{k+a} - S_k), \quad (a \le n)$$

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and Z_n be the largest integer for which $I(n, Z_n) = Z_n$, i.e. Z_n is the length of the longest run of heads in a Bernoulli trial up to n.

A characterization of the sequence $\{Z_n\}$ was given by Erdős and Révész at 1976 ([3]).

Theorem 1 (Révész [6], Theorem 7.2)**.** *Assume that* $\sum_{n=1}^{\infty} 2^{-a_n} < \infty$. *Then*

$$Z_n \le a_n \quad \text{a.s.} \tag{1.1}$$

if n is large enough.

Assume that $\sum_{n=1}^{\infty} 2^{-a_n} = \infty$. *Then*

$$Z_n \ge a_n \quad \text{a.s. i.o.} \tag{1.2}$$

$$Z_n \le \log n - \log_3 n + \log_2 e - 1 + \varepsilon \quad \text{a.s. i.o.} \tag{1.3}$$

$$Z_n > \log n - \log_3 n + \log_2 e - 2 - \varepsilon \quad \text{a.s.} \tag{1.4}$$

if n is large enough where log *is the logarithm with base* 2, $\log_p$ *is the p-th iterated logarithm, ε is an arbitrary positive number.*

Slightly stronger versions of (1.3) and (1.4) are given in [4] and [7]. Note that (1.1) and (1.2) imply

$$Z_n \le \log n + \log_2 n + (1+\varepsilon)\log_3 n \quad \text{a.s.}$$

if n is large enough and

$$Z_n \ge \log n + \log_2 n + \log_3 n \quad \text{a.s. i.o.}$$

In [3] we also investigated the following question: what is length of the longest run containing at most one (or at most T ($T = 1, 2, \dots$)) 0's. Let $Z_n^{(T)}$ be the largest integer for which

$$I(n, Z_n^{(T)}) \ge Z_n^{(T)} - T.$$

A generalization of Theorem 1 is the following:

Theorem 2 (Révész [6], Theorem 7.4)**.** *Let $\{a_n\}$ be a sequence of positive numbers and let*

$$A_T(\{a_n\}) = \sum_{n=1}^{\infty} a_n^T 2^{-a_n}.$$

Assume that

$$A_T(\{a_n\}) < \infty.$$

Then

$$Z_n^{(T)} \le a_n \quad \text{a.s.}$$

if n is large enough.

Assume that

$$A_T(\{a_n\}) = \infty.$$

Then

$$Z_n^{(T)} \ge a_n \quad \text{a.s. i.o.}$$

Further

$$Z_n^{(T)} \le \log n + T\log_2 n - \log_3 n - \log T! + \log_2 e - 1 + \varepsilon \quad \text{a.s. i.o.}$$

$$Z_n^{(T)} \ge \log n + T\log_2 n - \log_3 n - \log T! + \log_2 e - 2 - \varepsilon \quad \text{a.s.}$$

Given an array $\{X_n^{(j)} : j,\ n = 0,1,2,\dots\}$ of i.i.d.r.v.'s with distribution

$$\mathbf{P}\{X_0^{(0)} = 1\} = \mathbf{P}\{X_0^{(0)} = 0\} = \frac{1}{2}$$

and for each n an independent Poisson process $\{\psi_n^{(j)}\}$ $(j \ge 0)$ of rate 1. Define

$$X_n(t) = X_n^{(j)} \quad \text{for} \quad \psi_n^{(j-1)} \le t < \psi_n^{(j)} \quad (j = 1,2,\dots)$$

where $\psi_n^{(0)} = 0$ for each n.

Let

$$S_n(t) = \sum_{k=1}^{n} X_k(t), \qquad I(n,a,t) = \max_{0\le k\le n-a} (S_{k+a}(t) - S_k(t)), \quad (0 \le a \le n)$$

$Z_n(t)$ be the largest integer for which $I(n, Z_n(t), t) = Z_n(t)$ i.e. $Z_n(t)$ is the length of the longest head-run at t up to n.

Similarly let $Z_n(t,T)$ be the largest integer for which $I(n, Z_n(t,T), t) \ge Z_n(t,T) - T$.

A characterization of $\{Z_n(t),\ n = 1,2,\dots,\ 0 \le t \le 1\}$ is given by Benjamini *et al.* (2003 [1]). The properties of $\{Z_n(t,T),\ n = 1,2,\dots,\ 0 \le t \le 1,\ T = 1,2,\dots\}$ were studied by Khoshnevisan *et al.* (2007 [5]).

Theorem 3 (Benjamini *et al.* [1] Theorem 1.4). *Assume that* $\sum_{n=1}^{\infty} a_n 2^{-a_n} < \infty$. *Then*

$$\forall t \in (0,1) \text{ we have } Z_n(t) \le a_n \quad \text{a.s. if } n \text{ is large enough.} \tag{1.5}$$

Assume that $\sum_{n=1}^{\infty} a_n 2^{-a_n} = \infty$. *Then*

$$\exists t \in (0,1) \text{ such that } Z_n(t) \ge a_n \quad \text{a.s. i.o.} \tag{1.6}$$

Note that (1.5) and (1.6) imply

$$Z_n(t) \le \log n + 2\log_2 n + (1+\varepsilon)\log_3 n \quad \text{a.s.}$$

for any $t \in (0,1)$ if n is large enough and $\exists t \in (0,1)$ such that

$$Z_n(t) \ge \log n + 2\log_2 n + \log_3 n \quad \text{a.s. i.o.}$$

Theorem 4 (Khoshnevisan *et al.* [5])**.** *Assume that*

$$A_{T+1}(\{a_n\}) < \infty.$$

Then

$$Z_n(t,T) \le a_n \quad \text{a.s.}$$

for any $0 \le t \le 1$ *if* n *large enough.*

Assume that

$$A_{T+1}(\{a_n\}) = \infty.$$

Then there exists a t $(0 \le t \le 1)$ *such that*

$$Z_n(t,T) \ge a_n \quad \text{a.s. i.o.}$$

The main goal of this paper is to prove the following:

Theorem 5. *For any* $0 \le t \le 1 \le 1/8$ *we have*

$$Z_n(t) \ge \log n - \log_3 n - C \quad \text{a.s.}$$

if n *is large enough and*

$$C > 6 + \log \frac{4}{e-1}.$$

2. Combinatorial lemmas

Let $\nu(2), \nu(3), \dots, \nu(K)$ be i.i.d.r.v.'s with

$$\mathbf{P}\{\nu(2) = i\} = \frac{1}{K} \quad (i = 1, 2, \dots K).$$

Let $X(1,1), X(1,2), \dots, X(1,K), Y_2, Y_3, \dots, Y_K$ be i.i.d.r.v.'s with

$$\mathbf{P}\{X(1,1) = 1\} = \mathbf{P}\{X(1,1) = 0\} = \frac{1}{2}$$

being independent on $\nu(2),\nu(3),\dots,\nu(K)$.

Let

$$X(2,i)=\begin{cases}X(1,i) & \text{if } \nu(2)\neq i,\\ Y_2 & \text{if } \nu(2)=i,\end{cases}\qquad X(3,i)=\begin{cases}X(2,i) & \text{if } \nu(3)\neq i,\\ Y_3 & \text{if } \nu(3)=i,\end{cases}$$

$$X(k,i)=\begin{cases}X(k-1,i) & \text{if } \nu(k)\neq i,\\ Y_k & \text{if } \nu(k)=i,\end{cases}\quad \text{where } (i=1,2,\dots,K,\ k=4,5,\dots,K).$$

$$B(k)=\bigcap_{i=1}^{K}\{X(k,i)=1\},\qquad p(K)=\mathbf{P}\Big\{\bigcup_{k=1}^{K}B(k)\Big\}.$$

Lemma 1.

$$\frac{(e-1)K}{e2^{K+1}}\le p(K)\le\frac{K}{2^K}. \tag{2.1}$$

Proof. The upper part of (2.1) is trivial. Now we turn to its lower part. Clearly

$$p(K)=\mathbf{P}\{B(1)\}+\mathbf{P}\{\overline{B(1)}\,B(2)\}+\mathbf{P}\{\overline{B(1)}\,\overline{B(2)}\,B(3)\}\\ +\dots+\mathbf{P}\{\overline{B(1)}\,\overline{B(2)}\dots\overline{B(K-1)}\,B(K)\},$$

(where $\overline{A}$ is the complement of A),

$$\mathbf{P}\{B(1)\}=\frac{1}{2^K},\qquad bp\{\overline{B(1)}B(2)\}=\frac{1}{2^{K+1}},$$

$$\mathbf{P}\{\overline{B(1)}\,\overline{B(2)}\,B(3)\}=\mathbf{P}\big\{\nu(3)\neq\nu(2),\ X(3,\nu(3))=X(2,\nu(2))=1,\\ X(1,\nu(3))=0,\ X(1,k)=1 \text{ if } \{k\neq\nu(2) \text{ and } k\neq\nu(3)\}\big\}+\mathbf{P}\big\{\nu(3)=\nu(2),\\ X(3,\nu(3))=1,\ X(2,\nu(2))=X(1,\nu(2))=0,\ X(1,k)=1 \text{ if } k\neq\nu(2)\big\}\\ =\mathbf{P}\{\nu(3)\neq\nu(2)\}\frac{1}{2^{K+1}}+\mathbf{P}\{\nu(3)=\nu(2)\}\frac{1}{2^{K+2}}.$$

Since

$$\mathbf{P}\{\nu(2)\neq\nu(3)\}=1-\mathbf{P}\{\nu(2)=\nu(3)\}=1-\frac{1}{K}$$

we have

$$\mathbf{P}\{\overline{B(1)}\,\overline{B(2)}\,B(3)\}=\frac{1}{2^{K+1}}\left(1-\frac{1}{2K}\right).$$

Observe that

$$\big\{X(1,\nu(k))=0,\ X(k,m)=1\ (m=1,2,\dots,K),\ \nu(j)\neq\nu(k)\ (j=2,3,\dots,k-1)\big\}\\ \subset\overline{B(1)}\,\overline{B(2)}\dots\overline{B(k-1)}B(k).$$

Since

$$\mathbf{P}\big\{X(1,\nu(k))=0,\ X(k,m)=1\ (m=1,2,\dots,K),\ \nu(j)\neq\nu(k)\\ (j=2,3,\dots,k-1)\big\}=\frac{1}{2^{K+1}}\left(1-\frac{1}{K}\right)^{k-2},$$

we have

$$\mathbf{P}\{\overline{B(1)}\,\overline{B(2)}\dots\overline{B(k-1)}\,B(k)\}\geq\frac{1}{2^{K+1}}\left(1-\frac{1}{K}\right)^{k-2}\qquad(k=2,3,\dots,K)$$

and

$$p(K)\geq\frac{1}{2^K}+\sum_{k=2}^{K}\frac{1}{2^{K+1}}\left(1-\frac{1}{K}\right)^{k-2}=\frac{1}{2^K}+\frac{K}{2^{K+1}}\left(1-\left(1-\frac{1}{K}\right)^{K-1}\right)\\ \geq\frac{e-1}{e}\frac{K}{2^{K+1}}.$$

Hence Lemma 1 is proved.

Let $\mu(2),\mu(3),\dots,\mu(2K)$ be i.i.d.r.v.'s with

$$\mathbf{P}\{\mu(2)=i\}=\frac{1}{2K}\qquad(i=1,2,\dots,2K).$$

Let $Z(1,1),Z(1,2),\dots,Z(1,2K),U_2,U_3,\dots,U_{2K}$ be i.i.d.r.v.'s with

$$\mathbf{P}\{Z(1,1)=1\}=\mathbf{P}\{Z(1,1)=0\}=\frac{1}{2}$$

being independent on $\mu(2),\mu(3),\dots,\mu(2K)$.

Let

$$Z(2,i)=\begin{cases}Z(1,i) & \text{if } \mu(2)\neq i,\\ U_2 & \text{if } \mu(2)=i,\end{cases}\qquad Z(k,i)=\begin{cases}Z(k-1,i) & \text{if } \mu(k)\neq i,\\ U_k & \text{if } \mu(k)=i\end{cases}$$

$(k=3,4,\dots,2K,\ i=1,2,\dots,2K)$. □

Lemma 2. *Let*

$$q(2K)=\mathbf{P}\{\exists k,j:1\leq k\leq 2K,\ 1\leq j\leq K \textit{such that } Z(k,j)=\dots\\ =Z(k,j+K-1)=1\}.$$

Then

$$\Delta\frac{K^2}{2^K}\leq q(2K)\leq\frac{K^2}{2^K}\tag{2.2}$$

where

$$\Delta=\frac{e-1}{4e^2}=0.0581\dots.$$

Proof. The upper part of (2.2) is trivial. Now we turn to its lower part. Let

$$A_j = \{\exists k : 1 \le k \le 2K,\ Z(k,j) = Z(k,j+1) = \dots = Z(k,j+K-1) = 1\},$$
$$C_j = \{Z(1,j) = Z(2,j) = \dots = Z(2K,j) = 0\} \quad (j = 1,2,\dots,K).$$

Then

$$\begin{aligned} q(2K) = \mathbf{P}\left\{\bigcup_{j=1}^{K} A_j\right\} &= \mathbf{P}\{A_1\} + \mathbf{P}\{\overline{A_1}A_2\} + \dots + \mathbf{P}\{\overline{A_1}\,\overline{A_2}\dots\overline{A_{K-1}}A_K\} \\ &\ge \mathbf{P}\{A_1\} + \mathbf{P}\{C_1\overline{A_1}A_2\} + \dots + \mathbf{P}\{C_{K-1}\overline{A_1}\,\overline{A_2}\dots\overline{A_{K-1}}A_K\} \\ &= \mathbf{P}\{A_1\} + \mathbf{P}\{C_1A_2\} + \dots + \mathbf{P}\{C_{K-1}A_K\} = \mathbf{P}\{A_1\} + (K-1)\mathbf{P}\{C_1A_2\}. \end{aligned}$$

Clearly

$$\{Z(1,1) = 0,\ \mu_2 > 1,\ \mu_3 > 1, \dots, \mu_{2K} > 1\} \subset C_1,$$

$$\mathbf{P}\{C_1\} \ge \frac{1}{2}\left(1 - \frac{1}{2K}\right)^{2K-1} \ge \frac{1}{2e}.$$

Let

$$V_{2K} = \#\{k :\ k \le 2K,\ \mu_k \le K\}.$$

Then

$$V_{2K} \ge (1 - o(1))K \quad \text{a.s.}$$

Hence by Lemma 1 we have

$$\mathbf{P}\{A_1\} = \mathbf{P}\{A_2\} \ge \frac{e-1}{e}\frac{K}{2^{K+1}}$$

which implies (2.2). □

3. Proof of Theorem 5

Let

$$\Psi(K) = \{\psi_n^{(j)},\ n = 1,2,\dots,2K,\ j = 1,2,\dots\}.$$

Let $\tau_1 < \tau_2 < \dots$ be the ordered elements of $\Psi(K)$ i.e. τ_1 is the smallest element of $\Psi(K)$, τ_2 is the second smallest and so on. Note $\tau = \{\tau_1, \tau_2, \dots\}$ is a Poisson process of parameter $2K$.

Define n_1 by the equation

$$\psi_{n_1}^{(1)} = \tau_1.$$

Clearly

$$\mathbf{P}\{n_1 = i\} = \frac{1}{2K} \quad (i = 1, 2, \dots, 2K).$$

Similarly define $(n_2, j_2), (n_3, j_3), \dots$ by the equations

$$\psi_{n_2}^{(j_2)} = \tau_2, \quad \psi_{n_3}^{(j_3)} = \tau_3, \dots \tag{3.1}$$

Clearly $(n_2, j_2), (n_3, j_3), \dots$ are uniquely defined a.s. by (3.1) and we have

$$\mathbf{P}\{n_\ell = i\} = \frac{1}{2K} \quad (i = 1, 2, \dots, 2K,\ \ell = 1, 2, \dots, 2K).$$

Introduce the following definitions:

(i) Let $J(K)$ be the set of those j's ($j \leq 2K$) for which there exists an $n \in [1, K]$ such that for any $t \in [\tau_j, \tau_{j+1})$ we have $X_m(t) = 1$ for each $m \in [n, n+K-1]$.

(ii) Let $F(K)$ be the event that the set $J(K) \neq \emptyset$.

(iii) For any $t > 0$ let $j(t)$ be the largest integer for which $\tau_j \leq t$.

If $J(K)$ consists of exactly one element then for any $j \leq K$ we have

$$\mathbf{P}\{j \in J(K)\} = \frac{1}{K}, \tag{3.2}$$

if it consists of more than one element then for any $j \leq K$ we have

$$\mathbf{P}\{j \in J(K)\} \geq \frac{1}{K}. \tag{3.3}$$

Lemma 3. *Let $j \leq K$ and $\tau_j \leq t < \tau_{j+1}$. Then we have*

$$\mathbf{P}\left\{\tau_{j(t)+1} - \tau_{j(t)} \geq \tau_{j(t)+1} - t \geq \frac{1}{2K}\right\} = \frac{1}{e}, \tag{3.4}$$

$$\mathbf{P}\{Z_{2K}(t) \geq K \mid F(K)\} = \mathbf{P}\{j(t) \in J(K) \mid F(K)\} \geq \frac{1}{K}. \tag{3.5}$$

Consequently

$$\mathbf{P}\left\{Z_{2K}(t) \geq K,\ \tau_{j(t)+1} - t \geq \frac{1}{2K} \mid F(K)\right\} \geq \frac{1}{eK}. \tag{3.6}$$

Proof. Since τ is a Poisson process of parameter $1/2K$, we have (3.4). (3.5) follows from (3.2) and (3.3). (3.4) and (3.5) combined imply (3.6). □

Lemma 4.

$$\mathbf{P}\{F(K)\} \geq \Delta K^2 2^{-K}. \tag{3.7}$$

Proof. It is a trivial consequence of Lemma 2. □

Lemma 5. *Let $j \leq K$ and $\tau_j \leq t < \tau_{j+1}$. Then we have*

$$\mathbf{P}\left\{\tau_{j(t)+1} - t \geq \frac{1}{2K},\ Z_{2K}(t) \geq K\right\} \geq \frac{\Delta}{e} K 2^{-K}. \tag{3.8}$$

Proof. Clearly the r.v.'s $Z_{2K}(t)$ and $\tau_{j(t)+1} - \tau_{j(t)}$ are independent. Hence

$$\begin{aligned}
&\mathbf{P}\left\{\tau_{j(t)+1} - t \geq \frac{1}{2K},\ Z_{2K}(t) \geq K\right\} \\
&\quad = \mathbf{P}\left\{\tau_{j(t)+1} - t \geq \frac{1}{2K},\ Z_{2K}(t) \geq K,\ F(K)\right\} \\
&\quad = \mathbf{P}\left\{\tau_{j(t)+1} - t \geq \frac{1}{2K}\right\} \mathbf{P}\{Z_{2K}(t) \geq K \mid F(K)\} \mathbf{P}\{F(K)\}.
\end{aligned}$$

Hence we have (3.8) by (3.4), (3.5) and (3.6). □

Introduce two further definitions:

(iv) Let $Z(\ell, K, t)$ be the largest integer for which there exists an integer

$$b \in [2\ell K, (2\ell + 1)K)$$

such that

$$S_{b+K}(t) - S_b(t) = Z(\ell, K, t)$$

that is $Z(\ell, K, t)$ is the length of the longest head-run at

$$t \in [2\ell K, (2\ell + 2)K).$$

(v) Define $\Psi(\ell, K)$, $\tau_i(\ell)$, $F(\ell, K)$, $j(\ell, t)$ just like above using the block

$$[2\ell K, (2\ell + 2)K)$$

instead of the block $(0, 2K)$.

Let

$$Q := Q(K, N) = \#\{\ell : \ell \leq [N/2K] \text{ for which } F(\ell, K) \text{ holds true}\}$$

and let

$$K = \log N - \log_3 N - C.$$

Lemma 6.

$$\mathbf{P}\{Q(K,N) \le \Delta 2^{C-2} \log N \log_2 N\} \le N^{-O(1)\log_2 N}. \tag{3.9}$$

Proof. Apply the large deviation theorem (Révész [6], Theorem 2.3) with

$$n = [N/2K], \quad p = \frac{\Delta K^2}{2^K}, \quad \varepsilon = \frac{p}{2}.$$

Since

$$\frac{np}{2} = \Delta 2^{C-2} \log N \log_2 N, \qquad \frac{n\varepsilon^2}{2pq\left(1+\frac{\varepsilon}{2pq}\right)^2} = O(1) \log N \log_2 N,$$

we have (3.9). □

Let $D(t) = D(t,N,K)$ be the event that there exists an $\ell = \ell(t)$ $(0 \le \ell \le [N/2K])$ for which $Z(\ell,K,t) \ge K$ and $\tau_{j(t)+1}(\ell) - t \ge 1/2K$.

Lemma 7.

$$\mathbf{P}\{\overline{D(t)}\} \le \frac{2}{(\log N)^{\Delta 2^{C-2} e^{-1}}}. \tag{3.10}$$

Proof. By (3.6) and Lemma 6 we have

$$\begin{aligned}
bp\{\overline{D(t)}\} &= \mathbf{EP}\{\overline{D(t)} \mid Q\} \ge \mathbf{E}\Big(1 - \frac{1}{2K}\Big)^Q \\
&= \mathbf{E}\bigg(\Big(1-\frac{1}{eK}\Big)^Q \mid Q \le \Delta 2^{C-2} \log N \log_2 N\bigg) \mathbf{P}\{Q \le \Delta 2^{C-2} \log N \log_2 N\} \\
&\quad + \mathbf{E}\bigg(\Big(1-\frac{1}{eK}\Big)^Q \mid Q > \Delta 2^{C-2} \log N \log_2 N\bigg) \mathbf{P}\{Q > \Delta 2^{C-2} \log N \log_2 N\} \\
&\le \mathbf{P}\{Q \le \Delta 2^{C-2} \log N \log_2 N\} + \Big(1 - \frac{1}{eK}\Big)^{\Delta 2^{C-2} \log N \log_2 N} \le \\
&\le N^{-O(1)\log_2 N} + \exp\Big(-\frac{\Delta 2^{C-2}}{e} \log_2 N\Big) \le 2(\log N)^{-\Delta 2^{C-2} e^{-1}}.
\end{aligned}$$

Hence we have (3.10). □

Lemma 8.

$$\mathbf{P}\left\{\tau_{2K} < \frac{1}{8}\right\} \le e^{-BK}. \tag{3.11}$$

Consequently

$$\mathbf{P}\left\{\min_{\ell \le [N/2K]} \tau_{2K}(\ell) < \frac{1}{8}\right\} \le e^{-BK} \tag{3.12}$$

for a $B > 0$.

Proof. Since

$$\frac{d}{dx}\mathbf{P}\{2K\tau_K < x\} = \frac{x^{K-1}e^{-x}}{(K-1)!},$$

by an easy calculation we have (3.11). □

Lemma 9. *The event*

$$\bigcap_{m=1}^{2K} D\left(\frac{m}{2K}\right)$$

implies that for any $m \le 2K$ there exists an $\ell \le [N/2K]$ such that

$$\tau_{j(m/2K)+1}(\ell) - \frac{m}{2K} \ge \frac{1}{2K}$$

and

$$Z(\ell, K, t) \ge K$$

if

$$t \le \min_{\ell \le [N/2K]} \tau_{2K}(\ell).$$

Proof. It follows straight from the definition of $D(\cdot)$. □

Proof of Theorem 5. By Lemma 7 we have

$$\mathbf{P}\left\{\bigcap_{j=1}^{2K} D\left(\frac{j}{2K}\right)\right\} = 1 - \mathbf{P}\left\{\bigcup_{j=1}^{2K} \overline{D\left(\frac{j}{2K}\right)}\right\} \ge 1 - 2K\mathbf{P}\{\overline{D(t)}\}$$

$$\ge 1 - 2K2(\log N)^{\Delta 2^{C-2}e^{-1}} \ge 1 - 4(\log N)^{-\Delta 2^{C-2}e^{-1}+1}.$$

Let

$$C > 3 + \log\frac{e}{\Delta}$$

and

$$N_m = 2^m \quad (m = 1, 2, \dots).$$

Then

$$\Delta 2^{C-2}e^{-1} - 1 > 1$$

and the event

$$\bigcap_{j=1}^{2K} D\left(\frac{j}{2K}\right) \quad (K = K_m = \log N_m - \log_3 N_m - C)$$

occurs with probability 1 if m is large enough. Consequently for any $t \le 1/8$

$$Z_{N_m}(t) \ge \log N_m - \log_3 N_m - C$$

if m is large enough which easily implies Theorem 5. □

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PÁL RÉVÉSZ
INSTITUT FÜR STATISTIK
UND WAHRSCHEINLICHKEITSTHEORIE
TECHNISCHE UNIVERSITÄT WIEN
WIEDNER HAUPTSTRASSE 8-10/107
A-1040 WIEN
AUSTRIA

E-mail: revesz@ci.tuwien.ac.at

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