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Long-range dependence and asymptotic self-similarity in third order

By GYÖRGY TERDIK (Debrecen)

Dedicated to the 100th anniversary of the birthday of Béla Gyires

Abstract. The object of this paper is studying the topic of long-range dependence and asymptotic self-similarity from the viewpoint of third-order time series analysis in frequency domain. The long-range dependent (LRD) time series in third-order is defined by the bispectrum and by the bicovariances. A Tauber type connection between these two definitions is shown.

1. Introduction

Processes with long-range dependence have attracted a great deal of work in theory and applications. Applications include measurements from hydrology, soil science, signal processing, musics, network traffic etc., see [Cox84], [BI98], [TG09]. Although most of these measurements are non-Gaussian the theory concerns the second order structures of the processes which is sufficient only for Gaussian case, see [Ber92], [DOT03]. Another common property of measurements is that the probability structure does not change too much when the process is aggregated, in other words they are self-similar. Models for self-similar processes are well developed for stable-processes and processes connected to Gaussian through nonlinear functionals, see [ST94], [Maj81]. Barndorff-Nielsen and Leonenko [BNL05] consider second order long-range and self-similar processes with several

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(non-Gaussian) infinitely divisible marginal distributions each of which can be the subject of long-range dependence in higher order. There are only a few results capturing both properties long-range dependence and self-similarity, in particular for the non-Gaussian case, see [Taq86].

The object of this paper is studying the topic of long-range dependence and asymptotic self-similarity from the viewpoint of third-order time series analysis in frequency domain. In Section 2 we define the long-range dependent (LRD) time series in third-order by the bispectrum and by the bicovariances. A Tauber type connection between these two definitions is shown. The self-similarity in third-order is introduced in Section 3. Moreover we point out that a third order LRD time series is asymptotically self-similar in third order. The Appendix contains some technical details.

1.1. Long-range dependence. A stationary time series X_ℓ, $\ell = 0, \pm 1, \pm 2 \dots \pm n$ is called *long-range dependent* if its spectrum $S_2(\omega)$ behaves like $|\omega|^{-2h}$ at zero, more precisely

$$\lim_{\omega \to 0} \frac{S_2(\omega)}{|\omega|^{-2h} L(1/|\omega|)} = \mathsf{q}_s, \tag{1}$$

where $h \in (0, 1/2)$ and $L(\cdot)$ is a slowly varying function at infinity. This definition of long-range dependence can be stated in terms of the autocorrelation function as well, since (1) is equivalent to:

$$\lim_{k \to \infty} \frac{\mathrm{Cov}(X_{\ell+k}, X_\ell)}{|k|^{2h-1} L(k)} = \mathsf{q}_c, \quad h \in (0, 1/2). \tag{2}$$

provided $L(\cdot)$ is a quasi monotone slowly varying function. Note here the connection between these two constants $\mathsf{q}_s/\mathsf{q}_c = 2^{1-2h}\pi^{-2h}\Gamma(2h)\cos(\pi h)$. In other words, the autocorrelation function decays hyperbolically. In fact although the spectrum is in L_1 its Fourier coefficients $\mathrm{Cov}(X_0, X_k)$ are not in L_1 any more. The equivalence of (1) and (2) is studied in the theory of regular varying functions in details, see Theorems, 4.3.2 and 4.10.1 of [BGT87].

1.2. Bispectrum and bicovariances. Let the process X_ℓ be centered and stationary in third order, then its third order cumulants are

$$\mathrm{Cum}(X_{\ell+k_1}, X_{\ell+k_2}, X_\ell) = \mathsf{E} X_{\ell+k_1} X_{\ell+k_2} X_\ell = C_3(k_{1:2}),$$
$$k_1, k_2 = 0, \pm 1, \pm 2, \dots .$$

where $k_{1:2} = (k_1, k_2)$. The third order cumulants are called bicovariances as well. An easy consequence of this definition is the following properties

$$C_3(k_{1:2}) = C_3(k_2, k_1) = C_3(-k_1, k_2 - k_1).$$

These equations provide the symmetry of the third order cumulants. The plain is divided into six equivalent parts, each of them is sufficient for determining the third order cumulants on the whole plane. One of these parts, called principal domain for the third order cumulants, is where $0 \leq k_2 \leq k_1$ i.e., the lower half of the right upper quarter of the lattice with integer coordinates.

The bispectrum S_3 is a complex valued function of two variables with Fourier coefficients $C_3(k_{1:2})$, i.e.

$$C_3(k_{1:2}) = \iint\limits_{[-1/2,1/2]^2} e^{i2\pi(\omega_1 k_1 + \omega_2 k_2)} S_3(\omega_{1:2}) d\omega_{1:2}.$$

While the spectrum is real and nonnegative the bispectrum is generally complex valued and since $C_3(k_{1:2})$ is real, we have $S_3(\omega_{1:2}) = \overline{S_3}(-\omega_{1:2})$. The bispectrum S_3 is periodic, i.e. $S_3(\omega_{1:2}) = S_3(\omega_{1:2} + m_{1:2})$, $m_1, m_2 = \pm 1, \pm 2, \ldots$, and it has the following symmetry

$$S_3(\omega_{1:2}) = S_3(\omega_2, \omega_1) = S_3(\omega_1, \omega_3),$$

where $\omega_3 \mathrel{\mathring{=}} -\omega_1 - \omega_2$. These symmetries imply twelve equivalent domains for the bispectrum, the principal domain, among these, traditionally is the triangle $\triangle_1$ with vertices $(0,0)$, $(1/2,0)$ and $(1/3,1/3)$, see [Ter99].

2. Long-range dependence in third order

The relations (1) and (2) concern the behavior of the covariances at infinity and the spectrum at zero. Similar results in 2D are available for the isotropic case only, see Stein–Weiss [SW71], Ch. VII. Theorem 2.17. Since there is no isotropic bispectrum (except constant) we have to deal with more general 2D Fourier transforms then the isotropic one. Examples show, see [Ter08], that the bispectrum $S_3(\omega_{1:2})$ and the third order cumulants are connected in some particular way. Let $\alpha_{\omega_2/\omega_1} = \arctan(\omega_2/\omega_1)$, i.e. α corresponds to the angle of the unit vector $\omega_{1:2}/|\omega_{1:2}|$ and the ω_1-axis. When either the radius $|\omega_{1:2}| = \sqrt{\omega_1^2 + \omega_2^2}$ tends to zero and $\alpha_{\omega_2/\omega_1}$ is fixed or $\alpha_{\omega_2/\omega_1}$ tends to zero and $|\omega_{1:2}|$ is fixed then the bispectrum might have singularity. Hence we assume the bispectrum has the following form:

$$S_3(\omega_{1:2}) = \mathtt{c}|\omega_{1:2}|^{-3g_0} \alpha_{\omega_2/\omega_1}^{-2g_1} L\big(\omega_{1:2}|^{-1}, \alpha_{\omega_2/\omega_1}^{-1}\big), \quad \omega_{1:2} \in \triangle_1, \tag{3}$$

where $L(\cdot,\cdot)$ is a slowly varying function in either variable when the other one is fixed and from now on $\mathtt{c}$ denotes a general constant.

Definition 1. The time series X_ℓ is long-range dependent in third order with radial exponent g_0 and angular exponent g_1 if the bispectrum S_3 is factorized as (3) on the principal domain $\triangle_1$ and $0 < g_0 < 2/3$, $g_1 \in [0, 1/2)$.

On the principal domain let the third order cumulants be given asymptotically in the form

$$C_3(k_{1:2}) \simeq \mathsf{c}|k_{1:2}|^{3g_0-2} L\big(|k_{1:2}|\big) K_{\beta_{k_2/k_1}}\big(|k_{1:2}|\big),\ |k_{1:2}| \to \infty \tag{4}$$

where $\beta_{k_2/k_1} = \arctan(k_2/k_1)$, $\beta_{k_2/k_1} \in (0, \pi/4)$, and $K_\beta(\cdot)$ has a finite, continuous in β limit K_β, when $|k_{1:2}| \to \infty$, $L(|k_{1:2}|)$ is a slowly varying function. In addition,

$$K_\beta = \mathsf{c}\beta_{k_2/k_1}^{2g_2-1}(\pi/4 - \beta_{k_2/k_1})^{2g_3-1} L\big(\beta_{k_2/k_1}^{-1}(\pi/4 - \beta_{k_2/k_1})^{-1}\big). \tag{5}$$

The form (4)–(5) is corresponding to (3). We shall assume in sequel that the third order cumulants are τ-*slowly varying*, namely let $\tau(a)$ on $\mathbb{R}_+$ be regularly varying at infinity with index $d > -2$, and for all $k_{1:2}$ ($k_1 > k_2$), and for all series $k_{1:2}(a)$, which $k_{1:2}(a) \to k_{1:2}$, when $a \to \infty$, we have

$$\frac{C_3(ak_{1:2}(a)) - C_3(ak_{1:2})}{\tau(a)/a^2} \longrightarrow 0, \quad a \to \infty.$$

Definition 2. The time series X_ℓ is long-range dependent in third order with radial exponent g_0 and angular exponent g_2, if the third order cumulants are asymptotically of the factorized form (4) over the principal domain and (5) fulfils, moreover $0 < g_0 < 2/3$, $g_2 \in (0, 1/2]$.

Now, suppose that the bispectrum S_3 is third order LRD, i.e. it is written in the form (3) inside the principal domain $\triangle_1$. Since $|\omega_{1:2}| \cos \alpha_{\omega_2/\omega_1} = \omega_1$, $|\omega_{1:2}| \sin \alpha_{\omega_2/\omega_1} = \omega_2$ and $-\sqrt{2}|\omega_{1:2}| \sin(\pi/4 + \alpha_{\omega_2/\omega_1}) = \omega_3$, there are different possibilities of rewriting S_3 in terms of frequencies. So we may always consider S_3 in the following general form

$$S_3(\omega_{1:2}) = \mathsf{c} \operatorname*{sym}_{\omega_{1:3}} \left(|\omega_{1:2}|^{-h_0} \omega_1^{-h_1} \omega_2^{-h_2} \omega_3^{-h_3} L\big(|\omega_{1:2}|^{-1}, \alpha_{\omega_2/\omega_1}^{-1}\big)\right), \tag{6}$$

where $\sum_0^3 h_k > 0$.

Theorem 1. *Suppose that the time series X_ℓ is long-range dependent in third order with radial exponent g_0 and angular exponent g_1 and $0 < g_0 < 2/3$, $g_1 \in [0, 1/2)$. If the bispectrum S_3 is factorized as (6) and the third order cumulants are slowly varying, where $h_1 \geq h_2 \geq h_3 \geq 0$ then $g_0 = (h_0 + h_1 + h_2 + h_3)/3$ and $g_1 = (h_1 + h_2 + h_3)/2$, moreover the third order cumulants have the form (4), (5) with exponents*

$$2g_2 = \begin{cases} 0 & \text{if } h_1 = 0, \\ h_0 & \text{if } h_0 > 0,\ h_2 = 0,\ h_1 > 0, \\ h_1 & \text{if } h_0 = h_2 = 0,\ h_1 > 0, \\ h_0 + h_2 + h_3 & \text{otherwise.} \end{cases}$$

$$g_3 = \begin{cases} g_2 & \text{if } h_1 = 0, \\ g_2 & \text{if } h_1 = h_2 = h_3 = h > 0 \\ 0 & \text{otherwise.} \end{cases}$$

See Appendix A.1 for the proof. The case when $h_0 = 0$, $h_1 = h_2 = h_3 = h > 0$ corresponds to the linear process, it has been used for model fitting, see [BI98].

3. Self-similarity

A stationary time series Y_t is called self-similar if the distribution of the scaled sum of Y_t over the interval $[tn, (t+1)n)$ is the same as the distribution of Y_t for any n, see [ST94]. More precisely, let

$$Y_t^{(n^\alpha)} = \frac{1}{n^\alpha} \sum_{j=tn}^{(t+1)n-1} Y_j, \quad t \in \mathbb{Z}, \tag{7}$$

where α is an appropriate constant, then $Y_t^{(n^\alpha)} \overset{d}{=} Y_t$, for all $n = 1, 2, \dots$. The only Gaussian self-similar stationary time series ($\alpha = h + 1/2 < 1$), is

$$Z_t = \int_{-\infty}^{\infty} e^{i2\pi\omega t} \mathfrak{e}_I(\omega)(i2\pi\omega)^{-h} \mathcal{W}(d\omega),$$

where $\mathcal{W}(d\omega)$ is a Gaussian stochastic spectral measure and

$$\mathfrak{e}_I(\omega) = \frac{e^{i2\pi\omega} - 1}{i2\pi\omega},$$

see [Sin76], it is defined in continuous time but one samples it in discrete time points getting a time series. A pathological discrete self-similar stationary time series has been considered in [GVM03]. An other example for the self-similar time series is the following non-Gaussian process,

$$Y_t = \int_{\mathbb{R}^2} e^{i2\pi t(\omega_1+\omega_2)} \frac{\mathfrak{e}_I(\omega_1+\omega_2)\mathcal{W}(d\omega_{1:2})}{(i2\pi\omega_1)^h(i2\pi\omega_2)^h}, \tag{8}$$

where the integral is multiple Wiener–Ito integral. It looks very likely that, under some regularity conditions, Y_t is also a unique self-similar process in the L_2 space defined by the stochastic spectral measure $\mathcal{W}(d\omega_{1:2})$. Y_t is called also as Rosenblatt process, again a continuous time process sampled in discrete time. Indeed

$$\sum_{j=tn}^{(t+1)n-1} Y_j = n\int_{\mathbb{R}^2} e^{i2\pi tn(\omega_1+\omega_2)} \frac{\mathfrak{e}_I(n[\omega_1+\omega_2])\mathcal{W}(d\omega_{1:2})}{(i2\pi\omega_1)^h(i2\pi\omega_2)^h},$$

now changing the variables $\lambda_1 = n\omega_1$, $\lambda_2 = n\omega_2$, we obtain

$$\frac{1}{n^\alpha}\sum_{j=tn}^{(t+1)n-1} Y_j = n^{2h-\alpha}\int_{\mathbb{R}^2} e^{i2\pi t(\omega_1+\omega_2)} \frac{\mathfrak{e}_I(\omega_1+\omega_2)\mathcal{W}(d\omega_{1:2})}{(i2\pi\omega_1)^h(i2\pi\omega_2)^h},$$

since $E|\mathcal{W}(nd\omega_{1:2})|^2 = nd\omega_{1:2}$. If the parameter α in (7) is chosen to be $2h$ then $Y_t^{(n^\alpha)} \stackrel{d}{=} Y_t$, for all $n = 1, 2, \ldots$. Although there are self-similar time series defined by higher order multiple Wiener–Ito integrals similarly to the Rosenblatt process, see [Maj81], nevertheless we are interested in a larger class of time series having the self-similar property asymptotically. A simple example comes directly from the fact that the time series Y_t defined in (8), is the subject of some non-central limit theorem ([DM79], [Taq79]). For instance the weak limit of the weighted aggregated series

$$\frac{1}{n^{2h}}\sum_{j=tn}^{(t+1)n-1} X_j,$$

of

$$X_t = \int_{\mathbb{R}^2} e^{i2\pi t(\omega_1+\omega_2)} \frac{\mathfrak{e}_I(\omega_1)\mathfrak{e}_I(\omega_2)\mathcal{W}(d\omega_{1:2})}{(i2\pi\omega_1)^h(i2\pi\omega_2)^h},$$

by the non-central limit theorem, provided $1/4 < h < 1/2$, mentioned above, is Y_t. Actually $X_t = H_2(Z_t)$ is a homogenous Hermite process with order 2, hence it might be called H_2-process as well. We are interested in the correspondence between the correlation of the aggregated series and the correlation of the limit time series Y_t. Consider

$$X_t^{(n)} = \frac{1}{n}\sum_{j=tn}^{(t+1)n-1} X_j = \int_{\mathbb{R}^2} e^{i2\pi tn(\omega_1+\omega_2)} \frac{\mathfrak{e}_I(\omega_1)\mathfrak{e}_I(\omega_2)}{\mathfrak{e}_I(\omega_1+\omega_2)} \frac{\mathfrak{e}_I(n[\omega_1+\omega_2])\mathcal{W}(d\omega_{1:2})}{(i2\pi\omega_1)^h(i2\pi\omega_2)^h}. \tag{9}$$

The covariance of $X_t^{(n)}$ is straightforward:

$$
\begin{aligned}
&n^{2-4h}\operatorname{Cov}\left(X_t^{(n)}, X_{t+k}^{(n)}\right)\\
&\qquad = 2\int_{\mathbb{R}^2} e^{i2\pi k(\lambda_1+\lambda_2)}\left|\frac{\mathfrak{e}_I(\lambda_1/n)\mathfrak{e}_I(\lambda_2/n)}{\mathfrak{e}_I(\lambda_1/n+\lambda_2/n)}\right|^2 \frac{|\mathfrak{e}_I(\lambda_1+\lambda_2)|^2 d\lambda_{1:2}}{|2\pi\lambda_1|^{2h}|2\pi\lambda_2|^{2h}}.
\end{aligned}
$$

We see that the function $|\mathfrak{e}_I(\lambda_1/n)\mathfrak{e}_I(\lambda_2/n)/\mathfrak{e}_I(\lambda_1/n+\lambda_2/n)|^2$ is slowly varying in n and proceed

$$
n^{2-4h}\operatorname{Cov}(X_t^{(n)}, X_{t+k}^{(n)}) \simeq \operatorname{Cov}\left(Y_{t+k}, Y_t\right), \quad \text{as } n\to\infty.
$$

Note that if $h<1/4$ then X_t is not LRD and $X_t^{(n)}$ tends to an uncorrelated time series.

Now we turn our attention to the bicovariances and checking the limit behavior similarly to the covariances. We consider the bicovariances for the aggregated series $X_t^{(n)}$ of $X_t = H_2(Z_t)$, see (9)

$$
\begin{aligned}
n^{3-6h}C_3^{(n)}(k_{1:2}) &= 8\int_{\mathbb{R}^3} e^{i2\pi[(\lambda_1+\lambda_2)k_1+(\lambda_3-\lambda_1)k_2]}\\
&\times \frac{|\mathfrak{e}_I(\lambda_1/n)\mathfrak{e}_I(\lambda_2/n)\mathfrak{e}_I(\lambda_3/n)|^2}{\mathfrak{e}_I(\lambda_1/n+\lambda_2/n)\mathfrak{e}_I(\lambda_3/n-\lambda_1/n)\mathfrak{e}_I(-\lambda_2/n-\lambda_3/n)}\\
&\times \frac{\mathfrak{e}_I(\lambda_1+\lambda_2)\mathfrak{e}_I(\lambda_3-\lambda_1)\mathfrak{e}_I(-\lambda_2-\lambda_3)}{|2\pi\lambda_1|^{2h}|2\pi\lambda_2|^{2h}|2\pi\lambda_3|^{2h}} d\lambda_{1:3}\\
&\simeq 8c_{2h}^3\int_{[0,1]^3} |k_1-k_2+u_1-u_2|^{2h-1}|k_1+u_1-u_3|^{2h-1}|k_2+u_2-u_3|^{2h-1}du_{1:3}.
\end{aligned}
$$

where $C_3^{(n)}(k_{1:2}) = \operatorname{Cum}(X_{t+k_1}^{(n)}, X_{t+k_2}^{(n)}, X_t^{(n)})$ and see (13) for c_{2h}. From this last expression we conclude that in the limit $n^{1-2h}X_t^{(n)}$ has the same third order cumulants as Y_t. Put $k_1 = r\cos\beta$ and $k_2 = r\sin\beta$, $(\beta\neq\pi/4,0)$ then we obtain for large n and large $|k_{1:2}|$

$$
\begin{aligned}
\lim_{r\to\infty} r^{3-6h}\lim_{n\to\infty} n^{3-6h}C_3^{(n)}(k_{1:2}) &= 8c_{2h}^3|\sqrt{2}\cos\beta\sin\beta\sin(\pi/4-\beta)|^{2h-1}\\
&= 2^{h-5/2}c_{2h}^3\beta^{2h-1}(\pi/4-\beta)^{2h-1}L\big(\beta^{-1}(\pi/4-\beta)^{-1}\big).
\end{aligned}
$$

In other words $n^{1-2h}X_t^{(n)}$ asymptotically has a third order cumulant structure of Y_t, also like a third order LRD process, compare to Definition 2.

3.1. Asymptotic self-similarity. A property expressing the fractal-like feature of the aggregated process

$$X_k^{(n)} = \frac{1}{n} \sum_{j=kn}^{(k+1)n-1} X_j, \quad k \in \mathbb{Z}, \tag{10}$$

closely related to long-range dependence, is asymptotic second order self-similarity. This means that, as $n \to \infty$, the series of the autocorrelation functions $R^{(n)}(k)$ of the processes arising by averaging X_t over the intervals $[tn, (t+1)n)$, $t \in \mathbb{R}$, converges to an autocorrelation function $R^{(\infty)}(k)$, moreover $R^{(\infty)}$ and R_X are *equivalent* at infinity, i.e., $R^{(\infty)}(m)$ and $R_X(m)$ converge to zero as $m \to \infty$ in the same order.

Let us start with the Gaussian case, assume

$$S_{X,2}(\lambda) = |2\pi\lambda|^{-2h} L\big(|\lambda|^{-1}\big), \tag{11}$$

then the correlation of X_t is (up to some weak Tauberian conditions, see [Pal07]) necessarily $R(k) = \mathsf{c}|k|^{2h-1}L(k)$. Since $\operatorname{Var} X_t^{(n)} \simeq n^{2h-1}L(n^{-1})2\sigma_{2h} \neq 0$, see (14) for σ_{2h}, we consider the limit of the correlation

$$R^{(n)}(k) = \frac{\operatorname{Cov}(X_{t+k}^{(n)}, X_t^{(n)})}{\operatorname{Var} X_t^{(n)}}.$$

We have

$$R^{(\infty)}(k) = \lim_{n \to \infty} R^{(n)}(k) = \frac{1}{2}\Delta_{1/2}^2 k^{2h+1}, \tag{12}$$

which is called *asymptotic self-similarity (* [Cox84], [Cox91], [WPRT03]), see (15) for $\Delta_{1/2}$. The justification of this definition is based on Cox's observation [Cox84], namely, a stationary Gaussian time series is characterized not only either by the covariances or the spectrum but also the sequence of variances of the associated aggregated series as well. More specifically, let us introduce the sequence of variances $V_2(n) = \operatorname{Var} X_t^{(n)}$, then it is evident that $V_2(n)$ is determined by the covariance function $\operatorname{Cov}(X_{t+k}, X_t)$, but the inversion

$$\operatorname{Cov}(X_{t+k}, X_t) = \frac{1}{2}\Delta_{1/2}^2 (k^2 V_2(k)),$$

is also valid, see Lemma 2 in the Appendix. It follows that $\Delta_{1/2}^2 k^{2h+1}/2$ corresponds to the variance $V_2(k) = k^{2h-1}$ hence the order of the convergence of covariances is k^{2h-1}. The fractal property shows up in the limit i.e., if $a > 0$,

$$R^{(\infty)}(ak) \simeq a^{2h-1} h(2h+1) k^{2h-1} = a^{2h-1} R^{(\infty)}(k),$$

as $k \to \infty$. This type of self-similarity (mentioned by Cox in [Cox91] and Bladt [Bla94] generalized it for multiple time series in [Bla94]), is the weakest, i.e., the most general among the various other self-similarity concepts. It is the one that always follows from long-range dependence.

3.2. Asymptotic self-similarity and third order properties. The third order standardized cumulants

$$R^{(n)}(k_{1:2}) = \frac{\operatorname{Cum}(X^{(n)}_{t+k_1}, X^{(n)}_{t+k_2}, X^{(n)}_t)}{(\operatorname{Var} X^{(n)}_t)^{3/2}},$$

are not appropriate for the analysis of the third order asymptotic self-similarity. The main reason is that the denominator is second order and there are time series which are LRD in third order but they are not in second order, see [Ter08]) The definition (12) can be changed in terms of covariance function. Namely, the covariance function of the aggregated series in the limit does not change except some properly normed slowly varying function.

Definition 3. The third order stationary time series X_t with bicovariances $C_3(k_{1:2})$ is *asymptotically self-similar in third order* if the bicovarince $C_3^{(n)}(k_{1:2})$ of the aggregated series $X_t^{(n)}$ has the limit

$$\lim_{n\to\infty} C_3^{(n)}(k_{1:2}) = \widetilde{C}_3(k_{1:2}),$$

where $\widetilde{C}_3(k_{1:2})$ differs from $C_3(k_{1:2})$ by some slowly varying function at most.

This notion of self-similarity is general enough to include the LRD similarly to the second order case.

Theorem 2. *A third order LRD time series is asymptotically self-similar in third order.*

See Appendix A.2 for the outline of the proof.

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A. Appendix

Some constants mentioned in the paper:

$$c_{2h} = \frac{\Gamma(1-2h)}{\Gamma(h)\Gamma(1-h)}, \tag{13}$$

$$\sigma_{2h} = \frac{c_{2h}}{2h(2h+1)}. \tag{14}$$

A.1. Proof of Theorem 1. Before we prove the theorem let us make a note. If the bispectrum S_3 is given in the principal domain $\triangle_1$ then for the definition on the whole square $[0,1]^2$ one uses elementary transformations based on the symmetry of S_3 and including complex conjugation. Let us consider the integral

$$c_3(\ell_1,\ell_2) = \int_{\triangle_1} e^{i2\pi(\ell_1\omega_1+\ell_2\omega_2)} S_3(\omega_{1:2})d\omega_{1:2},$$

then it is easy to see that transforming S_3 is equivalent to transforming (ℓ_1,ℓ_2) in c_3.

Lemma 1. *The cumulant function can be calculated in terms of the integral c_3, namely*

$$C_3(\ell_1,\ell_2) = 4\operatorname{Re}\Big(\operatorname*{sym}_{\ell_{1:2}}[c_3(\ell_1,\ell_2)+c_3(\ell_2-\ell_1,-\ell_1)+c_3(-\ell_1\ell_2-\ell_1)]\Big),$$

where $\operatorname{sym}_{\ell_{1:2}}$ *denotes the sum according to all possible permutations of $\ell_{1:2}$ divided by the number of the (two) terms.*

Proof of Theorem 1. We consider

$$S_3(\omega_{1:2}) = \mathsf{c}\operatorname*{sym}_{\omega_{1:3}} |\omega_{1:2}|^{-h_0}\omega_1^{-h_1}\omega_2^{-h_2}\omega_3^{-h_3} L\big(|\omega_{1:2}|^{-1},\alpha^{-1}_{\omega_2/\omega_1}\big),$$

where $h_1 \geq h_2 \geq h_3 \geq 0$, and $\sum_0^3 h_k > 0$.

1. In case $h_1 = 0$, (6) rewrites into the form

$$S_3(\omega_{1:2}) = \mathsf{c}\operatorname*{sym}_{\omega_{1:3}} |\omega_{1:2}|^{-h_0} L\big(|\omega_{1:2}|^{-1},\alpha^{-1}_{\omega_2/\omega_1}\big),$$

and

$$\begin{aligned} C_3(k_{1:2}) &= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} e^{i2\pi(k_1\omega_1+k_2\omega_2)} S_3(\omega_{1:2})d\omega_{1:2} \\ &= \mathsf{c}\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} e^{i2\pi(k_1\omega_1+k_2\omega_2)}|\omega_{1:2}|^{-h_0} L\big(|\omega_{1:2}|^{-1},\alpha^{-1}_{\omega_2/\omega_1}\big)d\omega_{1:2} \\ &= \mathsf{c}r^{h_0-2}\int_0^{2\pi}\int_0^{p_\alpha} e^{i2\pi\rho\cos(\alpha-\beta)}\rho^{1-h_0} L\big((\rho/r)^{-1},\alpha^{-1}\big)d\rho d\alpha \\ &= \mathsf{c}r^{h_0-2}\int_0^{2\pi}\int_0^{p_\alpha} e^{i2\pi\rho\cos\alpha}\rho^{1-h_0} L\big((\rho/r)^{-1},(\alpha+\beta)^{-1}\big)d\rho d\alpha, \end{aligned}$$

where $k_1 = r\cos\beta$ and $k_2 = r\sin\beta$, $\beta \in (0,\pi/4)$, hence the result, see Stein–Weiss [SW71], Ch. IV. Theorem 4.1.

2. Case $h_2 = 0$, $h_1 > 0$. We have

$$C_3(k_{1:2}) = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} e^{i2\pi(k_1\omega_1+k_2\omega_2)} S_3(\omega_{1:2}) d\omega_{1:2}.$$

Consider first

$$\begin{aligned}
c_3(k_{1:2}) &= \mathsf{c}\int_{\triangle_1} e^{i2\pi(k_1\omega_1+k_2\omega_2)} |\omega_{1:2}|^{-h_0}\omega_1^{-h_1} L\big(|\omega_{1:2}|^{-1}, \alpha^{-1}_{\omega_2/\omega_1}\big) d\omega_{1:2}\\
&= \mathsf{c} r^{h_0+h_1-2}\cos^{h_1-1}\beta\sin^{-1}\beta|\sin\beta|^{h_0}\\
&\quad\times\int_{\triangle_1} e^{i2\pi(\omega_1+\omega_2)}\left|\sqrt{\omega_1^2\tan^2\beta+\omega_2^2}\right|^{-h_0}\\
&\quad\times\omega_1^{-h_1} L\big(|\omega_{1:2}|^{-1}, \alpha^{-1}_{\omega_2/\omega_1}\big) d\omega_{1:2}\\
&\simeq \mathsf{c} r^{h_0+h_1-2}\cos^{h_1-1}\beta\sin^{-1}\beta\sin^{h_0}\beta\\
&\quad\times\int_{\triangle_1} e^{i2\pi(\omega_1+\omega_2)}\omega_2^{-h_0}\omega_1^{-h_1} L\big(|\omega_{1:2}|^{-1}, \alpha^{-1}_{\omega_2/\omega_1}\big) d\omega_{1:2}\\
&\simeq \mathsf{c} r^{h_0+h_1-2}\beta^{h_0-1}.
\end{aligned}$$

We proceed by changing $k_{1:2}$ into (k_2, k_1), $(k_2 - k_1, -k_1)$ etc. hence follow the singularities β^{h_0-1}, $\beta^{h_0+h_1-1}$, $(\pi/4-\beta)^{h_0-1}$, $(\pi/4-\beta)^{h_0+h_1-1}$, provided $h_0 > 0$.

3. $h_1 \geq h_2 \geq h_3$, $h_3 = 0$, $h_2 > 0$

$$\begin{aligned}
c_3(k_{1:2}) &= \mathsf{c}\int_{\triangle_1} e^{i2\pi(k_1\omega_1+k_2\omega_2)} |\omega_{1:2}|^{-h_0}\omega_1^{-h_1}\omega_2^{-h_2} L\big(|\omega_{1:2}|^{-1}, \alpha^{-1}_{\omega_2/\omega_1}\big) d\omega_{1:2}\\
&= \mathsf{c} r^{h_0+h_1+h_2-2}\cos^{h_1-1}\beta\sin^{-1}\beta|\sin\beta|^{h_0+h_2}\\
&\quad\times\int_{\triangle_1} e^{i2\pi(\omega_1+\omega_2)}\left|\sqrt{\omega_1^2\tan^2\beta+\omega_2^2}\right|^{-h_0}\\
&\quad\times\omega_1^{-h_1}\omega_2^{-h_2} L\big(|\omega_{1:2}|^{-1}, \alpha^{-1}_{\omega_2/\omega_1}\big) d\omega_{1:2}\\
&\simeq \mathsf{c} r^{h_0+h_1+h_2-2}\cos^{h_1-1}\beta\sin^{-1}\beta|\sin\beta|^{h_0+h_2}\\
&\quad\times\int_{\triangle_1} e^{i2\pi(\omega_1+\omega_2)}\omega_2^{-h_0-h_2}\omega_1^{-h_1} L\big(|\omega_{1:2}|^{-1}, \alpha^{-1}_{\omega_2/\omega_1}\big) d\omega_{1:2}\\
&\simeq \mathsf{c} r^{h_0+h_1+h_2-2}\beta^{h_0+h_2-1},
\end{aligned}$$

hence follow the singularities $\beta^{h_0+h_2-1}$ and $(\pi/4-\beta)^{h_0+h_2-1}$

4. If $h_1 = h_2 = h_3 = h > 0$, then we have only a β^{h_0+2h-1} singularity. If $h_3 > 0$, $h_1 \neq h_3$, then $\beta^{h_0+h_2+h_3-1}$ and $(\pi/4-\beta)^{h_0+h_2+h_3-1}$ are the singularities.

□

A.2. Proof of Theorem 2. Let us introduce the following operators:

1. $\Delta_{1/2}$ is a one lag central difference operator
$$\Delta_{1/2} f(k) = f(k+1/2) - f(k-1/2). \tag{15}$$
2. For the difference operator $\Delta_{1,1}$ and $\Delta_{2,2}$ in 2D we use the following definitions
$$\begin{aligned}\Delta_{2,2} f(k_1,k_2) &= \Delta_{1,1} f(k_1+1,k_2+1) - \Delta_{1,1} f(k_1,k_2)\\ \Delta_{1,1} f(k_1,k_2) &= f(k_1,k_2) - f(k_1-1,k_2) - f(k_1,k_2-1)\\ &\quad + f(k_1-1,k_2-1).\end{aligned} \tag{16}$$

Lemma 2. *Suppose that the time series X_j is third order stationary with covariance function $C_2(k) = \operatorname{Cov}(X_{t+k}, X_t)$, bicovariance function $C_3(k,\ell) = \operatorname{Cum}(X_{t+k}, X_{t+\ell}, X_t)$, and variance sequence $V_2(k) = \operatorname{Var} X_t^{(k)}$ and bicovariance sequence $V_3(k,\ell) = \operatorname{Cum}(X_0^{(k)}, X_0^{(\ell)}, X_0^{(\ell)})$. Then*
$$\begin{aligned}C_2(k) &= \frac{1}{2}\Delta_{1/2}^2(k^2 V_2(k)), \qquad k>1,\\ C_3(k,\ell) &= \frac{1}{2}\Delta_{2,2}(k\ell^2 V_3(k,\ell)) \quad k \ge \ell > 1.\end{aligned}$$
The initial values are $C_2(0) = V_2(1)$, $C_3(0,0) = V_3(1,1)$, $C_3(k,0) = \Delta(kV_3(k,1)) = (k+1)V_3(k+1,1) - kV_3(k,1)$.

Proof. We prove the third order formula. If
$$X_t = \int_{-1/2}^{1/2} e^{i2\pi t\omega} \mathcal{M}(d\omega),$$
then
$$X_t^{(k)} = \frac{1}{k}\int_{-1/2}^{1/2} e^{i2\pi tk\omega} \frac{e^{i2\pi k\omega}-1}{e^{i2\pi\omega}-1}\mathcal{M}(d\omega) = \int_{-1/2}^{1/2} e^{i2\pi tk\omega}\mathfrak{e}_I(k\omega)\frac{i2\pi\omega}{e^{i2\pi\omega}-1}\mathcal{M}(d\omega),$$
If $k \ge \ell > 1$ then
$$V_3(k,\ell) = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \mathfrak{e}_I(k\omega_1)\mathfrak{e}_I(\ell\omega_2)\mathfrak{e}_I(\ell\omega_3)\Psi^{-1}(\omega_{1:2})S_3(\omega_{1:2})d\omega_{1:2},$$
where $\omega_3 = -\omega_1 - \omega_2$, and
$$\Psi(\omega_{1:2}) = \prod_{j=1}^{3}\mathfrak{e}_I(\omega_j).$$

Now, we rewrite

$$k\ell^2\mathfrak{e}_I(k\omega_1)\mathfrak{e}_I(\ell\omega_2)\mathfrak{e}_I(\ell\omega_3) = \int_0^k\int_0^\ell\int_0^\ell e^{i2\pi(\lambda_1\omega_1+\lambda_2\omega_2+\lambda_3\omega_3)}d\lambda_{1:3},$$

and prove the identity

$$\Delta_{2,2}(k\ell^2\mathfrak{e}_I(k\omega_1)\mathfrak{e}_I(\ell\omega_2)\mathfrak{e}_I(\ell\omega_3)) = (e^{i2\pi(k\omega_1+\ell\omega_2)} + e^{i2\pi(k\omega_1+\ell\omega_3)})\mathfrak{e}_I(\omega_1)\mathfrak{e}_I(\omega_2)\mathfrak{e}_I(\omega_3). \tag{17}$$

We start with the first order difference

$$\begin{aligned}
&k\ell^2\Delta_{1,1}(\mathfrak{e}_I((k+1)\omega_1)\mathfrak{e}_I((\ell+1)\omega_2)\mathfrak{e}_I((\ell+1)\omega_3))\\
&= \left[\int_0^{k+1}\int_0^{\ell+1}\int_0^{\ell+1} - \int_0^{k}\int_0^{\ell+1}\int_0^{\ell+1} - \int_0^{k+1}\int_0^{\ell}\int_0^{\ell} + \int_0^{k}\int_0^{\ell}\int_0^{\ell}\right]\\
&\quad\times e^{i2\pi(\lambda_1\omega_1+\lambda_2\omega_2+\lambda_3\omega_3)}d\lambda_{1:3}\\
&= \left[\int_k^{k+1}\left(\int_0^{\ell+1}\int_0^{\ell+1} - \int_0^{\ell}\int_0^{\ell}\right)\right]e^{i2\pi(\lambda_1\omega_1+\lambda_2\omega_2+\lambda_3\omega_3)}d\lambda_{1:3}\\
&= \left[\int_k^{k+1}\left(\int_\ell^{\ell+1}\int_0^{\ell} + \int_0^{\ell}\int_\ell^{\ell+1} + \int_\ell^{\ell+1}\int_\ell^{\ell+1}\right)\right]e^{i2\pi(\lambda_1\omega_1+\lambda_2\omega_2+\lambda_3\omega_3)}d\lambda_{1:3}.
\end{aligned}$$

and proceed with the second order one

$$\begin{aligned}
&k\ell^2\Delta_{1,1}[\mathfrak{e}_I((k+1)\omega_1)\mathfrak{e}_I((\ell+1)\omega_2)\mathfrak{e}_I((\ell+1)\omega_3) - \mathfrak{e}_I(k\omega_1)\mathfrak{e}_I(\ell\omega_2)\mathfrak{e}_I(\ell\omega_3)]\\
&\quad= \int_k^{k+1}\left(\int_\ell^{\ell+1}\int_0^{1} + \int_0^{1}\int_\ell^{\ell+1}\right)e^{i2\pi(\lambda_1\omega_1+\lambda_2\omega_2+\lambda_3\omega_3)}d\lambda_{1:3}\\
&\quad= \left(e^{i2\pi(k\omega_1+\ell\omega_2)} + e^{i2\pi(k\omega_1+\ell\omega_3)}\right)\mathfrak{e}_I(\omega_1)\mathfrak{e}_I(\omega_2)\mathfrak{e}_I(\omega_3).
\end{aligned}$$

It follows

$$\Delta_{2,2}(k\ell^2V_3(k,\ell)) = 2C_3(k,\ell). \qquad \square$$

Outline of the proof of Theorem 2. The bicovariances $C_3^{(n)}(k_{1:2}) = \mathrm{Cum}(X_{t+k_1}^{(n)}, X_{t+k_2}^{(n)}, X_t^{(n)})$ of the aggregated series $X_t^{(n)}$ is calculated directly

$$\begin{aligned}
C_3^{(n)}(k_{1:2}) &= \frac{1}{n^3}\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} e^{i2\pi n(k_1\omega_1+k_2\omega_2)}\prod_{j=1}^{3}\frac{e^{i2\pi n\omega_j}-1}{e^{i2\pi\omega_j}-1}S_3(\omega_{1:2})d\omega_{1:2}\\
&= \frac{1}{n^2}\int_{-n/2}^{n/2}\int_{-n/2}^{n/2} e^{i2\pi(k_1\lambda_1+k_2\lambda_2)}\Psi(\lambda_{1:2})\Psi^{-1}(\lambda_{1:2}/n)S_3(\lambda_{1:2}/n)d\lambda_{1:2}.
\end{aligned}$$

The bispectrum S_3 has the form (3), put

$$L_3(\lambda_{1:2}, n) = L\big(|\lambda_{1:2}/n|^{-1}, \alpha^{-1}_{\lambda_2/\lambda_1}\big)\Psi^{-1}(\lambda_{1:2}/n),$$

such that $\Psi^{-1}(\lambda_{1:2})S_3(\lambda_{1:2}) = s_3(\lambda_{1:2})L_3(\lambda_{1:2})$ and apply Lemma 2

$$n^{2-3g_0}C_3^{(n)}(k_{1:2}) = \mathsf{c}\int_{-n/2}^{n/2}\int_{-n/2}^{n/2} e^{i2\pi(k_1\lambda_1+k_2\lambda_2)}\Psi(\lambda_{1:2})s_3(\lambda_{1:2})L_3(\lambda_{1:2}, n)d\lambda_{1:2}$$

$$\simeq \mathsf{c}\frac{1}{2}\Delta_{2,2}\left[k_1k_2^2\int_{\mathbb{R}^2}\mathfrak{e}_I(k_1\omega_1)\mathfrak{e}_I(k_2\omega_2)\mathfrak{e}_I(k_2\omega_3)s_3(\lambda_{1:2})L\big(\alpha^{-1}_{\omega_2/\omega_1}\big)d\omega_{1:2}\right] = \widetilde{C}_3(k_{1:2}),$$

see (17), where $3g_0 = h_0 + h_1 + h_2 + h_3$. Hence we have

$$\lim_{n\to\infty}\frac{\mathrm{Cum}(X^{(n)}_{t+k_1}, X^{(n)}_{t+k_2}, X^{(n)}_t)}{n^{3g_0-2}} = \widetilde{C}_3(k_{1:2}). \qquad \square$$

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GYÖRGY TERDIK
FACULTY OF INFORMATICS
UNIVERSITY OF DEBRECEN
4010, DEBRECEN P.O. BOX 12
HUNGARY

E-mail: terdik@delfin.unideb.hu

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