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Closed form results for BMAP/G/1 vacation model with binomial type disciplines

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Dedicated to the 100th anniversary of the birthday of Béla Gyires

Abstract. The paper deals with the analysis of BMAP/G/1 vacation models. We apply a formerly introduced two-step methodology separating the analysis into service discipline independent and service discipline dependent parts. In this paper we investigate the later analysis part for the binomial-gated and the binomial-exhaustive service disciplines, for which a specific form functional equation can be established for the vector probability generating function of the stationary number of customers at start of vacations. We provide new results for the model with these disciplines. These are the closed-form expressions of the vector probability generating function of the stationary number of customers at start of vacations, which are applied to express the mean of the stationary number of customers at an arbitrary moment.

1. Introduction

Queueing models with server vacation have been intensively studied in the past. These models are applied in the analysis of computer and manufacturing systems as well as in telecommunication models. In these models the server occasionally takes a vacation period, in which no customer is served. For details

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on vacation models and for their analysis we refer to the comprehensive survey of Doshi [1] and to the excellent book of Takagi [2].

The batch Markovian arrival process (BMAP) introduced by Lucantoni [3] enables more realistic and more accurate traffic modeling than the (batch) Poisson process. Consequently analysis of queueing models with BMAP attracted a great attention. The vast majority of the analyzed BMAP/G/1 queueing models exploit the underlying $M/G/1$-type structure of the model, i.e., that the embedded Markov chain at the customer departure epochs is of $M/G/1$-type [4], in which the block size in the transition probability matrix equals to the number of phases of the BMAP. Hence most of the analysis of BMAP/G/1 vacation models are based on the standard matrix analytic-method pioneered by Neuts [5] and further extended by many others (see e.g., [6]).

Chang and Takine [7] applied the factorization property (presented by Chang et al. [8]) to get analytical results for the vector probability generating function (vector GF) of the stationary queue length and its factorial moments for models with exhaustive discipline.

Due to the complexity of these models the results have usually a numeric part, i.e. no closed-form solutions are given.

The principal goal of this paper is to provide closed-form solution for BMAP/G/1 vacation model with binomial-gated and binomial-exhaustive service disciplines.

Our analysis is based on a two-step methodology introduced formerly by the authors [9]. This methodology separates the analysis into service discipline independent and service discipline dependent parts. In this paper we are dealing with the service discipline specific part of the analysis.

The contributions of this paper are the discipline specific analysis and the new results for the model with binomial-gated and binomial-exhaustive disciplines. To the best knowledge of the authors there are no results available for these models.

For the model with binomial-gated and binomial-exhaustive disciplines a specific form functional equation can be established for the vector GF of the stationary number of customers at start of vacations. We give a closed-form solution of this vector GF by applying a recursive method. This method is based on a convergence property of a class of matrix probability generating function (matrix GF) series, which we show in the Appendix. This property is a generalization of the corresponding convergence property of series of scalar probability generating functions (PGFs), see Kim, Chang and Chae [10]. Applying the closed-form solution in our previous result [9] leads to the expression for the mean of the stationary number of customers at an arbitrary moment.

The rest of this paper is organized as follows. In Section II we introduce the model and the notations. The summary of the necessary part of previous results follows in Section III. The system equations for the model with both binomial-gated and binomial-exhaustive disciplines are established in Section IV. The closed form solution for the model with binomial-gated and binomial-exhaustive disciplines are derived in Section V and Section VI, respectively. The consideration on the special case of the model with exhaustive discipline closes the paper in Section VII.

2. Model and notation

2.1. BMAP process. We give a brief summary on the BMAP related definitions and notations. For more details we refer to [3].

$\Lambda(t)$ denotes the number of arrivals in $(0,t]$. $J(t)$ is the state of a background continuous-time Markov chain (CTMC) at time t, which is referred to as phase and phase process, respectively. The BMAP batch arrival process is characterized by $\{(\Lambda(t), J(t)); t \geq 0\}$ bivariate CTMC on the state space $(\Lambda(t), J(t))$; where $(\Lambda(t) \in \{0,1,\dots\},\ J(t) \in \{1,2,\dots,L\})$. Its infinitesimal generator is:

$$\begin{pmatrix} \mathbf{D}_0 & \mathbf{D}_1 & \mathbf{D}_2 & \mathbf{D}_3 & \dots \\ \mathbf{0} & \mathbf{D}_0 & \mathbf{D}_1 & \mathbf{D}_2 & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{D}_0 & \mathbf{D}_1 & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D}_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $\mathbf{0}$ is an $L \times L$ matrix and $\{\mathbf{D}_k; k \geq 0\}$ is a set of $L \times L$ matrices.

$\mathbf{D}_0$ and $\{\mathbf{D}_k; k \geq 1\}$ govern the transitions corresponding to no arrivals and to batch arrivals with size k, respectively. The irreducible infinitesimal generator of the phase process is $\mathbf{D} = \sum_{k=0}^{\infty} \mathbf{D}_k$. Let $\boldsymbol{\pi}$ be the stationary probability vector of the phase process. Then $\boldsymbol{\pi}\mathbf{D} = \mathbf{0}$ and $\boldsymbol{\pi}\mathbf{e} = 1$ uniquely determine $\boldsymbol{\pi}$, where $\mathbf{e}$ is the column vector having all elements equal to one. $\widehat{\mathbf{D}}(z)$, the matrix generating function of $\mathbf{D}_k$ is defined as

$$\widehat{\mathbf{D}}(z) = \sum_{k=0}^{\infty} \mathbf{D}_k z^k, \quad |z| \leq 1.$$

The stationary arrival rate of the BMAP,

$$\lambda = \boldsymbol{\pi} \frac{d}{dz} \widehat{\mathbf{D}}(z)\Big|_{z=1} \mathbf{e} = \boldsymbol{\pi} \sum_{k=0}^{\infty} k \mathbf{D}_k \mathbf{e},$$

is supposed to be positive and finite.

2.2. The BMAP/G/1 queue with server vacation. Batch of customers arrive to the infinite buffer queue according to a BMAP process described by $\widehat{\mathbf{D}}(z)$. The service times are independent and identically distributed. B, $B(t)$, b denote the service time r.v., its cumulated distribution function and its first moment, respectively. The mean service time is positive and finite, $0 < b < \infty$.

The server occasionally takes vacations, in which no customer is served. After finishing the vacation the server continues to serve the queue. If the server finds the queue empty upon return from vacation then it immediately takes the next vacation. The vacation periods are independent and identically distributed. V, $V(t)$, v denote the vacation time r.v., its cumulated distribution function and its mean, respectively. The mean vacation time is positive and finite, $0 < v < \infty$. We define the *cycle time* as a service period and a vacation period together. The server utilization is $\rho = \lambda b$ and the system is stable. On the vacation model we impose the following assumptions:

A.1 Independence property: The arrival process, the customer service times and the length of the vacation periods are independent.

A.2 Nonpreemtive service property: The service is nonpreemtive. Hence the service of the actual customer is finished before the server goes to vacation.

In the following $[\mathbf{Y}]_{i,j}$ stands for the i,j-th element of matrix $\mathbf{Y}$. Similarly $[\mathbf{y}]_j$ denotes the j-th element of vector $\mathbf{y}$.

We define matrix $\mathbf{A}_k$, whose (i,j)-th element denotes the conditional probability that during a customer service time the number of arrivals is k and the final phase of the BMAP is j, given that the initial phase of the BMAP is i. That is, for $k \geq 0$, $1 \leq i,\ j \leq L$,

$$[\mathbf{A}_k]_{i,j} = P\{\Lambda(B) = k,\ J(B) = j | J(0) = i\}.$$

The matrix GF $\widehat{\mathbf{A}}(z)$ is defined as $\widehat{\mathbf{A}}(z) = \sum_{k=0}^{\infty} \mathbf{A}_k z^k$ and it can be expressed explicitly as ([3])

$$\widehat{\mathbf{A}}(z) = \int_{t=0}^{\infty} e^{\widehat{\mathbf{D}}(z)t} dB(t). \tag{1}$$

Matrix GF $\widehat{\mathbf{A}}(z)$ has the following properties:

- the matrices $\mathbf{A}_k$ are nonnegative,
- $\widehat{\mathbf{A}}(1)\mathbf{e} = \mathbf{e}$, i.e. $\widehat{\mathbf{A}}(1)$ is stochastic,
- $\boldsymbol{\pi}\widehat{\mathbf{A}}(1) = \boldsymbol{\pi}$, which can be shown by the help of Taylor expansion of (1) at $z = 1$.

Similar to the quantities associated with the service period, we define matrix $\mathbf{U}_k$, whose elements, for $k \geq 0$, $1 \leq i,\ j \leq L$, are

$$[\mathbf{U}_k]_{i,j} = P\{\Lambda(V) = k,\ J(V) = j | J(0) = i\}.$$

We also define the matrix GF as

$$\widehat{\mathbf{U}}(z) = \sum_{k=0}^{\infty} \mathbf{U}_k z^k = \int_{t=0}^{\infty} e^{\widehat{\mathbf{D}}(z)t} dV(t),$$

which has the following properties:

- the matrices $\mathbf{U}_k$ are nonnegative,
- $\widehat{\mathbf{U}}(1)\mathbf{e} = \mathbf{e}$, i.e. $\widehat{\mathbf{U}}(1)$ is stochastic,
- $\boldsymbol{\pi}\widehat{\mathbf{U}}(1) = \boldsymbol{\pi}$, which can be shown by the help of Taylor expansion of $\widehat{\mathbf{U}}(z)$ at $z = 1$.

Vacation models are distinguished by their (service) discipline that is the set of rules determining the end of the service. Commonly applied service disciplines are, e.g., the exhaustive, the gated, the binomial-exhaustive, the binomial-gated, the limited-N, etc. In case of exhaustive discipline, the server continues serving the customers until the queue is emptied. Under gated discipline only those customers are served, which are present at the beginning of the service period. In case of binomial-gated discipline, which was introduced by Levy [11], every customer present at the beginning of the service period is served with probability p, for $0 < p \le 1$. In the binomial-exhaustive discipline (Boxma [12]) the binomial limitation is applied on the busy periods associated to the customers presents at the beginning of the service period, and hence each of them occurs with probability q, for $0 < q \le 1$. In case of limited-N discipline at most N customers are served among the customers, which are present at the beginning of the service period.

2.3. Embedding matrix GFs. Let $\widehat{\mathbf{X}}(z) = \sum_{k=0}^{\infty} \mathbf{X}_k z^k,\ |z| \le 1$ be $L \times L$ matrix GF with the following properties

- matrices $\mathbf{X}_k$ are nonnegative,
- $\widehat{\mathbf{X}}(1)\mathbf{e} = \mathbf{e}$, i.e. $\widehat{\mathbf{X}}(1)$ is stochastic.

Let $\widehat{\mathbf{h}}(z) = \sum_{k=0}^{\infty} \mathbf{h}_k z^k,\ |z| \le 1$ be $1 \times L$ vector GF such that vectors $\mathbf{h}_k$ are nonnegative and $\widehat{\mathbf{h}}(1)\mathbf{e} = 1$. We introduce the notation

$$\widehat{\mathbf{h}}\big(\widehat{\mathbf{X}}(z)\big) = \sum_{k=0}^{\infty} \mathbf{h}_k \widehat{\mathbf{X}}(z)^k, \quad |z| \le 1. \tag{2}$$

Each element of $\widehat{\mathbf{h}}(\widehat{\mathbf{X}}(z))$ is also power series of z as in case of a scalar probability generating function. Therefore for the convergence of $\widehat{\mathbf{h}}(\widehat{\mathbf{X}}(z))$, for $|z| \le 1$, it is enough to show that $|\widehat{\mathbf{h}}(\widehat{\mathbf{X}}(z))|$ is upper limited. This can be seen as follows:

$$|\widehat{\mathbf{h}}\big(\widehat{\mathbf{X}}(z)\big)|\mathbf{e} \le \widehat{\mathbf{h}}\big(\widehat{\mathbf{X}}(|z|)\big)\mathbf{e} = \sum_{k=0}^{\infty} \mathbf{h}_k \widehat{\mathbf{X}}(|z|)^k \mathbf{e} \le \sum_{k=0}^{\infty} \mathbf{h}_k \widehat{\mathbf{X}}(1)^k \mathbf{e} = 1.$$

Additionally $\widehat{\mathbf{h}}(\widehat{\mathbf{X}}(z))$ is also a $1 \times L$ vector GF having the same properties as vector GF $\widehat{\mathbf{h}}(z)$.

Similarly let $\widehat{\mathbf{H}}(z) = \sum_{k=0}^{\infty} \mathbf{H}_k z^k,\ |z| \le 1$ be an $L \times L$ matrix GF with the following properties

- the matrices $\mathbf{H}_k$ are nonnegative,
- $\widehat{\mathbf{H}}(1)\mathbf{e} = \mathbf{e}$, i.e. $\widehat{\mathbf{H}}(1)$ is stochastic.

We introduce the notation

$$\widehat{\mathbf{H}}\big(\widehat{\mathbf{X}}(z)\big) = \sum_{k=0}^{\infty} \mathbf{H}_k \widehat{\mathbf{X}}(z)^k, \quad |z| \le 1. \tag{3}$$

Note that each element of $\widehat{\mathbf{H}}(\widehat{\mathbf{X}}(z))$ is also power series of z as in case of a scalar probability generating function. Therefore for the convergence of $\widehat{\mathbf{H}}(\widehat{\mathbf{X}}(z))$, for $|z| \le 1$, it is enough to show that $|\widehat{\mathbf{H}}(\widehat{\mathbf{X}}(z))|$ is upper limited. This can be seen as follows:

$$\left|\widehat{\mathbf{H}}\big(\widehat{\mathbf{X}}(z)\big)\right| \mathbf{e} \le \widehat{\mathbf{H}}\big(\widehat{\mathbf{X}}(|z|)\big)\mathbf{e} = \sum_{k=0}^{\infty} \mathbf{H}_k \widehat{\mathbf{X}}(|z|)^k \mathbf{e} \le \sum_{k=0}^{\infty} \mathbf{H}_k \widehat{\mathbf{X}}(1)^k \mathbf{e} = \mathbf{e}.$$

Furthermore $\widehat{\mathbf{H}}(\widehat{\mathbf{X}}(z))$ gives an $L \times L$ matrix GF having the same properties as matrix GF $\widehat{\mathbf{H}}(z)$.

3. Service discipline independent stationary relations

In this section we present general relations which do not depend on the specific service discipline.

Let $N(t)$ denote the number of customers in the system at time t. We define $\widehat{\mathbf{q}}(z)$ as the vector GF of the stationary number of customers by its element as

$$[\widehat{\mathbf{q}}(z)]_j = \lim_{t\to\infty} \sum_{n=0}^{\infty} P\{N(t) = n,\ J(t) = j\} z^n, \quad |z| \le 1.$$

Furthermore, t_k^m denotes the start of vacation in the k-th cycle. We define $\widehat{\mathbf{m}}(z)$ as the vector GF of the stationary number of customers at start of vacations by its elements as

$$[\widehat{\mathbf{m}}(z)]_j = \lim_{k\to\infty} \sum_{n=0}^{\infty} P\{N(t_k^m) = n,\ J(t_k^m) = j\} z^n, \quad |z| \le 1.$$

Additionally we introduce the notations $\mathbf{D}^{(i)}$, $\mathbf{A}^{(i)}$, $\mathbf{U}^{(i)}$, $\mathbf{q}^{(i)}$ and $\mathbf{m}^{(i)}$, $i \ge 1$ for the i-th derivatives of $\widehat{\mathbf{D}}(z)$, $\widehat{\mathbf{A}}(z)$, $\widehat{\mathbf{U}}(z)$, $\widehat{\mathbf{q}}(z)$ and $\widehat{\mathbf{m}}(z)$ at $z = 1$, respectively. We also use the notations $\mathbf{A} = \widehat{\mathbf{A}}(1)$, $\mathbf{U} = \widehat{\mathbf{U}}(1)$ and $\mathbf{m} = \widehat{\mathbf{m}}(1)$.

As usual, $\mathbf{I}$ stands for the identity matrix.

Theorem 1. *In the vacation model the following service discipline independent relations hold:*

- *the vector GF of the stationary number of customers at an arbitrary instant can be expressed as*

$$\widehat{\mathbf{q}}(z)\widehat{\mathbf{D}}(z)\big(z\mathbf{I} - \widehat{\mathbf{A}}(z)\big) = \frac{\widehat{\mathbf{m}}(z)(\widehat{\mathbf{U}}(z) - \mathbf{I})}{v}(1-\rho)(z-1)\widehat{\mathbf{A}}(z), \tag{4}$$

- *the mean of the stationary number of customers at an arbitrary instant is given by*

$$\begin{aligned}
\mathbf{q}^{(1)} = {} & \frac{\mathbf{m}^{(1)}}{\lambda v}\left(\mathbf{U}^{(1)}\mathbf{e}\boldsymbol{\pi} + (\mathbf{U}-\mathbf{I})\big(\mathbf{A}^{(1)} - \mathbf{A}(\mathbf{D}+\mathbf{e}\boldsymbol{\pi})^{-1}\mathbf{D}^{(1)}\big)\mathbf{e}\boldsymbol{\pi}\right) \\
& + \frac{\mathbf{m}}{\lambda v}\left(\frac{1}{2}\mathbf{U}^{(2)}\mathbf{e}\boldsymbol{\pi} + \frac{1}{2}(\mathbf{U}-\mathbf{I})\mathbf{A}^{(2)}\mathbf{e}\boldsymbol{\pi} + \mathbf{U}^{(1)}\mathbf{A}^{(1)}\mathbf{e}\boldsymbol{\pi}\right) \\
& - \frac{\mathbf{m}}{\lambda v}\left(\mathbf{U}^{(1)}\mathbf{A} + (\mathbf{U}-\mathbf{I})\mathbf{A}^{(1)}\right)(\mathbf{D}+\mathbf{e}\boldsymbol{\pi})^{-1}\mathbf{D}^{(1)}\mathbf{e}\boldsymbol{\pi} \\
& + \frac{\mathbf{m}}{\lambda v}\left(\mathbf{U}^{(1)}\mathbf{A}\mathbf{e}\boldsymbol{\pi} + (\mathbf{U}-\mathbf{I})\mathbf{A}^{(1)}\mathbf{e}\boldsymbol{\pi}\right)\left(\frac{\mathbf{C_2}\mathbf{e}\boldsymbol{\pi}}{\lambda} + (1-\rho)\mathbf{C_1}\right) \\
& + \frac{\mathbf{m}}{\lambda v}(\mathbf{U}-\mathbf{I})\mathbf{A}(\mathbf{D}+\mathbf{e}\boldsymbol{\pi})^{-1}\big(\lambda\mathbf{I} - \mathbf{D}^{(1)}\mathbf{e}\boldsymbol{\pi}\big)\left(\frac{\mathbf{C_2}\mathbf{e}\boldsymbol{\pi}}{\lambda} + (1-\rho)\mathbf{C_1}\right) \\
& + \boldsymbol{\pi}\left(\frac{\mathbf{A}^{(2)}\mathbf{e}\boldsymbol{\pi}}{2(1-\rho)} - \big(\mathbf{I} - \mathbf{A}^{(1)}\big)\mathbf{C_1}\right),
\end{aligned} \tag{5}$$

where matrices $\mathbf{C_1}$ and $\mathbf{C_2}$ are defined as

$$\mathbf{C_1} = (\mathbf{I} - \mathbf{A} + \mathbf{e}\boldsymbol{\pi})^{-1}\left(\frac{\mathbf{A}^{(1)}\mathbf{e}\boldsymbol{\pi}}{(1-\rho)} + \mathbf{I}\right), \quad \mathbf{C_2} = \mathbf{D}^{(1)}(\mathbf{D}+\mathbf{e}\boldsymbol{\pi})^{-1}\mathbf{D}^{(1)} - \frac{1}{2}\mathbf{D}^{(2)}.$$

Proof. The proof of the theorem can be found in [9]. □

Note that the contribution of the concrete service discipline to relations (4) and (5) is incorporated by the quantities $\widehat{\mathbf{m}}(z)$, $\mathbf{m}^{(1)}$ and $\mathbf{m}$.

4. System equations

To obtain the unknowns $\widehat{\mathbf{m}}(z)$, $\mathbf{m}$, $\mathbf{m}^{(1)}$ in (4) and (5), we setup the system equations for the model with binomial-gated and binomial-exhaustive disciplines.

Let t_k^f denote the end of vacation in the k-th cycle. The $1 \times L$ stationary probability vector $\mathbf{f}_n$ is defined by its elements as

$$[\mathbf{f}_n]_j = \lim_{k\to\infty} P\big\{N(t_k^f) = n,\ J(t_k^f) = j\big\}.$$

The corresponding vector GF, $\widehat{\mathbf{f}}(z)$, is defined as

$$\widehat{\mathbf{f}}(z) = \sum_{n=0}^{\infty} \mathbf{f}_n z^n, \quad |z| \le 1.$$

Theorem 2. *In the vacation model with binomial-gated discipline* $(0 < p \le 1)$ *the following system equation holds:*

$$\widehat{\mathbf{m}}(z) = \widehat{\mathbf{m}}\big(p\widehat{\mathbf{A}}(z) + (1-p)\mathbf{I}z\big)\widehat{\mathbf{U}}\big(p\widehat{\mathbf{A}}(z) + (1-p)\mathbf{I}z\big). \tag{6}$$

Proof. In the binomial-gated discipline each customer, who is present at the end of the vacation, gets service with probability p $(0 < p \le 1)$ independently from the other ones. Assuming, that the number of customers present at the end of the vacation is n, for $n \ge 0$, the probability that $0 \le k \le n$ customers get service is $\binom{n}{k} p^k (1-p)^{n-k}$. Each of the k customers getting service generates a random population of customers arriving during its service time, whose matrix GF is $\widehat{\mathbf{A}}(z)$. Each of the other $n-k$ customers remains present at start of the next vacation, thus each of them can be described by matrix GF $\mathbf{I}z$. Since $\mathbf{I}$ and $\widehat{\mathbf{A}}(z)$ commute, independently of the selection order of the customers the matrix GF of the number of customers at start of the next vacation is $\widehat{\mathbf{A}}^k(z)(\mathbf{I}z)^{n-k}$. Using it and applying the binomial theorem yields to the matrix GF of the number of customers at start of the vacation, given that there is n customers present at end of the previous vacation, as

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \widehat{\mathbf{A}}^k(z)(\mathbf{I}z)^{n-k} = \big(p\widehat{\mathbf{A}}(z) + (1-p)\mathbf{I}z\big)^n.$$

Unconditioning gives the the governing relation for transition $f \to m$ of the vacation model as

$$\widehat{\mathbf{m}}(z) = \sum_{n=0}^{\infty} \mathbf{f}_n \big(p\widehat{\mathbf{A}}(z) + (1-p)\mathbf{I}z\big)^n = \widehat{\mathbf{f}}\big(p\widehat{\mathbf{A}}(z) + (1-p)\mathbf{I}z\big). \tag{7}$$

The number of customers at end of the vacation equals the sum of those present at start of the vacation and those who arrived during the vacation period. Utilizing the independence property **A.1** the governing relation for transition $m \to f$ of the vacation model can be expressed as

$$\widehat{\mathbf{f}}(z) = \widehat{\mathbf{m}}(z)\widehat{\mathbf{U}}(z). \tag{8}$$

Combining (7) and (8) results in the statement. $\square$

We define matrix $\mathbf{G}$ as the minimal non-negative solution of $\mathbf{G} = \sum_{k=0}^{\infty} \mathbf{A}_k \mathbf{G}^k$. To give the stochastic interpretation of this matrix we introduce the homogenous bivariate Markov chain $\{(N(t_\ell^d), J(t_\ell^d)); \ell \in \{1, \dots\}\}$ on the state space $\{0, 1, \dots\} \times \{1, 2, \dots, L\}$, where t_ℓ^d denotes the ℓ-th customer departure epoch of the corresponding BMAP/GI/1 queue (having the same arrival and departure processes as the vacation model) for $\ell \geq 1$. $N(t_\ell^d)$ is referred to as level of the process, for $\ell \geq 1$. Then the (i, j)-th element of matrix $\mathbf{G}$ is interpreted as the probability that starting from state $(n+1, i)$ at the first passage to one level down the Markov chain hits the state (n, j), $n \in 0, 1, 2, \dots$, $1 \leq i, j \leq L$. Matrix $\mathbf{G}$ can be computed from its defining equation $\mathbf{G} = \sum_{k=0}^{\infty} \mathbf{A}_k \mathbf{G}^k$, e.g., by applying the standard algorithm (Lucantoni [3]). Matrix $\mathbf{G}$ is stochastic if $\rho < 1$. For more details on matrix $\mathbf{G}$ we refer to Neuts [4].

Theorem 3. *In the vacation model with binomial-exhaustive discipline* $(0 < q \leq 1)$ *the following system equation holds:*

$$\widehat{\mathbf{m}}(z) = \widehat{\mathbf{m}}(q\mathbf{G} + (1-q)\mathbf{I}z)\widehat{\mathbf{U}}(q\mathbf{G} + (1-q)\mathbf{I}z). \tag{9}$$

Proof. In the binomial-exhaustive discipline busy period can be associated to each customer, who is present at end of the vacation. Each busy period occurs with probability q $(0 < q \leq 1)$ independently from the other ones. If a busy period occurs the above introduced Markov chain goes one level down. It can be described by a matrix (GF) $\mathbf{G}$, since none of the customers belonging to the busy period remains at start of the next vacation. We apply a similar argument as in case of the binomial-gated discipline. Thus assuming, that the number of customers present at end of the vacation is n, for $n \geq 0$, the matrix GF of number of customers at start of the vacation is given as

$$\sum_{k=0}^{n} \binom{n}{k} q^k (1-q)^{n-k} \mathbf{G}^k (\mathbf{I}z)^{n-k} = (q\mathbf{G} + (1-q)\mathbf{I}z)^n.$$

Unconditioning gives the the governing relation for transition $f \to m$ of the vacation model as

$$\widehat{\mathbf{m}}(z) = \sum_{n=0}^{\infty} \mathbf{f}_n (q\mathbf{G} + (1-q)\mathbf{I}z)^n = \widehat{\mathbf{f}}(q\mathbf{G} + (1-q)\mathbf{I}z). \tag{10}$$

The governing relation for transition $m \to f$ of the vacation model is the same as for the binomial-gated discipline, since it is discipline independent. Therefore it is given by (8) again. Combining (10) and (8) results in the statement. □

5. Closed form result for vacation model with binomial-gated discipline

In order to express the unknowns $\widehat{\mathbf{m}}(z)$, $\mathbf{m}$, $\mathbf{m}^{(1)}$ in (4) and (5) we solve the system equation (6).

Let $\mathbf{e_i}$ stand for the row vector, whose i-th element equals to 1 and its other elements are 0. In addition let $\mathbf{Y} \mid\mid \mathbf{x}$ denote the matrix $\mathbf{Y}$ with the last column replaced by the column vector $\mathbf{x}$.

Theorem 4. *In the stable vacation model with binomial-gated discipline* $(0 < p \leq 1)$ *the closed-form expression of the vector GF of stationary number of customers at start of vacations,* $\widehat{\mathbf{m}}(z)$, *is given as:*

$$\widehat{\mathbf{m}}(z) = \widehat{\mathbf{m}}(\mathbf{G}) \prod_{r=1}^{\infty} \widehat{\mathbf{U}}\left(\sum_{l=0}^{r-1} (1-p)^l p \widehat{\mathbf{A}}(\widehat{\mathbf{\Delta}}_{r-1-l}(z)) + (1-p)^r \mathbf{I}z \right), \tag{11}$$

where $\widehat{\mathbf{m}}(\mathbf{G})$ *is a solution of a system of linear equation*

$$\widehat{\mathbf{m}}(\mathbf{G}) = \mathbf{e_L}\Big((\mathbf{I} - \widehat{\mathbf{U}}(\mathbf{G})) \mid\mid \mathbf{e} \Big)^{-1}, \tag{12}$$

and the series of matrix GFs $\widehat{\mathbf{\Delta}}_k(z)$ *is defined recursively as*

$$\begin{aligned} \widehat{\mathbf{\Delta}}_0(z) &= \mathbf{I}z, \quad |z| \leq 1, \\ \widehat{\mathbf{\Delta}}_{k+1}(z) &= p\widehat{\mathbf{A}}(\widehat{\mathbf{\Delta}}_k(z)) + (1-p)\mathbf{I}\widehat{\mathbf{\Delta}}_k(z), \quad k \geq 0. \end{aligned} \tag{13}$$

Proof. Due to the convergence of (2) and (3) for $|z| \leq 1$ system equation (6) remains valid when z is substituted by a matrix GF $\widehat{\mathbf{X}}(z)$.

Now we apply the series of matrix GFs, $\widehat{\mathbf{\Delta}}_k(z)$, for $k \geq 0$, defined in (13).

Substituting z by matrix GF $\widehat{\mathbf{\Delta}}_k(z)$, for $k \geq 0$ in (6) leads to

$$\widehat{\mathbf{m}}(\widehat{\mathbf{\Delta}}_k(z)) = \widehat{\mathbf{m}}\big(\widehat{\mathbf{\Delta}}_{k+1}(z)\big)\widehat{\mathbf{U}}(\widehat{\mathbf{\Delta}}_{k+1}(z)), \quad k \geq 0. \tag{14}$$

Solving (14) by recursive substitution for $k \geq 0$ yields

$$\widehat{\mathbf{m}}(z) = \widehat{\mathbf{m}}\Big(\lim_{k\to\infty} \widehat{\mathbf{\Delta}}_k(z)\Big) \prod_{r=1}^{\infty} \widehat{\mathbf{U}}(\widehat{\mathbf{\Delta}}_r(z)). \tag{15}$$

Applying recursive substitution in (13) results in

$$\begin{aligned}
\widehat{\mathbf{\Delta}}_1(z) &= p\widehat{\mathbf{A}}(z) + (1-p)\mathbf{I}z,\\
\widehat{\mathbf{\Delta}}_2(z) &= p\widehat{\mathbf{A}}(\widehat{\mathbf{\Delta}}_1(z)) + (1-p)p\widehat{\mathbf{A}}(z) + (1-p)^2\mathbf{I}z,\\
&\ \vdots\\
\widehat{\mathbf{\Delta}}_k(z) &= \sum_{l=0}^{k-1}(1-p)^l p\widehat{\mathbf{A}}(\widehat{\mathbf{\Delta}}_{k-1-l}(z)) + (1-p)^k\mathbf{I}z.
\end{aligned} \tag{16}$$

We apply lemma 1 (see in Apendix) for determining $\lim_{k\to\infty} \widehat{\mathbf{\Delta}}_k(z)$, since the properties of $\widehat{\mathbf{A}}(z)$, $0 < p$ and due to stability of the model $\rho < 1$ ensure that its assumptions hold. Note, that matrix $\mathbf{G}$ in (34), in the context of the vacation model with binomial-gated discipline (in the whole Section 5), is the minimal non-negative solution of $\mathbf{G} = p\widehat{\mathbf{A}}(\mathbf{G}) + (1-p)\mathbf{G}$ and it can be computed from this defining equation. Applying (34) and (16) in (15) gives (11).

Next we determine the term $\widehat{\mathbf{m}}(\mathbf{G})$ in (11). Taking the limit $k \to \infty$ on (14) results in

$$\widehat{\mathbf{m}}\Big(\lim_{k\to\infty} \widehat{\mathbf{\Delta}}_k(z)\Big) = \widehat{\mathbf{m}}\Big(\lim_{k\to\infty} \widehat{\mathbf{\Delta}}_k(z)\Big)\widehat{\mathbf{U}}\Big(\lim_{k\to\infty} \widehat{\mathbf{\Delta}}_k(z)\Big). \tag{17}$$

Applying (34) in (17) and rearranging yields

$$\widehat{\mathbf{m}}(\mathbf{G})(\mathbf{I} - \widehat{\mathbf{U}}(\mathbf{G})) = 0. \tag{18}$$

The same system of linear equation holds for the phase probability vector at start of vacations ($\mathbf{m}$) in the corresponding vacation model with exhaustive discipline (see (30)). Due to stability of the model, (30) and $\mathbf{me} = 1$ uniquely determine $\mathbf{m}$, which implies that matrix $(\mathbf{I} - \widehat{\mathbf{U}}(\mathbf{G}))$ has rank $L - 1$.

Therefore an additional relation is required to make the system of linear equation (18) complete. For this purpose we add the normalization condition

$$\widehat{\mathbf{m}}(\mathbf{G})\mathbf{e} = 1. \tag{19}$$

The solution of system of linear equation (18) and (19) for $\widehat{\mathbf{m}}(\mathbf{G})$ is given by (12), which completes the proof. □

We point out here that the essence of the method of series of matrix GFs, which is applied in the proof of Theorem 4, is the use of the argument of $\widehat{\mathbf{m}}(\,)$ on the r.h.s. of the system equation (6) in the recursive definition of $\widehat{\mathbf{\Delta}}_{k+1}(z)$ in (13). This appropriate definition implies that $\widehat{\mathbf{m}}(\widehat{\mathbf{\Delta}}_k(z))$ can be expressed in terms of $\widehat{\mathbf{m}}(\widehat{\mathbf{\Delta}}_{k+1}(z))$ (for $k \geq 0$ in (14)), which allows the recursive solution of $\widehat{\mathbf{m}}(z)$.

The unknowns $\mathbf{m}$, $\mathbf{m}^{(1)}$ in (5) can be determined by setting $z = 1$ in (11) and taking the first derivative of (11) at $z = 1$, respectively, which result in

$$\mathbf{m} = \widehat{\mathbf{m}}(\mathbf{G}) \prod_{r=1}^{\infty} \widehat{\mathbf{U}}\Big(\sum_{l=0}^{r-1} (1-p)^l p \widehat{\mathbf{A}}\big(\widehat{\mathbf{\Delta}}_{r-1-l}(1)\big) + (1-p)^r \mathbf{I} \Big),$$

$$\begin{aligned} \mathbf{m}^{(1)} = \widehat{\mathbf{m}}(\mathbf{G}) \sum_{i=1}^{\infty} \prod_{k=1}^{i-1} \widehat{\mathbf{U}}\Big(\sum_{l=0}^{k-1} (1-p)^l p \widehat{\mathbf{A}}\big(\widehat{\mathbf{\Delta}}_{k-1-l}(1)\big) + (1-p)^k \mathbf{I} \Big) \\ \frac{d\widehat{\mathbf{U}}\big(\sum_{l=0}^{i-1} (1-p)^l p \widehat{\mathbf{A}}(\widehat{\mathbf{\Delta}}_{i-1-l}(z)) + (1-p)^i \mathbf{I} z \big)}{dz}\Bigg|_{z=1} \\ \prod_{r=i+1}^{\infty} \widehat{\mathbf{U}}\Big(\sum_{l=0}^{r-1} (1-p)^l p \widehat{\mathbf{A}}\big(\widehat{\mathbf{\Delta}}_{r-1-l}(1)\big) + (1-p)^r \mathbf{I} \Big), \end{aligned} \tag{20}$$

where the empty product equals to 1 as usual.

6. Closed form result for vacation model with binomial-exhaustive discipline

Similarly to the case of binomial-gated discipline we solve the system equation (9) for $\widehat{\mathbf{m}}(z)$, however here we start with a proposition.

Proposition 1. *Let us consider the stable vacation model with binomial-exhaustive discipline having parameter* $0 < q < 1$*. A series of matrix GFs is defined recursively as*

$$\begin{aligned} \widehat{\mathbf{\Theta}}_0(z) &= \mathbf{I} z, \quad |z| \leq 1, \\ \widehat{\mathbf{\Theta}}_{k+1}(z) &= q\mathbf{G} + (1-q)\mathbf{I}\widehat{\mathbf{\Theta}}_k(z), \quad k \geq 0. \end{aligned} \tag{21}$$

The series of matrix GFs $\widehat{\mathbf{\Theta}}_k(z)$ *uniquely converges to stochastic matrix* $\mathbf{G}$*, which is independent of* z*:*

$$\lim_{k\to\infty} \widehat{\mathbf{\Theta}}_k(z) = \mathbf{G}. \tag{22}$$

Proof. Applying recursive substitution in (21) results in

$$\begin{aligned}\widehat{\mathbf{\Theta}}_1(z) &= q\mathbf{G} + (1-q)\mathbf{I}z,\\ \widehat{\mathbf{\Theta}}_2(z) &= q\mathbf{G} + (1-q)q\mathbf{G} + (1-q)^2\mathbf{I}z,\\ &\vdots\\ \widehat{\mathbf{\Theta}}_k(z) &= \sum_{l=0}^{k-1}(1-q)^l\ q\mathbf{G} + (1-q)^k\mathbf{I}z. \end{aligned}\tag{23}$$

Utilizing $0 < q < 1$ and applying the form for the limit of the sum of geometric series in (23) it leads to

$$\lim_{k\to\infty}\widehat{\mathbf{\Theta}}_k(z) = \frac{1}{1-(1-q)}\ q\mathbf{G} = \mathbf{G}. \tag{24}$$

Due to stability of the model $\rho < 1$ and hence matrix $\mathbf{G}$ is stochastic. Additionally (24) holds for $|z| \le 1$ and hence the proof is completed. $\square$

Theorem 5. *In the stable vacation model with binomial-exhaustive discipline* $(0 < q < 1)$ *the closed-form expression of the vector GF of stationary number of customers at start of vacations,* $\widehat{\mathbf{m}}(z)$, *is given as:*

$$\widehat{\mathbf{m}}(z) = \widehat{\mathbf{m}}(\mathbf{G})\ \prod_{r=1}^{\infty}\widehat{\mathbf{U}}\Big(\sum_{l=0}^{r-1}(1-q)^l\ q\mathbf{G} + (1-q)^r\mathbf{I}z\Big), \tag{25}$$

where $\widehat{\mathbf{m}}(\mathbf{G})$ *is a solution of a system of linear equation and it is given as*

$$\widehat{\mathbf{m}}(\mathbf{G}) = \mathbf{e_L}\big((\mathbf{I} - \widehat{\mathbf{U}}(\mathbf{G}))\ ||\ \mathbf{e}\big)^{-1}. \tag{26}$$

Proof. We apply again the method of series of matrix GFs as in the proof of Theorem 4. Now we need the series of matrix GFs $\widehat{\mathbf{\Theta}}_k,\ k \ge 0$, defined recursively in (21). Substituting z by matrix GF $\widehat{\mathbf{\Theta}}_k(z)$, for $k \ge 0$ in system equation (9) leads to

$$\widehat{\mathbf{m}}\big(\widehat{\mathbf{\Theta}}_k(z)\big) = \widehat{\mathbf{m}}\big(\widehat{\mathbf{\Theta}}_{k+1}(z)\big)\widehat{\mathbf{U}}(\widehat{\mathbf{\Theta}}_{k+1}(z)), \quad k \ge 0. \tag{27}$$

Solving (27) by recursive substitution for $k \ge 0$ yields

$$\widehat{\mathbf{m}}(z) = \widehat{\mathbf{m}}\Big(\lim_{k\to\infty}\widehat{\mathbf{\Theta}}_k(z)\Big)\prod_{r=1}^{\infty}\widehat{\mathbf{U}}\big(\widehat{\mathbf{\Theta}}_r(z)\big). \tag{28}$$

Applying (22) and (23) in (28) gives (25).

Starting from (27) and applying the same arguments as in the proof of Theorem 4 leads to the same system of linear equation for $\widehat{\mathbf{m}}(\mathbf{G})$ as there. Therefore the solution for $\widehat{\mathbf{m}}(\mathbf{G})$ is also the same as there and hence (26) is the same as (12). $\square$

The unknowns $\mathbf{m}$, $\mathbf{m}^{(1)}$ in (5) can be determined by setting $z = 1$ in (25) and taking the first derivative of (25) at $z = 1$, respectively, which result in

$$\mathbf{m} = \widehat{\mathbf{m}}(\mathbf{G}) \prod_{r=1}^{\infty} \widehat{\mathbf{U}}\Big(\sum_{l=0}^{r-1} (1-q)^l \; q\mathbf{G} + (1-q)^r \mathbf{I} \Big),$$

$$\begin{aligned}\mathbf{m}^{(1)} = \widehat{\mathbf{m}}(\mathbf{G}) \sum_{i=1}^{\infty} \prod_{k=1}^{i-1} \widehat{\mathbf{U}}\Big(\sum_{l=0}^{k-1} (1-q)^l \; q\mathbf{G} + (1-q)^k \mathbf{I} \Big) \\ \frac{d\widehat{\mathbf{U}}\big(\sum_{l=0}^{i-1} (1-q)^l \; q\mathbf{G} + (1-q)^i \mathbf{I} z \big)}{dz}\Big|_{z=1} \\ \prod_{r=i+1}^{\infty} \widehat{\mathbf{U}}\Big(\sum_{l=0}^{r-1} (1-q)^l \; q\mathbf{G} + (1-q)^r \mathbf{I} \Big).\end{aligned} \tag{29}$$

7. Special case of the model with the exhaustive discipline

By setting $q = 1$ in the binomial-exhaustive discipline results in the exhaustive discipline as a special case. In case of this discipline no customer present at start of vacation, i.e. $\widehat{\mathbf{m}}(z) = \widehat{\mathbf{m}}(0) = \widehat{\mathbf{m}}(1)$. Using it, setting $q = 1$ in (9) and applying the notation $\mathbf{m} = \widehat{\mathbf{m}}(1)$ results in the system equation for the vacation model with exhaustive discipline as

$$\mathbf{m} = \mathbf{m}\widehat{\mathbf{U}}(\mathbf{G}). \tag{30}$$

Due to stability of the model the system of linear equation (30) (together with the normalization condition $\mathbf{me} = 1$) can be solved for $\mathbf{m}$ by applying the same method as in the proof of Theorem 4 for solving $\widehat{\mathbf{m}}(\mathbf{G})$ and it yields

$$\mathbf{m} = \mathbf{e_L}\big((\mathbf{I} - \widehat{\mathbf{U}}(\mathbf{G})) \mid\mid \mathbf{e} \big)^{-1}. \tag{31}$$

For this model $\mathbf{m}^{(1)} = 0$, since no customer present at start of vacation. Applying it in (5) results in the mean of the stationary number of customers at an arbitrary instant in the vacation model with exhaustive discipline as

$$\begin{aligned}\mathbf{q}^{(1)} = \frac{\mathbf{m}}{\lambda v} \left(\frac{1}{2}\mathbf{U}^{(2)}\mathbf{e}\boldsymbol{\pi} + \frac{1}{2}(\mathbf{U}-\mathbf{I})\mathbf{A}^{(2)}\mathbf{e}\boldsymbol{\pi} + \mathbf{U}^{(1)}\mathbf{A}^{(1)}\mathbf{e}\boldsymbol{\pi} \right) \\ - \frac{\mathbf{m}}{\lambda v}\big(\mathbf{U}^{(1)}\mathbf{A} + (\mathbf{U}-\mathbf{I})\mathbf{A}^{(1)}\big)(\mathbf{D}+\mathbf{e}\boldsymbol{\pi})^{-1}\mathbf{D}^{(1)}\mathbf{e}\boldsymbol{\pi} \\ + \frac{\mathbf{m}}{\lambda v}\big(\mathbf{U}^{(1)}\mathbf{A}\mathbf{e}\boldsymbol{\pi} + (\mathbf{U}-\mathbf{I})\mathbf{A}^{(1)}\mathbf{e}\boldsymbol{\pi}\big) \left(\frac{\mathbf{C_2}\mathbf{e}\boldsymbol{\pi}}{\lambda} + (1-\rho)\mathbf{C_1} \right)\end{aligned}$$

$$+\frac{\mathbf{m}}{\lambda v}(\mathbf{U}-\mathbf{I})\mathbf{A}(\mathbf{D}+\mathbf{e}\boldsymbol{\pi})^{-1}\big(\lambda\mathbf{I}-\mathbf{D}^{(1)}\mathbf{e}\boldsymbol{\pi}\big)\left(\frac{\mathbf{C_2}\mathbf{e}\boldsymbol{\pi}}{\lambda}+(1-\rho)\mathbf{C_1}\right)$$
$$+\boldsymbol{\pi}\left(\frac{\mathbf{A}^{(2)}\mathbf{e}\boldsymbol{\pi}}{2(1-\rho)}-(\mathbf{I}-\mathbf{A}^{(1)})\mathbf{C_1}\right). \tag{32}$$

Appendix

In the following we give a generalization of a convergence property of series of scalar PGFs (see [10]) for the series of matrix GFs.

Appendix A. Convergence of a class of matrix GF series

Lemma 1. *Let $\widehat{\mathbf{H}}(z)=\sum_{i=0}^{\infty}\mathbf{H}_i z^i,\ |z|\le 1$ be an $L\times L$ matrix GF with the following assumptions*

(1) *the matrices $\mathbf{H}_i$ are nonnegative,*

(2) *$\widehat{\mathbf{H}}(1)\mathbf{e}=\mathbf{e}$, i.e. $\widehat{\mathbf{H}}(1)$ is stochastic,*

(3) *the discrete-time Markov chain (DTMC) with the transition matrix $\widehat{\mathbf{H}}(1)=\sum_{i=0}^{\infty}\mathbf{H}_i$ is irreducible and*

(4) *$0<\sigma$, where $\sigma=\boldsymbol{\pi}^{\mathbf{H}}\frac{\widehat{H}(z)}{dz}\big|_{z=1}\mathbf{e}$ and $\boldsymbol{\pi}^{\mathbf{H}}$ is the unique (due to assumption (3)) eigenvector of $\widehat{\mathbf{H}}(1)$ belonging to eigenvalue 1, for which $\boldsymbol{\pi}^{\mathbf{H}}\mathbf{e}=1$.*

Additionally a series of matrix GFs is defined recursively as

$$\widehat{\mathbf{X}}_0(z)=\mathbf{I}z,\ |z|\le 1,$$
$$\widehat{\mathbf{X}}_{k+1}(z)=\widehat{\mathbf{H}}(\widehat{\mathbf{X}}_k(z)),\quad k\ge 0. \tag{33}$$

If $\sigma<1$ then the series of matrix GFs $\widehat{\mathbf{X}}_k(z)$ converges to a stochastic matrix, which is independent of z,

$$\lim_{k\to\infty}\widehat{\mathbf{X}}_k(z)=\mathbf{G}, \tag{34}$$

where $\mathbf{G}$ is the minimal non-negative solution of $\mathbf{G}=\sum_{k=0}^{\infty}\mathbf{H}_k\mathbf{G}^k$.

Proof. Let $\{(\Gamma_\ell, J_\ell); \ell \geq 1\}$ be a bivariate DTMC on the state space ($\Gamma_\ell \in \{0, 1, \dots\}, J_\ell \in \{1, 2, \dots, L\}$) with the following transition matrix:

$$\mathbf{\Pi} = \begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots \\ \mathbf{H}_0 & \mathbf{H}_1 & \mathbf{H}_2 & \mathbf{H}_3 & \dots \\ \mathbf{0} & \mathbf{H}_0 & \mathbf{H}_1 & \mathbf{H}_2 & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{H}_0 & \mathbf{H}_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{35}$$

Γ_ℓ is called the level of the chain at time ℓ and J_ℓ is the phase of the chain. In the context of queuing models with BMAP arrival the level of the chain corresponds to the number of customers in the system and the phase of the chain is the phase of the BMAP.

$\widehat{\mathbf{X}}_k(z)$ in (33) is matrix GF, which implies that $\widehat{\mathbf{X}}_k^\ell(z)$ is also matrix GF, for $k \geq 1$ and $\ell \geq 0$. The matrix $\mathbf{X}_{k,i}^{(\ell)}$, for $k \geq 1$ and $i, \ell \geq 0$, is defined as the i-th coefficient of the power series form of the matrix GF $\widehat{\mathbf{X}}_k^\ell(z)$. Hence $\widehat{\mathbf{X}}_k^\ell(z)$ can be written as

$$\widehat{\mathbf{X}}_k^\ell(z) = \sum_{i=0}^{\infty} \mathbf{X}_{k,i}^{(\ell)} z^i, \ |z| \leq 1.$$

We also use the notation $\mathbf{X}_{k,i} = \mathbf{X}_{k,i}^{(1)}$, for $k \geq 1$ and $i \geq 0$. Note that matrices $\mathbf{X}_{k,i}^{(\ell)}$, for $k \geq 1$ and $i, \ell \geq 0$, are nonnegative and matrices $\widehat{\mathbf{X}}_k(1)$, for $k \geq 1$, are stochastic.

Analogously to the infinite state Markov chain with transition matrix in (35) a sequence of infinite state Markov chains is defined, where the transition matrix of the k-th Markov chain, for $k \geq 1$, is given as

$$\mathbf{\Pi}_k = \begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots \\ \mathbf{X}_{k,0} & \mathbf{X}_{k,1} & \mathbf{X}_{k,2} & \mathbf{X}_{k,3} & \dots \\ \mathbf{0} & \mathbf{X}_{k,0} & \mathbf{X}_{k,1} & \mathbf{X}_{k,2} & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{X}_{k,0} & \mathbf{X}_{k,1} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{36}$$

We refer to the Markov chain with transition matrix $\mathbf{\Pi}_k$ as Markov chain $\mathbf{\Pi}_k$. It can be directly seen from (36) that, for $k \geq 1$, the matrix GF $\widehat{\mathbf{X}}_k(z)$ describes the random population after one step transition starting from the population of 1 initial customer, i.e. starting from level 1. Introducing matrix $\mathbf{E}_i$ of size $(L \times \infty)$, whose i-th $(L \times L)$ block is $\mathbf{I}$ and all other entries equals to 0 (e.g., the first block row of $\mathbf{\Pi}_k$ is $\mathbf{E}_0$) and matrix $\mathbf{Z}$ of size $(\infty \times L)$, whose i-th $(L \times L)$ block is $\mathbf{I}z^i$, for $i \geq 0$, (i.e., $\mathbf{Z}^T = (\mathbf{I}, \mathbf{I}z, \mathbf{I}z^2, \ldots)$, where $\mathbf{Z}^T$ stands for the transpose of $\mathbf{Z}$), we have $\widehat{\mathbf{X}}_k(z) = \mathbf{E}_1 \mathbf{\Pi}_k \mathbf{Z}$.

We apply induction to show that the matrix GF describing the random population after n step transition of the Markov chain $\mathbf{\Pi}_k$ ($\mathbf{\Pi}_k^n$) starting from level $c+n$ is

$$\mathbf{E}_{c+n}\mathbf{\Pi}_k^n \mathbf{Z} = z^c \widehat{\mathbf{X}}_k^n(z), \quad n \geq 0,\ c \geq 0,\ k \geq 1. \tag{37}$$

For $n = 0$ and $n = 1$ (37) follows directly from the transition matrix (36) as

$$\mathbf{E}_c \mathbf{\Pi}_k^0 \mathbf{Z} = \mathbf{I}z^c = z^c \widehat{\mathbf{X}}_k^0(z), \qquad \mathbf{E}_{c+1}\mathbf{\Pi}_k \mathbf{Z} = z^c \widehat{\mathbf{X}}_k(z). \tag{38}$$

For $n \geq 2$ we apply induction. We show that the matrix GF of the random population after ℓ step transition of the Markov chain $\mathbf{\Pi}_k$ starting from level $c+n$ is given as

$$\mathbf{E}_{c+n}\mathbf{\Pi}_k^\ell \mathbf{Z} = z^{c+n-\ell} \widehat{\mathbf{X}}_k^\ell(z), \quad 1 \leq \ell < n,\ c \geq 0,\ k \geq 1. \tag{39}$$

For $l = 1$ (39) follows from (38) as

$$\mathbf{E}_{c+n}\mathbf{\Pi}_k \mathbf{Z} = z^{c+n-1} \widehat{\mathbf{X}}_k(z).$$

The inductive condition (39) can be rearranged by using $\mathbf{E}_i \mathbf{Z} = \mathbf{I}z^i$, for $i \geq 0$, as

$$\mathbf{E}_{c+n}\mathbf{\Pi}_k^\ell \mathbf{Z} = z^{c+n-\ell} \widehat{\mathbf{X}}_k^\ell(z) = z^{c+n-\ell} \sum_{i=0}^{\infty} \mathbf{X}_{k,i}^{(\ell)} \mathbf{I} z^i = \sum_{i=0}^{\infty} \mathbf{X}_{k,i}^{(\ell)} \mathbf{I} z^{c+n-l+i}$$
$$= \sum_{i=0}^{\infty} \mathbf{X}_{k,i}^{(\ell)} \mathbf{E}_{c+n-l+i} \mathbf{Z}.$$

Hence

$$\mathbf{E}_{c+n}\mathbf{\Pi}_k^\ell = \sum_{i=0}^{\infty} \mathbf{X}_{k,i}^{(\ell)} \mathbf{E}_{c+n-l+i}. \tag{40}$$

By using (40) and (38) the matrix GF of the random population after $\ell+1$ step transition, $\mathbf{E}_{c+n}\mathbf{\Pi}_k^{\ell+1}\mathbf{Z}$ can be expressed as

$$\mathbf{E}_{c+n}\mathbf{\Pi}_k^{\ell+1} \mathbf{Z} = \mathbf{E}_{c+n}\mathbf{\Pi}_k^\ell \mathbf{\Pi}_k \mathbf{Z} = \sum_{i=0}^{\infty} \mathbf{X}_{k,i}^{(\ell)} \mathbf{E}_{c+n-l+i} \mathbf{\Pi}_k \mathbf{Z}$$
$$= \sum_{i=0}^{\infty} \mathbf{X}_{k,i}^{(\ell)} z^{c+n-\ell+i-1} \widehat{\mathbf{X}}_k(z) = z^{c+n-(\ell+1)} \sum_{i=0}^{\infty} \mathbf{X}_{k,i}^{(\ell)} z^i \widehat{\mathbf{X}}_k(z) = z^{c+n-(\ell+1)} \widehat{\mathbf{X}}_k^{\ell+1}(z).$$

Therefore if the inductive condition holds for ℓ ($\mathbf{E}_{c+n}\mathbf{\Pi}_k^\ell \mathbf{Z} = z^{c+n-\ell}\widehat{\mathbf{X}}_k^\ell(z)$) then it follows that it also holds for $\ell+1$ ($\mathbf{E}_{c+n}\mathbf{\Pi}_k^{\ell+1}\mathbf{Z} = z^{c+n-(\ell+1)}\widehat{\mathbf{X}}_k^{\ell+1}(z)$). Applying the induction recursively for $\ell = 1, \ldots, n-1$ gives the matrix GF of the random population after n step transition as $\mathbf{E}_{c+n}\mathbf{\Pi}_k^n \mathbf{Z} = z^c \widehat{\mathbf{X}}_k^n(z)$.

Now let us turn back to the matrix GFs $\widehat{\mathbf{X}}_k(z)$, for $k \geq 1$. $\widehat{\mathbf{X}}_1(z) = \mathbf{E}_1\mathbf{\Pi}_1\mathbf{Z} = \widehat{\mathbf{H}}(z)$ describes the random population after one step transition of the Markov chain $\mathbf{\Pi}_1$ starting from level 1 ($\mathbf{E}_1$), where $\mathbf{\Pi}_1 = \mathbf{\Pi}$. In order to interpret $\widehat{\mathbf{X}}_{k+1}(z)$ we continue with the interpretation of $\mathbf{E}_n\mathbf{\Pi}_{k+1}\mathbf{Z}$, for which (38) implies that $\mathbf{E}_n\mathbf{\Pi}_{k+1}\mathbf{Z} = z^{n-1}\widehat{\mathbf{X}}_{k+1}(z)$, for $n \geq 1$. It follows from transition matrix (35) that

$$\mathbf{H}_i = \mathbf{E}_n\mathbf{\Pi}\mathbf{E}^T_{n-1+i}, \quad i \geq 0,\ n \geq 1. \tag{41}$$

Using the definition of $\widehat{\mathbf{X}}_{k+1}(z)$, the property (37) and (41) results in

$$\begin{aligned}\mathbf{E}_n\mathbf{\Pi}_{k+1}\mathbf{Z} &= z^{n-1}\widehat{\mathbf{X}}_{k+1}(z) = z^{n-1}\sum_{i=0}^{\infty}\mathbf{H}_i\widehat{\mathbf{X}}^i_k(z) = \sum_{i=0}^{\infty}\mathbf{H}_i z^{n-1}\widehat{\mathbf{X}}^i_k(z)\\ &= \sum_{i=0}^{\infty}\mathbf{H}_i\mathbf{E}_{n-1+i}\mathbf{\Pi}^i_k\mathbf{Z} = \mathbf{E}_n\mathbf{\Pi}\sum_{i=0}^{\infty}\mathbf{E}^T_{n-1+i}\mathbf{E}_{n-1+i}\mathbf{\Pi}^i_k\mathbf{Z}, \quad k \geq 1,\ n \geq 1. \end{aligned} \tag{42}$$

$\mathbf{E}^T_{n-1+i}$ in (42) represents the case when the next level after one step transition of the Markov chain $\mathbf{\Pi}$ starting from level n ($\mathbf{E}_n\mathbf{\Pi}$) is $n-1+i$, where $i \geq 0$ is random. Setting $n = 1$ in (41) shows that i in (42) can be also interpreted as the random number of customers after one step transition of the Markov chain $\mathbf{\Pi}$ starting from level 1. Hence (42) shows the interpretation of the matrix GF of the random population after one step transition of the Markov chain $\mathbf{\Pi}_{k+1}$ starting from level $n \geq 1$ ($\mathbf{E}_n$) as the matrix GF of the random population after one initial one step transition of the Markov chain $\mathbf{\Pi}$ starting from level n and followed by so many consecutive one step transitions of the Markov chain $\mathbf{\Pi}_k$ as the random number of customers (i) after one step transition of the Markov chain $\mathbf{\Pi}$ starting from level 1.

$\mathbf{E}_{n-1+i}\mathbf{\Pi}^i_k$ in (42) means no transition for $i = 0$ and i number of one step transitions of Markov chain $\mathbf{\Pi}_k$, for $i \geq 1$. The first transition of Markov chain $\mathbf{\Pi}_k$ starts from level $n-1+i$, the second one starts from level $n-1+i-1$ or greater (see transition matrix (36)) and so on. Thus each of the i consecutive transitions of Markov chain $\mathbf{\Pi}_k$ in (42) starts from level $n \geq 1$ or greater. Therefore the interpretation of (42) can be applied recursively. Applying the interpretation of (42) recursively for $k \geq 1$ shows that the random population characterized by $\mathbf{E}_n\mathbf{\Pi}_{k+1}\mathbf{Z}$ is originated from a random number of consecutively applied one step transitions of the Markov chain $\mathbf{\Pi}$ starting from level n.

In order to express quantitatively the number of consecutively applied one step transitions of the Markov chain $\mathbf{\Pi}$ we introduce several generic random variables. O denotes the number of customers after one step transition of the

Markov chain $\mathbf{\Pi}$ starting from level 1. Because we need to distinguish among the samples of this random variable we also use the notations O_{i_1}, O_{i_2,i_1}, $O_{\dots,i_2,i_1}$, for its individual samples. $T(k)$, for $k \geq 1$, denotes the number of consecutive one step transitions of the Markov chain $\mathbf{\Pi}$ starting from level $n \geq 1$, which leads to the random population after one step transition of the Markov chain $\mathbf{\Pi}_k$ starting from level n. Note that $T(k)$ is independent of n, since the power of $\mathbf{\Pi}$ and the power of $\mathbf{\Pi}_k$ in (42) are independent of n. Again we need to distinguish among the samples of this random variable hence we also use the notation $T_i(k)$ for its individual samples. Then using the above interpretation of $\mathbf{E}_n\mathbf{\Pi}_{k+1}\mathbf{Z}$ the number of one step transitions of the Markov chain $\mathbf{\Pi}$, which leads to the random population characterized by matrix GF $\mathbf{E}_n\mathbf{\Pi}_{k+1}\mathbf{Z}$, can be expressed as

$$T(1) = 1, \qquad T(k+1) = 1 + \sum_{i=1}^{O} T_i(k), \quad k \geq 1. \tag{43}$$

Applying recursive substitution in (43) results in

$$\begin{aligned}
&T(1) = 1,\\
&T(2) = 1 + O,\\
&T(3) = 1 + \sum_{i_1=1}^{O}(1 + O_{i_1}),\\
&T(4) = 1 + \sum_{i_2=1}^{O}\left(1 + \sum_{i_1=1}^{O_{i_2}}(1 + O_{i_2,i_1})\right),\\
&\qquad\vdots
\end{aligned} \tag{44}$$

Each time when any $O_{\dots,i_2,i_1,} = 0$, the Markov chain $\mathbf{\Pi}$ goes to the absorbing level 0. Thus the random realizations of the chain, in which the total number of transitions in (44) is finite, as $k \to \infty$, all go to the absorbing level 0. Hence they can be extended by infinite number of further one step transitions without resulting any change in the state of the Markov chain $\mathbf{\Pi}$. Therefore, as $k \to \infty$, the random population with matrix GF $z^{n-1}\widehat{\mathbf{X}}_k(z) = \mathbf{E}_n\mathbf{\Pi}_k\mathbf{Z}$, for $n \geq 1$, corresponds to infinite many one step transition of the Markov chain $\mathbf{\Pi}$ starting from level n.

The upper block Hessenberg form and the repetitive structure of the transition matrix $\mathbf{\Pi}$ ensures that $\{(\Gamma_\ell, J_\ell);\ \ell \geq 1\}$ is an $M/G/1$-type chain. If the assumptions of the lemma hold then starting from any level greater than 0 this Markov chain eventually gets absorbed at level 0, as the number of one step transitions goes to ∞. In case of starting from level 1 the corresponding phase transition matrix is $\mathbf{G}$ (see Neuts [4]). It follows that

$$\lim_{k\to\infty} \widehat{\mathbf{X}}_k(z) = \mathbf{G}, \tag{45}$$

which is stochastic if $\sigma < 1$. Furthermore (45) shows that $\lim_{k\to\infty} \widehat{\mathbf{X}}_k(z)$ is independent of z, which completes the proof. $\square$

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