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Annihilators on co-commutators with generalized derivations on Lie ideals

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Abstract. Let R be a prime ring of characteristic different from 2, U the Utumi quotient ring of R, $C = Z(U)$ the extended centroid of R, L a non-central Lie ideal of R, H and G non-zero generalized derivations of R. Suppose that there exists $0 \neq a \in R$ such that $a(H(u)u - uG(u)) = 0$, for all $u \in L$, then one of the following holds:

(1) there exist $b', c' \in U$ such that $H(x) = b'x + xc'$, $G(x) = c'x$ with $ab' = 0$;

(2) R satisfies s_4 and there exist $b', c', q' \in U$ such that $H(x) = b'x + xc'$, $G(x) = c'x + xq'$, with $a(b' - q') = 0$.

1. Introduction

Let R be a prime ring of characteristic different from 2 with center $Z(R)$ and extended centroid C. The standard polynomial of degree 4 is defined as $s_4(x_1, \dots, x_4) = \sum_{\sigma \in S_4} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(4)}$, where σ runs over S_4 the symmetric group of degree 4 and where $(-1)^\sigma$ is 1 or -1 according as σ is an even or odd permutation.

A well known result of Posner [18] states that if d is a derivation of R such that $[d(x), x] \in Z(R)$, for any $x \in R$, then either $d = 0$ or R is commutative. This theorem indicates that the global structure of a ring R is often tightly connected to the behaviour of additive mappings defined on R. Following this line of investigation, several authors generalized the Posner's Theorem. For instance in [2] Bresar proves that if d and δ are derivations of R such that

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$d(x)x - x\delta(x) \in Z(R)$, for all $x \in R$, then either $d = \delta = 0$ or R is commutative. Later in [12] Lee and Wong consider the case when $d(x)x - x\delta(x) \in Z(R)$, for all x in some non-central Lie ideal L of R. They prove that either $d = \delta = 0$ or R satisfies s_4, the standard identity of degree 4. Recently in [17] Niu and Wu study the left annihilator of the set $\{d(u)u - u\delta(u), u \in L\}$, where d and δ are derivations of R and L is a non-central Lie ideal of R. In case the annihilator is not zero, the conclusion is that R satisfies the standard identity s_4 and $d = -\delta$ are inner derivations. These facts in a prime ring are natural tests which evidence that the set $\{d(u)u - u\delta(u), u \in L\}$ is rather large in R.

Here we will consider the same situation in the case the derivations d and δ are replaced respectively by the generalized derivations H and G. More specifically an additive map $G : R \longrightarrow R$ is said to be a generalized derivation if there is a derivation d of R such that, for all $x, y \in R$, $G(xy) = G(x)y + xd(y)$. A significative example is a map of the form $G(x) = ax + xb$, for some $a, b \in R$; such generalized derivations are called inner. Generalized derivations have been primarily studied on operator algebras. Therefore any investigation from the algebraic point of view might be interesting (see for example [13]). Here our purpose is to prove the following theorem:

Theorem 1. *Let R be a prime ring of characteristic different from 2, U the Utumi quotient ring of R, $C = Z(U)$ the extended centroid of R, L a non-central Lie ideal of R, H and G non-zero generalized derivations of R. Suppose that there exists $0 \neq a \in R$ such that $a(H(u)u - uG(u)) = 0$, for all $u \in L$, then one of the following holds:*

(1) *there exist $b', c' \in U$ such that $H(x) = b'x + xc'$, $G(x) = c'x$ with $ab' = 0$;*
(2) *R satisfies s_4 and there exist $b', c', q' \in U$ such that $H(x) = b'x + xc'$, $G(x) = c'x + xq'$, with $a(b' - q') = 0$.*

In all that follows let R be a non-commutative prime ring of characteristic different from 2, U its Utumi quotient ring and $C = Z(U)$ the center of U. We refer the reader to [1] for the definitions and the related properties of these objects. In particular we make use of the following well known facts:

Fact 1. If I is a two-sided ideal of R, then R, I and U satisfies the same generalized polynomial identities with coefficients in U ([4]).

Fact 2. Every derivation d of R can be uniquely extended to a derivation of U (see Proposition 2.5.1 in [1]).

Fact 3. We denote by $\mathrm{Der}(U)$ the set of all derivations on U. By a derivation word we mean an additive map Δ of the form $\Delta = d_1 d_2 \dots d_m$, with each

$d_i \in \mathrm{Der}(U)$. Then a differential polynomial is a generalized polynomial, with coefficients in U, of the form $\Phi(^{\Delta_j} x_i)$ involving non-commutative indeterminates x_i on which the derivations words Δ_j act as unary operations. The differential polynomial $\Phi(^{\Delta_j} x_i)$ is said to be a differential identity on a subset T of U if it vanishes for any assignment of values from T to its indeterminates x_i.

Let D_{int} be the C-subspace of $\mathrm{Der}(U)$ consisting of all inner derivations on U and let d be a non-zero derivation on R. By Theorem 2 in [10] we have the following result (see also Theorem 1 in [14]): If $\Phi(x_1, \ldots, x_n, {}^d x_1, \ldots, {}^d x_n)$ is a differential identity on R, then one of the following holds:

(1) either $d \in D_{\mathrm{int}}$;

(2) or R satisfies the generalized polynomial identity $\Phi(x_1, \ldots, x_n, y_1, \ldots, y_n)$.

Fact 4. If I is a two-sided ideal of R, then R, I and U satisfies the same differential identities ([14]).

We refer the reader to Chapter 7 in [1] for a complete and detailed description of the theory of generalized polynomial identities involving derivations.

Fact 5. Since we assume that $\mathrm{char}(R) \neq 2$, then there exists a non-zero two-sided ideal I of R such that $0 \neq [I, R] \subseteq L$. In particular, if R is a simple ring it follows that $[R, R] \subseteq L$.

This follows from pp. 4–5 in [9], Lemma 2 and Proposition 1 in [6], Theorem 4 in [11].

2. The case of inner generalized derivations

We dedicate this section to prove the theorem in case both the generalized derivations H and G are inner, that is there exist $b, c, p, q \in U$ such that $H(x) = bx + xc$ and $G(x) = px + xq$, for all $x \in R$.

In light of Fact 5, since we suppose $\mathrm{char}(R) \neq 2$, there exists a non-central ideal I of R such that $[I, I] \subseteq L$. This implies that $a(b[r_1, r_2]^2 + [r_1, r_2](c-p)[r_1, r_2] - [r_1, r_2]^2 q) = 0$ for all $r_1, r_2 \in I$. Moreover by Fact 1, I and R satisfy the same generalized polynomial identities, thus $a(b[r_1, r_2]^2 + [r_1, r_2](c-p)[r_1, r_2] - [r_1, r_2]^2 q) = 0$ for all $r_1, r_2 \in R$. Hence in all that follows we assume that R satisfies the following generalized polynomial identity

$$P(x_1, x_2) = a\big(b[x_1, x_2]^2 + [x_1, x_2](c - p)[x_1, x_2] - [x_1, x_2]^2 q\big).$$

$P(x_1, x_2)$ is a generalized polynomial in the free product $U *_C C\{x_1, x_2\}$ of the C-algebra U and the free C-algebra $C\{x_1, x_2\}$.

We first prove the following:

Proposition 1. *If $a \in Z(R)$, then one of the following holds:*

(1) *either there exists $b' \in U$ such that $H(x) = xb'$ and $G(x) = b'x$, for all $x \in R$;*

(2) *or R satisfies s_4 and there exists $\alpha \in C$ such that $p = c - \alpha$ and $q = b + \alpha$, that is $H(x) = bx + xc$ and $G(x) = cx + xb$.*

Proof. Since $a \in Z(R)$, then a is not a zero-divisor, then by main assumption it follows that $(H(u)u - uG(u)) = 0$, for all $u \in [R, R]$. In this case, it is proved in [15] that either there exists $b' \in U$ such that $H(x) = xb'$ and $G(x) = b'x$, for all $x \in R$, or R satisfies s_4. In this last case R is PI-ring, moreover U satisfies the same generalized polynomial identities of R. Therefore U is a central simple algebra of dimension at most 4 over its center, and it is known that in this case $[r, s]^2 \in Z(U) = C$ for all $r, s \in U$. Moreover U satisfies

$$b[x_1, x_2]^2 + [x_1, x_2](c - p)[x_1, x_2] - [x_1, x_2]^2 q.$$

Since the polynomial $[x_1, x_2]^2$ is central valued in U, then U satisfies

$$(b - q)[x_1, x_2]^2 + [x_1, x_2](c - p)[x_1, x_2]. \tag{1}$$

Denote e_{ij} the usual matrix unit, with 1 in the (i, j)-entry and zero elsewhere, and write $w = (c - p) = \sum_{rs} w_{rs} e_{rs}$, for suitable $w_{rs} \in C$. Therefore for any $i \neq j$, let $r_1 = e_{ii}$, $r_2 = e_{ij}$ and $[r_1, r_2] = e_{ij}$. It follows by (1) that $e_{ij} w e_{ij} = 0$ for all $i \neq j$, that is $w_{ji} = 0$ and w is a diagonal matrix in $M_2(C)$. Moreover, for all $\varphi \in \mathrm{Aut}_F(M_2(C))$, U satisfies

$$\varphi\big((b - q)[x_1, x_2]^2 + [x_1, x_2](c - p)[x_1, x_2]\big)$$

which is

$$\big(\varphi(b - q)[x_1, x_2]^2 + [x_1, x_2]\varphi(c - p)[x_1, x_2]\big)$$

since the set of all the evaluation of $[x_1, x_2]$ is invariant under the action of any element of $\mathrm{Aut}_F(M_2(C))$. By the above argument, $\varphi(c-p)$ must be diagonal. In particular, let $r \neq s$ and $\varphi(x) = (1 + e_{rs})x(1 - e_{rs})$, hence

$$\varphi(c - p) = \sum_t w_{tt} e_{tt} + w_{ss} e_{rs} - w_{rr} e_{rs}$$

which implies $w_{rr} = w_{ss}$, for all $r \neq s$. Thus $c - p$ is a central matrix, namely $c - p = \alpha$. By (1) we get that U satisfies $(b - q + \alpha)[x_1, x_2]^2$, and since $0 \neq [U, U]^2 \subseteq C$, we also have $q - b = \alpha = c - p$. Thus we conclude that, in case R satisfies s_4, $p = c - \alpha$ and $q = b + \alpha$. □

Proposition 2. *If $a \notin Z(R)$ then either $P(x_1, x_2)$ is a non-trivial generalized polynomial identity for R or $H(x) = b'x + xc'$, $G(x) = c'x$ for some $b', c' \in U$ satisfying $ab' = 0$.*

Proof. Suppose now that R does not satisfy any non-trivial generalized polynomial identity. Let $T = U *_C C\{X\}$ be the free product over C of the C-algebra U and the free C-algebra $C\{X\}$, with X the countable set consisting of non-commuting indeterminates $x_1, x_2, \dots, x_n, \dots$.

For brevity we write $P(X)$ instead of $P(x_1, x_2)$ and $f(X)$ instead of $[x_1, x_2]$.

Now consider the generalized polynomial $P(X) \in U *_C C\{X\}$. By our hypothesis, R satisfies the following generalized polynomial identity:

$$P(X) = abf(X)^2 + af(X)(c-p)f(X) - af(X)^2 q = 0 \in T.$$

Since R does not satisfy non-trivial GPIs, by [4], the coefficients $\{ab, a\}$ must be linearly C-dependent. Therefore there exist $\beta_1, \beta_2 \in C$ such that $\beta_1(ab) + \beta_2 a = 0$, with $\beta_1 \neq 0$ since $a \notin C$. Hence we may write $ab = \lambda a$, for a suitable $\lambda \in C$. In this situation R satisfies

$$a\big(\lambda f(X)^2 + f(X)(c-p)f(X) - f(X)^2 q\big)$$

that is

$$\lambda f(X)^2 + f(X)(c-p)f(X) - f(X)^2 q = 0 \in T.$$

Again since R does not satisfy any non-trivial generalized polynomial identity, $\{1, q\}$ must be linearly C-dependent, that is $q \in C$. This implies that $G(x) = (p+q)x$ and also that R satisfies

$$f(X)\big(\lambda + (c-p) - q\big)f(X)$$

which implies $\lambda + (c-p) - q = 0$, that is $H(x) = bx + x(p+q-\lambda) = (b-\lambda)x + x(p+q)$, and we obtain the required conclusion, for $b' = b - \lambda$ and $c' = p + q$. □

Lemma 1. *Let $R = M_m(F)$ be the ring of all $m \times m$ matrices over a field F of characteristic different from 2. If a is not central in R then there exists $\alpha \in F$ such that $p = c - \alpha \cdot I_m$, where I_m is the identity matrix of order m, and one of the following holds:*

(1) *$q \in Z(R)$ and there exists $\gamma \in F$ such that $p + q = c + \gamma \cdot I_m$, that is $H(x) = bx + xc$, $G(x) = (c + \gamma \cdot I_m)x$; moreover $a(b - \gamma \cdot I_m) = 0$;*

(2) *R satisfies s_4 and there exists $q' \in R$ such that $G(x) = cx + xq'$, with $a(b - q') = 0$.*

Proof. Denote $a = \sum_{rs} e_{rs}a_{rs}$, $q = \sum_{rs} e_{rs}q_{rs}$, $c - p = w = \sum_{rs} e_{rs}w_{rs}$, for suitable $a_{rs}, q_{rs}, w_{rs} \in F$. By the main assumption, R satisfies

$$a\bigl(b[x_1,x_2]^2 + [x_1,x_2](c-p)[x_1,x_2] - [x_1,x_2]^2 q\bigr). \tag{2}$$

Fix $[x_1,x_2] = e_{ij}$, for any $i \neq j$. In this case from (2) we have

$$ae_{ij}(c-p)e_{ij} = 0 \tag{3}$$

that is

$$\text{either} \quad a_{ki} = 0 \quad \forall k \qquad \text{or} \quad w_{ji} = 0 \qquad (3').$$

Here we first prove that w is a diagonal matrix. In order to do this, we suppose that there exists some non-zero off-diagonal entry of w and divide the proof into two cases:

Case 1: $m = 2$.

Suppose $w_{21} \neq 0$, then by (3) it follows $a_{11} = a_{21} = 0$. Of course, since we suppose $a \neq 0$, we must assume now $w_{12} = 0$.

Choose $[x_1,x_2] = [e_{12}, e_{21}] = e_{11} - e_{22}$ and by (2) we have

$$0 = Y = a\bigl(b(e_{11}-e_{22})^2 + (e_{11}-e_{22})(c-p)(e_{11}-e_{22}) - (e_{11}-e_{22})^2 q\bigr)$$

in particular the $(1,1)$-entry of the matrix Y is $a_{12}(b_{21} - w_{21} - q_{21}) = 0$ and the $(2,1)$-one is $a_{22}(b_{21} - w_{21} - q_{21}) = 0$. Therefore, from $a \neq 0$ follows

$$b_{21} - w_{21} - q_{21} = 0. \tag{4}$$

In the same way, for $[x_1,x_2] = [e_{12} - e_{21}, e_{22}] = e_{12} + e_{21}$ in (2) we have

$$0 = T = a\bigl(b(e_{12}+e_{21})^2 + (e_{12}+e_{21})(c-p)(e_{12}+e_{21}) - (e_{12}+e_{21})^2 q\bigr).$$

The $(1,1)$-entry of the matrix T is $a_{12}(b_{21} - q_{21}) = 0$ and the $(2,1)$-one is $a_{22}(b_{21} - q_{21}) = 0$. Since $a \neq 0$ we get

$$b_{21} - q_{21} = 0. \tag{5}$$

Thus by (5) and (4) we obtain the contradiction $w_{21} = 0$.

Case 2: $m \geq 3$.

Also in this case we suppose that there exists $w_{ji} \neq 0$ for some $i \neq j$, so that $a_{ki} = 0$ for all k, that is the i-th column of a is zero.

Let now $q \neq i, j$ and fix $[x_1, x_2] = [e_{ij} + e_{qj}, e_{jj}] = e_{ij} + e_{qj}$. Then (2) implies $a(e_{ij} + e_{qj})w(e_{ij} + e_{qj}) = 0$ and since $a_{ki} = 0$ for all k, it follows that $ae_{qj}w(e_{ij} + e_{qj}) = 0$. Moreover, by (3), we get $ae_{qj}we_{qj} = 0$, which implies that $ae_{qj}we_{ij} = 0$. The assumption $w_{ji} \neq 0$ implies that $a_{kq} = 0$ for all k, that is a has just one non-zero column, the j-th one: $a = \sum_r a_{rj}e_{rj}$.

Notice that if $w_{tj} \neq 0$ for some $t \neq j$, by the same argument we get that a has just the t-th column non-zero, that is $a = 0$. Thus we may assume that $w_{tj} = 0$ for all $t \neq j$.

Let $t \neq i, j$ and denote by σ_t and τ_t the following automorphisms of R:

$$\sigma_t(x) = (1 + e_{jt})x(1 - e_{jt}) = x + e_{jt}x - xe_{jt} - e_{jt}xe_{jt}$$

$$\tau_t(x) = (1 - e_{jt})x(1 + e_{jt}) = x - e_{jt}x + xe_{jt} - e_{jt}xe_{jt}$$

and say $\sigma_t(w) = \sum \sigma_{rs}e_{rs}$, $\tau_t(w) = \sum \tau_{rs}e_{rs}$ where $\sigma_{rs}, \tau_{rs} \in F$. We have

$$\sigma_{ji} = w_{ji} + w_{ti} \quad \text{and} \quad \tau_{ji} = w_{ji} - w_{ti}.$$

If there exists t such that $\sigma_{ji} = w_{ji} + w_{ti} = 0$ or $\tau_{ji} = w_{ji} - w_{ti} = 0$ then $w_{ti} = -w_{ji} \neq 0$ or $w_{ti} = w_{ji} \neq 0$. Therefore $w_{ji} \neq 0$ and $w_{ti} \neq 0$, and so, by using (3), $a = 0$.

Hence assume that $\sigma_{ji} \neq 0$ and $\tau_{ji} \neq 0$, for all $t \neq i, j$, and recall that, for any F-automorphism φ of R, the following holds

$$\varphi(a)\big(\varphi(b)[x_1, x_2]^2 + [x_1, x_2]\varphi(c - p)[x_1, x_2] - [x_1, x_2]^2\varphi(q)\big).$$

Thus in this case by (3), for any $t \neq i, j$, the non-zero entries of the matrices $\sigma_t(a)$ and $\tau_t(a)$ are just in the j-th column. In particular, since

$$\sigma_t(a) = a + e_{jt}a - ae_{jt} - e_{jt}ae_{jt} = \sum_r a_{rj}e_{rj} - \sum_r a_{rj}e_{rt} + a_{tj}e_{jj} - a_{tj}e_{jt}$$

$$\tau_t(a) = a - e_{jt}a + ae_{jt} - e_{jt}ae_{jt} = \sum_r a_{rj}e_{rj} + \sum_r a_{rj}e_{rt} - a_{tj}e_{jj} - a_{tj}e_{jt}$$

then both the above matrices have zero in the (j, t) entry that is

$$-a_{jj} - a_{tj} = 0 \quad \text{for} \quad \sigma_t(a)$$

$$a_{jj} - a_{tj} = 0 \quad \text{for} \quad \tau_t(a).$$

By $\operatorname{char}(R) \neq 2$ we obtain $a_{jj} = a_{tj} = 0$ for all $t \neq i$, that is $a = a_{ij}e_{ij}$.

Denote now by φ and χ the following automorphisms of R:

$$\varphi(x) = (1 + e_{ji})x(1 - e_{ji}) = x + e_{ji}x - xe_{ji} - e_{ji}xe_{ji},$$

$$\chi(x) = (1 - e_{ji})x(1 + e_{ji}) = x - e_{ji}x + xe_{ji} - e_{ji}xe_{ji}$$

and say $\varphi(w) = \sum \varphi_{rs}e_{rs}$, $\chi(w) = \sum \chi_{rs}e_{rs}$ where $\varphi_{rs}, \chi_{rs} \in F$. Since, by (3′), $w_{ij} \neq 0$ implies $a = 0$, we assume that $w_{ij} = 0$. Then we have

$$\varphi_{ji} = w_{ji} - w_{jj} + w_{ii} \quad \text{and} \quad \chi_{ji} = w_{ji} + w_{jj} - w_{ii}$$

If $\varphi_{ji} = \chi_{ji} = 0$, then we get the contradiction $w_{ji} = 0$.

If at least one of φ_{ji} and χ_{ji} is not zero, then, by (3), one of $\varphi(a)$ and $\chi(a)$ has zero in all the entries of the i-th column. In particular notice that

$$\varphi(a) = a_{ij}e_{ij} - a_{ij}e_{ii} + a_{ij}e_{jj} - a_{ij}e_{ji},$$

$$\chi(a) = a_{ij}e_{ij} + a_{ij}e_{ii} - a_{ij}e_{jj} - a_{ij}e_{ji}$$

which means that in any case the (j, i)-entry is $a_{ij} = 0$, a contradiction again.

All the previous arguments say that if a is not zero, then w must be a diagonal matrix, $w = \sum_t w_t e_{tt}$.

Moreover, for all $\lambda \in \mathrm{Aut}_F(M_m(F))$, since $\lambda(a) \neq 0$ and R satisfies

$$\lambda(a)\big(\lambda(b)[x_1, x_2]^2 + [x_1, x_2]\lambda(c - p)[x_1, x_2] - [x_1, x_2]^2\lambda(q)\big),$$

we also have that $\lambda(c - p)$ is diagonal. In particular, let $r \neq s$ and $\lambda(x) = (1 + e_{rs})x(1 - e_{rs})$, hence

$$\lambda(c - p) = \sum_t w_t e_{tt} + w_s e_{rs} - w_r e_{rs}$$

is diagonal implying $w_r = w_s = \alpha$, for all $r \neq s$. Thus $c - p$ is a central matrix, namely $c - p = \alpha \cdot I_m$. Therefore R satisfies

$$ab[x_1, x_2]^2 + a[x_1, x_2]^2(\alpha - q).$$

Denote by G the additive subgroup of R generated by all the evaluations of the polynomial $[x_1, x_2]^2$. By [3], since $\mathrm{char}(R) \neq 2$, either $[R, R] \subseteq G$ or $[x_1, x_2]^2$ is central valued on R that is R satisfies s_4.

In the first case R satisfies

$$ab[x_1, x_2] + a[x_1, x_2](\alpha - q).$$

Let $\alpha - q = u = \sum_{r,s} u_{rs} e_{rs}$, with $u_{rs} \in F$. For $[x_1, x_2] = e_{ij}$, with any $i \neq j$, it follows $abe_{ij} + ae_{ij}(\alpha - q) = 0$. By right multiplying for any e_{qq}, with $q \neq j$, we have $ae_{ij}(\alpha - q)e_{qq} = 0$ that is

$$\text{either} \quad a_{ki} = 0 \quad \forall k \qquad \text{or} \quad u_{jq} = 0 \quad \forall q \neq j.$$

In particular

$$\text{either} \quad a_{ki} = 0 \quad \forall k \qquad \text{or} \quad u_{ji} = 0 \qquad (3'').$$

Notice that $(3'')$ has the same flavour of $(3')$. By the same argument as above, in case $a \neq 0$ we have that $u = \alpha - q$ is a central matrix, and so $a(b+u)[r_1, r_2] = 0$, for all $r_1, r_2 \in R$. This implies $a(b+u) = 0$, which is the conclusion 1 of Lemma 1, for $\gamma = -u$.

Consider finally the case when $[x_1, x_2]^2$ is central valued on R. Here R satisfies $a(b + \alpha - q)[x_1, x_2]^2$, moreover there exists $0 \neq [r_1, r_2]^2 \in F \cdot I_m$, which implies $a(b + \alpha - q) = 0$, the conclusion 2 of Lemma 1, for $q' = q - \alpha$. □

Lemma 2. *Let R be a prime ring of characteristic different from 2. If a is not central in R then $c - p = \alpha \in C$ and one of the following holds:*

(1) *$q \in C$ and there exist $\lambda \in C$, $b' = b - \lambda$, $c' = p + q$ such that $H(x) = b'x + xc'$, $G(x) = c'x$, with $ab' = 0$;*

(2) *$q \in C$ and there exists $\gamma = q - \alpha \in C$ such that $p + q = c + \gamma$, that is $H(x) = bx + xc$, $G(x) = (c + \gamma)x$, with $a(b - \gamma) = 0$;*

(3) *R satisfies s_4 and there exists $q' = q - \alpha$ such that $G(x) = cx + xq'$, with $a(b - q') = 0$.*

Proof. As above we denote for brevity $P(x_1, x_2)$ by $P(X)$ and $[x_1, x_2]$ by $f(X)$ and consider the generalized polynomial

$$P(X) = abf(X)^2 + af(X)(c - p)f(X) - af(X)^2 q.$$

Since U and R satisfy the same generalized polynomial identities with coefficients in U (see Fact 1),then $P(X)$ is also a generalized identity for U.

Suppose first that U does not satisfy any non-trivial generalized polynomial identity. Therefore by Proposition 2 we get conclusion 1.

Hence we may suppose now that U satisfies some non-trivial generalized polynomial identity. By [16] U is primitive having a non-zero socle $\mathrm{Soc}(U)$ with C as the associated division ring and by Jacobson's Theorem (p. 75 in [8]) U is isomorphic to a dense ring of linear transformations of some vector space V over C.

If V is finite-dimensional over C, it follows that $R \subseteq U = M_k(C)$, for $k = \dim_C V$. In this case we get the required conclusions by Lemma 1.

Let $\dim_C V = \infty$. Denote $\mathrm{End}_C V$ the ring of endomorphisms of ${}_C V$ and recall that the range of a polynomial $f(X) \in C\{x_1, x_2\}$ is defined as follows

$$r(f;U) = \{f(x_1,x_2) \in \mathrm{End}_C V : x_1, x_2 \in U\}.$$

In [19] (Lemma) it is proved that, if U is a dense subring of $\mathrm{End}_C V$ and $\dim_C V = \infty$, then $r(f;U)$ is a dense subset of $\mathrm{End}_C V$ and this implies that U satisfies the generalized polynomial identity

$$abx^2 + ax(c-p)x - ax^2q. \tag{6}$$

Suppose that there exists a minimal idempotent element e of $\mathrm{Soc}(U)$ such that $e(c-p)(1-e) \neq 0$. Replace in (6) x by $(1-e)re$ for any $r \in U$, then it follows that $a(1-e)re(c-p)(1-e)re = 0$, which implies $a(1-e) = 0$, since $e(c-p)(1-e) \neq 0$. This means that $a = ae$.

On the other hand, if in (6) we replace x by ere for any $r \in U$, we get $ab(ere)^2 + aere(c-p)ere - a(ere)^2q = 0$, and by right multiplying by $(1-e)$ one has $-ae(ere)^2q(1-e) = 0$. Since $0 \neq a = ae$, we have $eq(1-e) = 0$, that is $eq = eqe$.

Finally replace in (6) x by $x+y$. It follows that U satisfies:

$$ab(xy) + ab(yx) + ax(c-p)y + ay(c-p)x - a(xy)q - a(yx)q$$

and for any $x = re$ and $y = (1-e)s$, with $r, s \in U$, we get

$$ab(1-e)sre + are(c-p)(1-e)s + a(1-e)s(c-p)re - a(1-e)sreq = 0.$$

By right multiplying by $(1-e)$ and since $eq(1-e) = 0$, we have $are\,(c-p)(1-e)$ $s(1-e) = 0$, for all $r, s \in U$. By the primeness of U and by the assumption that $e(c-p)(1-e) \neq 0$, the contradiction $a = 0$ follows.

Therefore $e(c-p)(1-e) = 0$, for any idempotent element $e \in \mathrm{Soc}(U)$ of rank 1. Hence $[c-p, e] = 0$, for any idempotent of rank 1, and $[c-p, \mathrm{Soc}(U)] = 0$, since $\mathrm{Soc}(U)$ is generated by these idempotent elements. This argument gives $c - p \in C$, and as a consequence of (6), U satisfies the generalized polynomial identity

$$abx^2 + ax^2(c-p-q). \tag{7}$$

As above, suppose that there exists a minimal idempotent element e of $\mathrm{Soc}(U)$ such that $(1-e)(c-p-q)e \neq 0$. If we replace in (7) x by $(1-e)r(1-e)$

for any $r \in U$ and multiply by e on the right, then we get $a((1-e)r(1-e))^2(c-p-q)e = 0$, that is $a(1-e) = 0$, since $(1-e)(c-p-q)e \neq 0$.

Now by (7), for $x = t + y$, it follows that U satisfies

$$abty + abyt + aty(c-p-q) + ayt(c-p-q).$$

Finally for $t = (1-e)$, $y = z(1-e)$ and by right multiplying by e, we have $az(1-e)(c-p-q)e = 0$, that is $a = 0$, a contradiction. Therefore $(1-e)(c-p-q)e = 0$, for any idempotent element $e \in \mathrm{Soc}(U)$ of rank 1, that is as above $c-p-q \in C$, which implies $q \in C$. Therefore U satisfies $a(b+c-p-q)x^2$, that is $a(b+c-p-q) = 0$, with $c-p \in C$ and $q \in C$, which is the conclusion 2 of Lemma 2. □

3. The general case

We consider now the more general situation and prove the main Theorem of the paper. As in Section 1, since we suppose $\mathrm{char}(R) \neq 2$, by Fact 5 we may assume that there exists a non-zero ideal I of R such that

$$a\big(H([r_1,r_2])[r_1,r_2] - [r_1,r_2]G([r_1,r_2])\big) = 0$$

for all $r_1, r_2 \in I$. Under these assumptions we have that:

Theorem 2. *If R is a prime ring of characteristic different from 2, then one of the following holds:*

(1) *there exist $b', c' \in U$ such that $H(x) = b'x + xc'$, $G(x) = c'x$ with $ab' = 0$;*

(2) *R satisfies s_4 and there exist $b', c', q' \in U$ such that $H(x) = b'x + xc'$, $G(x) = c'x + xq'$, with $a(b'-q') = 0$.*

Proof. By Theorem 3 in [13] every generalized derivation g on a dense right ideal of R can be uniquely extended to the Utumi quotient ring U of R, and thus we can think of any generalized derivation of R to be defined on the whole U and to be of the form $g(x) = bx + d(x)$ for some $b \in U$ and d a derivation on U. Thus we may assume that there exist $b, p \in U$ and d, δ derivations on U such that

$$H(x) = bx + d(x) \quad \text{and} \quad G(x) = px + \delta(x).$$

Since I, R and U satisfy the same differential identities [14], then without loss of generality, in order to prove our results we may assume that

$$a\big(H([r_1,r_2])[r_1,r_2] - [r_1,r_2]G([r_1,r_2])\big) = 0$$

for all $r_1, r_2 \in U$. Hence U satisfies

$$a\big(\big(b[x_1,x_2]+d([x_1,x_2])\big)[x_1,x_2]-[x_1,x_2]\big(p[x_1,x_2]+\delta([x_1,x_2])\big)\big)$$

that is

$$\begin{aligned} a\big(\big(b[x_1,x_2]+[d(x_1),x_2]+[x_1,d(x_2)]\big)\big)[x_1,x_2]& \\ -[x_1,x_2]\big(p[x_1,x_2]+[\delta(x_1),x_2]+[x_1,\delta(x_2)]\big)\big)\big)& \qquad (8) \end{aligned}$$

where d, δ are derivations on U. We divide the proof into 3 cases:

Case 1: Let $d(x)=[c,x]$ and $\delta(x)=[q,x]$ be both inner derivations in U, so that $H(x)=bx+[c,x]=(b+c)x+x(-c)$ and $G(x)=px+[q,x]=(p+q)x+x(-q)$, for suitable elements $c,q \in U$. In this case H and G are both inner generalized derivations in U. We notice that, if $a \in C$, then by Proposition 1 we have that either there exists $b' \in U$ such that $H(x)=xb'$ and $G(x)=b'x$ for all $x \in R$ (conclusion 1); or R satisfies s_4 and there exist $b', c' \in U$ such that $H(x)=b'x+xc'$, $G(x)=c'x+xb'$ (which is a particular case of conclusion 2). In what follows we assume that $a \notin C$.

Thus by Lemma 2 one of the following holds:

1. By conclusion 1 of Lemma 2 we get: $-c-p-q=\alpha \in C$ and $q \in C$, $a(b+c-\lambda)=0$, $c'=p$ such that $H(x)=(b+c-\lambda)x+xc'$ and $G(x)=c'x$, which is the conclusion 1 of the Theorem.

2. By conclusion 2 of Lemma 2 it follows: $-c-p-q=\alpha \in C$ and $q \in C$, $\gamma=-q-\alpha \in C$, $p=-c+\gamma$ such that $H(x)=(b+c)x+x(-c)$ and $G(x)=(-c+\gamma)x$ with $a(b+c-\gamma)=0$. By rewriting $H(x)=(b+c-\gamma)x+x(\gamma-c)$, we obtain conclusion 1 of the Theorem.

3. By conclusion 3 of Lemma 2 it follows: $-c-p-q=\alpha \in C$, R satisfies s_4 and $q'=-q-\alpha$ such that $H(x)=(b+c)x+x(-c)$ and $G(x)=-cx+xq'$ with $a(b+c-q')=0$, which is the conclusion 2 of the Theorem.

Case 2: Assume now that both d and δ are not inner derivations. Suppose first that d and δ are linearly C-independent modulo X-inner derivations. In this case, by KHARCHENKO's Theorem in [10] (see Fact 3), by (8) we have that U satisfies

$$a\big(\big(b[x_1,x_2]+[t_1,x_2]+[x_1,t_2]\big)[x_1,x_2]-[x_1,x_2]\big(b[x_1,x_2]+[z_1,x_2]+[x_1,z_2]\big)\big)$$

and in particular U satisfies the blended component

$$a\big([[x_1,t_2],[x_1,z_2]]\big).$$

By Lemma 3 in [5], since we suppose $a \neq 0$, U must satisfy $\big[[x_1,t_2],[x_1,z_2]\big]$. In this case it is well known by Posner's Theorem that there exists a suitable field F such that U and $M_m(F)$, the ring of $m \times m$ matrices over F, satisfy the same polynomial identities. In particular, for $m \geq 2$, we get the contradiction that

$$0 = \big[[e_{12},e_{22}],[e_{12},e_{21}]\big] = -2e_{12} \neq 0.$$

Consider now the case when there exist $\alpha,\beta \in C$ such that $\alpha d + \beta\delta = ad(q)$, the inner derivation induced by some $q \in U$. Of course both α and β are not zero, since d and δ are not inner derivations. So, if denote $\lambda = -\alpha\beta^{-1}$ and $\mu = \beta^{-1}$, it follows that $\delta = \lambda d + \mu ad(q)$. Thus by (8) we have

$$\begin{aligned} a\Big(\big(b[x_1,x_2] + [d(x_1),x_2] + [x_1,d(x_2)]\big)[x_1,x_2] \\ -[x_1,x_2]\big(p[x_1,x_2]+\lambda[d(x_1),x_2]+\lambda[x_1,d(x_2)]+\mu[[q,x_1],x_2]+\mu[x_1,[q,x_2]]\big)\Big) \end{aligned} \tag{9}$$

From (9) and applying Kharchenko's result, it follows that R satisfies

$$\begin{aligned} a\Big(\big(b[x_1,x_2] + [t_1,x_2] + [x_1,t_2]\big)[x_1,x_2] \\ -[x_1,x_2]\big(p[x_1,x_2] + \lambda[t_1,x_2] + \lambda[x_1,t_2] + \mu[[q,x_1],x_2] + \mu[x_1,[q,x_2]]\big)\Big) \end{aligned}$$

and in particular R satisfies the blended component

$$a\big([x_1,t_2][x_1,x_2] - \lambda[x_1,x_2][x_1,t_2]\big).$$

As above by Lemma 3 in [5], since $a \neq 0$, R satisfies the polynomial identity $[x_1,t_2][x_1,x_2] - \lambda[x_1,x_2][x_1,t_2]$. Since R is a PI-ring, then there exists a field F such that R, U and $M_m(F)$ satisfy the same polynomial identities. In particular $M_m(F)$ satisfies

$$[x_1,t_2][x_1,x_2] - \lambda[x_1,x_2][x_1,t_2] \tag{10}$$

Consider $m \geq 2$. In (10) choose $x_1 = e_{12}$, $x_2 = e_{21}$ and $t_2 = e_{22}$. By calculations it follows $-(1+\lambda)e_{12} = 0$, which means $\lambda = -1$.

On the other hand, for $x_1 = e_{12}$ and $x_2 = t_2 = e_{21}$, by (10) we have $(1-\lambda)(e_{11}+e_{22}) = 0$, which implies $\lambda = 1$, that is a contradiction, since $\mathrm{char}(R) \neq 2$.

Case 3: Finally assume that either d or δ is an inner derivation on U. Without loss of generality we may assume that $d(x) = [c,x]$, for a suitable $c \in U$ and let δ be an outer derivation of U. By (8) and Kharchenko's result, we get that U satisfies

$$a\Big(\big(b[x_1,x_2] + c[x_1,x_2] - [x_1,x_2]c\big)[x_1,x_2] - [x_1,x_2]\big(p[x_1,x_2] + [z_1,x_2] + [x_1,z_2]\big)\Big)$$

and in particular U satisfies the component

$$a\big(-[x_1,x_2][x_1,z_2]\big).$$

As above, by Lemma 3 in [5] and since $a \neq 0$, it follows that U satisfies the polynomial identity $[x_1,x_2][x_1,z_2]$. Let $M_m(F)$ be the ring of $m \times m$ matrices over a field F, which satisfies the same identities of U. This implies the following contradiction:

$$0 = [e_{12},e_{22}][e_{12},e_{21}] = -e_{12} \neq 0.$$

Notice that in the case δ is inner and d is outer, we may obtain the same contradiction by using the same argument as above. □

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