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Finsler metrizability and Landsberg spaces in dimension two

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Abstract. The problems of Finsler metrizability in dimension two is analyzed and solved completely. Thereafter the famous Landsberg problem concerning the existence of Landsberg spaces whose the sprays do not correspond to linear connection is considered. It is argued in dimension two that the answer to this question is affirmative and it is explained why it is difficult to produce a concrete example. Some of the results are extended to arbitrary dimension and finally a comparison is made with recent alternative investigations into the Landsberg problems.

1. Introduction

We consider the problems of Finsler metrizability in dimension two; in other words given a spray in the sense that its right hand sides are homogeneous of degree two in velocities, is there a Finsler function L of degree two so that its Euler-Lagrange equations engender the given spray? We will provide a complete solution to this problem in dimension two in Section two. In Section three we change point of view and consider Landsberg spaces again in dimension two. A Landsberg space is a Finsler function L which in addition satisfies a compatability condition: namely, the Finsler metric, which is the Hessian of L, is parallel with respect to the natural Berwald connection. This condition is a natural generalization of the well known compatability connection that expresses the fact that a Riemannian metric is parallel with respect to its Levi-Cevita connection. The Landsberg condition is given below as equation (3.1). We show that in dimension two at least, this Landsberg condition reduces to a single fourth order partial

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differential equation and we find it explicitly. We compare this equation with another fifth order equation which arises in the context of the famous Landsberg problem; namely, are there examples of Landsberg spaces such that the spray does *not* correspond to a genuine linear connection? We argue again in dimension two that the answer to this question is affirmative but we explain why it is unlikely that a concrete example will ever be found.

In Section four we consider the possibility of extending our results to n dimensions. Although some of the arguments are repeated the general case can be difficult without a sound understanding of the two-dimensional case. Again the inverse Finsler problem is solved in the generic case. Finally in Section five a comparison of our results with various recent investigations into the Landsberg problem is made.

Recently Muzsnay [6] has written a very elegant article that considers much the same topic using Fröhlicher–Nijenhuis and Spencer theory. However, Muzsnay considers generally structures in dimension n and apparently there is almost not much overlap with results presented here which is presented in a much more pedestrian manner except for Section 5 in [6]. We refer to [6] for further references besides the ones given here.

Below the term "spray" will be taken to mean that the right hand sides of the second order equations are homogeneous of degree two in velocities. When we need to refer to the geodesics of a linear connection we shall refer to a "quadratic spray". Furthermore in Finsler geometry it is usually assumed that the the Finsler metric is positive definite. We have not included this condition here but it can easily be added to the main results. Finally there is a slight notational difficulty in terms of denoting partial derivatives and whether to include a comma in the notation. It seems to be too fastidious to insist on using a comma on every occasion so we have suppressed them wherever possible; however, the formulas are sufficiently well known that there should be no difficulty in interpreting them in practice.

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2. Finsler metrizability

We shall use local coordinates x and y on a manifold of dimension two. The corresponding velocities will be denoted by u and v rather than $\dot{x}$ and $\dot{y}$. We define z to be $\frac{u}{v}$ and write our spray in the following form:

$$\dot{u} = v^2 Q, \quad \dot{v} = v^2 R \tag{2.1}$$

where Q and R are functions of x, y and z. We shall also write our putative Finsler function in the form

$$L = v^2 M \tag{2.2}$$

where M is a function of x, y and z. In order for L to give rise to equation (2.1) as its Finsler geodesics it is necessary and sufficient that the following conditions should be satisfied:

$$\frac{M_z(zM_{zx} + M_{yz} - M_x) - (zM_x + M_y)M_{zz}}{2MM_{zz} - M_z^2} = R \tag{2.3}$$

$$\frac{2M(M_x - M_{yz} - zM_{xz}) + (zM_x + M_y)M_z}{2MM_{zz} - M_z^2} = Q - zR. \tag{2.4}$$

The last two equations can be rewritten as

$$zM_x + M_y = R(zM_z - 2M) - QM_z \tag{2.5}$$

$$zM_{zx} + M_{yz} - M_x = R(zM_{zz} - M_z) - QM_{zz}. \tag{2.6}$$

We can regard the last pair of equations as conditions on the unknown function M given the spray in terms of Q and R.

We now proceed as follows: differentiating equation (2.5) with respect to z gives

$$zM_{zx} + M_{yz} + M_x = R_z(zM_z - 2M) + R(zM_{zz} - M_z) - QM_{zz} - Q_zM_z. \tag{2.7}$$

Now we obtain from equations (2.5), 2.6) and (2.7):

$$2M_x = R_z(zM_z - 2M) - Q_zM_z \tag{2.8}$$

$$2M_{xz} = R_{zz}(zM_z - 2M) + R_z(zM_{zz} - M_z) - Q_{zz}M_z - Q_zM_{zz}. \tag{2.9}$$

and

$$2M_y = (2R - zR_z)(zM_z - 2M) + M_z(zQ_z - 2Q). \tag{2.10}$$

The following Lemma is easily verified.

Lemma 2.1. *The pair of equations* 2.5 *and* 2.6 *is equivalent to the pair* 2.8 *and* 2.10.

Now from the equations 2.8 and 2.10 we obtain as a compatibility condition by equating M_{xy} and M_{yx}

$$\begin{aligned}2(zR_{zx} + R_{zy} - 2R_x)(zM_z - 2M) - 2(Q_{zy} + zQ_{zx} - 2Q_x)M_z + ((zR_z - Q_z)R_z \\ + 2(Q - zR)R_{zz})(zM_z - 2M) - (2RQ_z - 2R_zQ + 2(Q - zR)Q_{zz} \\ - Q_z(Q_z - zR_z))M_z = 0. \qquad (2.11)\end{aligned}$$

In the generic case we now have expressions for all three derivatives M_x, M_y, M_z of M. In fact this will be the case when the coefficient A of M_z in equation (2.10)

$$\begin{aligned}A = 2z(zR_{zx} + R_{zy} - 2R_x) - 2RQ_z + 2R_zQ - 2(Q - zR)Q_{zz} + Q_z(Q_z - zR_z) \\ - 2Q_{zy} - 2zQ_{zx} + 4Q_x + z((zR_z - Q_z)R_z + 2(Q - zR)R_{zz}) \qquad (2.12)\end{aligned}$$

is not zero. Let us define the coefficient of M in equation (2.10) to be B so that

$$B = -2(zR_z - Q_z)R_z - 4(Q - zR)R_{zz} - 4zR_{zx} - 4R_{zy} + 8R_x.$$

We can now state the solution to the Finsler inverse problem in the generic case as follows:

Theorem 2.1. *If the function A is nowhere zero then the necessary and sufficient condition for a Finsler function to be compatible with the system* 2.1 *is that the one-form*

$$\frac{((Q_z - zR_z)B - 2AR_z)dx + ((zB + 2A)(zR_z - 2R) + B(2Q - zQ_z))dy - Bdz}{2A}$$

should be exact, say, equal to dm. The only stipulation is that the Hessian should be regular, that is, that $2m_{zz} + m_z^2$ should be nowhere zero. In that case the Finsler function is given by $e^m v^2$ and is unique up to scaling by a constant.

In the non-generic case where $A = 0$ we must have also that B is zero in order to have a Lagrangian. Thus we have that:

$$2zR_{zx} + 2R_{zy} - 4R_x + 2(Q - zR)R_{zz} - (Q_z - zR_z)R_z = 0$$

and

$$2zQ_{zx} + 2Q_{zy} - 4Q_x + 2(Q - zR)Q_{zz} - (Q_z - zR_z)Q_z - 2QR_z + 2RQ_z = 0.$$

Since there is just one dependent variable M and we have a first order system and the compatibility conditions are satisfied, existence of M follows from ODE techniques alone – essentially the method of characteristics. Moreover the solution for M depends on a "single function of one variable".

As another way to treat the non-generic case we refer to the work of DOUGLAS [2]: see also [1], [3], [4]. For a general system of second order ODE in dimension n

$$\ddot{x}^i = f^i(x^j, \dot{x}^j) \tag{2.13}$$

there is a matrix called Φ and sometimes known as the Jacobi endomorphism defined as follows where u^i stands for $\dot{x}^i$:

$$\Phi^i_j = \frac{1}{2}\,\frac{d}{dt}\left(\frac{\partial f^i}{\partial u^j}\right) - \frac{\partial f^i}{\partial x^j} - \frac{1}{4}\,\frac{\partial f^i}{\partial u^k}\,\frac{\partial f^k}{\partial u^j}. \tag{2.14}$$

The matrix Φ is the principal tensorial invariant associated to the general inverse problem for second order ODE and it enables many special subcases to be defined. The simplest case of all, sometimes referred to as "Case 1", is defined by the condition that Φ should be a multiple of the identity matrix. In that case there certainly is a Lagrangian that depends on n "functions of $n+1$ variables". For the system (2.1) we may compute that the entries in the matrix Φ are given by:

$$\Phi^1_1 = \frac{1}{4}zQ_{zx} + 2Q_{zy} - 4Q_x + 2(Q - zR)Q_{zz} - (Q_z - zR_z)Q_z - 2QR_z + 2RQ_z$$

$$\Phi^1_2 = -\frac{z}{4}zQ_{zx} + 2Q_{zy} - 4Q_x + 2(Q - zR)Q_{zz} - (Q_z - zR_z)Q_z - 2QR_z + 2RQ_z$$

$$\Phi^2_1 = \frac{-z}{4} - (2zR_{zx} + 2R_{zy} - 4R_x + 2(Q - zR)R_{zz} - (Q_z - zR_z)R_z)$$

$$\Phi^2_2 = \frac{-z}{4}(2zR_{zx} + 2R_{zy} - 4R_x + 2(Q - zR)R_{zz} - (Q_z - zR_z)R_z).$$

Thus in the non-generic case it follows that Φ is zero and so the system (2.1) is certainly variational. It is not immediately apparent, however, from this approach that there is a Finsler function that engenders (2.1).

3. Landsberg spaces

We now change point of view and consider Landsberg spaces. As such we are *starting* with a Finsler function L so that the conditions derived in Section 2 are now to be regarded as identities. A Landsberg space has the property that the metric defined by the Hessian of L is parallel with respect to the Berwald connection or, in tensorial notation valid in arbitrary dimensions,

$$2g_{ij,x^k} + f^l_{,u^i}g_{ij,u^l} + f^l_{,u^iu^k}g_{lj} + f^l_{,u^iu^j}g_{lk} = 0 \tag{3.1}$$

where g_{ij} denotes the Hessian of L and the position and velocity coordinates are denoted by x^i and u^i, respectively, and the f^l denote the right hand sides of the geodesic equations or "forces" and comma denotes a derivative with respect to the indicated variables.

Coming back now to the case of dimension two we use the notation of the last Section. As such it turns out that g_{ij} can be written

$$g_{ij} = \begin{bmatrix} M_{zz} & M_z - zM_{zz} \\ M_z - zM_{zz} & 2M - 2zM_z + z^2 M_{zz} \end{bmatrix}. \tag{3.2}$$

Notice that since L is homogeneous of degree 2 the components of g_{ij} must be homogeneous of degree zero and so can be expressed in terms of x, y and z. When the parallel condition is imposed it turns out to be just a single condition:

Proposition 3.1. *The Landsberg parallel condition reduces to*

$$(Q_{zzz} - zR_{zzz})M_z + 2MR_{zzz} = 0 \tag{3.3}$$

where Q and R are defined by equations (2.4) *and* (2.5).

Proof. The proof is a matter of calculation. We understand that $x^1 = x$, $x^2 = y$, $u^1 = u$ and $u^2 = v$. In principle the Landsberg condition comprises six conditions. We note that three of them may be written as:

$$2\frac{\partial g_{jk}}{\partial x} + (Q_z - zR_z)\frac{\partial g_{jk}}{\partial z} + \frac{\partial(vQ_z)}{\partial u^k} g_{1j} + \frac{\partial(vQ_z)}{\partial u^j} g_{1k} + \frac{\partial(vR_z)}{\partial u^j} g_{2k} + \frac{\partial(vR_z)}{\partial u^k} g_{2j}$$

Now we choose successively $j = 1$, $k = 1$, $j = 1$, $k = 2$ and $j = 2$, $k = 2$. All terms except the first produce terms that involve derivatives with respect only to z. The first term involves just one x-derivative. These latter terms may be converted into z-derivatives using 2.8 and 2.10. In the case $j = 1, k = 1$, we obtain

$$\begin{aligned} &(Q_z - zR_z)M_{zzz} + 2Q_{zz}M_{zz} + 2R_{zz}(M_z - zM_{zz}) + (zM_z - 2M)R_{zzz} \\ &+(zM_{zz} - M_z)R_{zz} + zR_zM_{zzz} - M_zQ_{zzz} - 2M_{zz}Q_{zz} - M_{zzz}Q_z = 0. \end{aligned} \tag{3.4}$$

the last six terms having been obtained from converting $2M_{zzx}$ by means of equation (2.8). The only terms in equation (3.4) that do not cancel give equation (3.3). The other five cases are similar. □

Our principal concern is to decide whether there are any *proper* Landsberg spaces. Indeed for all known examples of Landsberg spaces it turns out that they all correspond to linear connections; in other words the functions Q and R are quadratic in z. Clearly if Q and R are quadratic in z then equation (3.3) is satisfied: the challenge is try to prove the converse, namely, does equation (3.3) somehow imply that Q and R are quadratic in z? A Landsberg space that does not correspond to a linear connection is a proper Landsberg space.

It might appear that equation (3.3) is a fifth order condition in z but in fact it is only fourth order. Indeed equation (3.3) is equivalent to the following condition:

$$\begin{aligned}\Delta(zM_{xzzz}+M_{yzzz})+(Q-zR)(\Delta M_{zzzz}-3MM_{zzz}^2)-R\Delta M_{zzz}\\ -3(M(zM_{xzz}+M_{yzz})-M_xM_z)M_{zzz}=0.\end{aligned} \tag{3.5}$$

where Δ denotes the determinant of the Hessian of L, namely, $2MM_{zz}-M_z^2$ and it is understood that Q and R are determined in terms of M from equations (2.3) and (2.4).

We now consider $R_{zzz}=0$ regarded as a condition on M. It is a fifth order exceedingly complicated condition in z. In fact its fifth order terms, after multiplication by a factor that can be assumed to be non-zero involving Δ and M_z, are given by

$$zM_{xzzzz}+M_{yzzzz}+(Q-zR)M_{zzzzz}+\cdots=0.$$

It is apparent that by differentiating equation (3.3) we can remove the fifth order terms from $R_{zzz}=0$ to obtain a fourth order condition that is equivalent in the presence of equation (3.3). It is:

$$\begin{aligned}(M_xM_z-M(zM_{xzz}+M_{yzz}))M_{zzzz}+(\Delta+zMM_{zzz})M_{xzzz}+MM_{zzz}M_{yzzz}\\ (M_xM_{zz}+3M_zM_{xz}-3MM_{xzz}-2M_z(zM_{xzz}+M_{yzz}))M_{zzz}\\ +((2zM_z-M)R-2M_zQ)M_{zzz}^2=0.\end{aligned} \tag{3.6}$$

The precise form of equation (3.6) was obtained by adding in a multiple of equation (3.3) itself and seemed to offer about the simplest alternative. In order to make clear what is going on we offer the following self-evident lemma.

Lemma 3.2. *The PDE system consisting of equation* (3.3) *and* $R_{zzz}=0$ *is equivalent for non-singular Lagrangians to the PDE system consisting of equation* (3.3) *and equation* (3.6).

In fact the PDE system consisting of equation (3.3) and equation (3.6) is an involutive system after we add in one more equation which is obtained by adding a multiple of the derivative of equation (3.3) to $R_{zzz}=0$. The Cartan characters are $s_1 = 10$, $s_2 = 3$ and $s_3 = 0$. However, it is clear that the latter PDE system is equivalent to the system consisting of $Q_{zzz} = 0$ and $R_{zzz} = 0$, that is, in the case where the Finsler function gives rise to a genuine linear connection. We have investigated the solutions of this system in terms of M in some detail in [10] so we shall not repeat the analysis here.

Let us now come back to the Landsberg problem. What is required to find a proper Landsberg space is to find a regular solution of equation (3.3), that is, such that $2MM_{zz} - M_z^2 \neq 0$ and that is *not* a solution of equation (3.6).

Theorem 3.1. *A proper Landsberg space is precisely one for which the function M satisfies condition* (3.3) *and for which the equation* (3.6) *is not valid and in addition for which the inequality $2FH - G^2 \neq 0$ holds.*

In the abstract such a solution should exist: in the analytic category we can solve equation (3.3) for M_{zzzz} and then use the Cauchy–Kowaleskaya theorem to find a solution depending on $F(x, y) = M(x, y, 0)$, $G(x, y) = M_z(x, y, 0)$, $H(x, y) = M_{zz}(x, y, 0)$ and $K(x, y) = M_{zzz}(x, y, 0)$ where F, G, H and K are arbitrary analytic functions of x and y. We will require F, G, H and K to satisfy $2FH - G^2 \neq 0$. If we use equation (3.3) to find M_{zzzz} and then substitute all the values of M and its partial derivatives into the left hand side of equation (3.6) we find the following quadratic condition on z:
$-G^5F_xK_y+4G^4F_x^2K+4G^2F_xKG_yFH-G^4F_xKG_y+4F^3H_yH^2K_y+FH_yG^4K_y-$
$6F^3H_y^2KH + 3F^2H_y^2KG^2 + 8K_xF^3H^2F_x - 8K_xF^3H^2G_y + 2K_xG^4FF_x -$
$2K_xG^4FG_y+4KF_x^2H^2F^2-3KG^4G_xF_y+2KG^4H_yF_y-6K^2F^2GG_y^2-6K^2F^2GF_x^2-$
$4F^2H_yHK_yG^2 + 4G^3F_xK_yFH - 10G^2F_x^2KFH - 4GF_xF^2H^2K_y +$
$6GF_xKF^2H_yH3G^3F_xKFH_y+6F^2H_yKGG_yH-3FH_yKG^3G_y+2F^2H_yKH^2F_y-$
$5FH_yKHF_yG^2 + K_xG^5F_y + 4K_xF^2H^2GF_y - 8K_xF^2HG^2F_x + 8K_xF^2HG^2G_y -$
$4K_xFHG^3F_y+4F^3KK_yHF_x-4F^3KK_yHG_y+2F^2KK_yHGF_y-2F^2KK_yG^2F_x+$
$2F^2KK_yG^2G_y - FKK_yG^3F_y - 4KF_xH^2F^2G_y - 6KG^3G_xFF_x + 6KG^3G_xFG_y -$
$12KF^3H_xHF_x + 12KF^3H_xHG_y + 6KF^2H_xG^2F_x - 6KF^2H_xG^2G_y +$
$3KFH_xG^3F_y - 2K^2G^3F_y^2 + 12KGG_xF^2HF_x - 12KGG_xF^2HG_y +$
$6KG^2G_xFHF_y - 6KF^2H_xHGF_y + 12K^2F^2GG_yF_x + 7K^2FG^2G_yF_y -$
$7K^2FG^2F_xF_y + 2K^2F^2HF_yF_x - 2K^2F^2HF_yG_y + K^2FHF_y^2G + (-4G^4F_xKG_x +$
$4F^3H_yH^2K_x + FH_yG^4K_x + 4F^3H_xH^2K_y + FH_xG^4K_y - 8K_xF^3H^2G_x -$
$2K_xG^4FG_x+6KG^3G_x^2F+2KG^4H_yF_x+2KG^4H_xF_y-7K^2G^2F_x^2F-4K^2G^3F_xF_y+$
$2K^2F^2HF_x^2 + 10G^2F_xKG_xFH - 4F^2H_yHK_xG^2 - 12F^3H_yKH_xH +$
$6F^2H_yKH_xG^2 + 6F^2H_yKGG_xH - 3FH_yKG^3G_x + 2F^2H_yKH^2F_x -$
$5FH_yKHF_xG^2 - 4F^2H_xHK_yG^2 + 6F^2H_xKGG_yH$
$-3FH_xKG^3G_y + 2F^2H_xKH^2F_y - 5FH_xKHF_yG^2 + 8K_xF^2HG^2G_x +$

$$
\begin{aligned}
&2K_xF^2KG^2G_y - K_xFKG^3F_y - 4F^3KK_yHG_x + 2F^2KK_yHGF_x + \\
&2F^2KK_yG^2G_x - FKK_yG^3F_x + 4K_xF^3KHF_x - 4K_xF^3KHG_y + 2K_xF^2KHGF_y - \\
&2K_xF^2KG^2F_x - 12KGG_x^2F^2H + 12KF^3H_xHG_x - 6KF^2H_xG^2G_x + \\
&7K^2G^2F_xFG_y - 12K^2F^2GG_yG_x + 12K^2F^2GF_xG_x + 7K^2FG^2G_xF_y - \\
&2K^2F^2HF_yG_x + 2K^2FHF_yGF_x - 2K^2F^2HF_xG_y - 4KF_xH^2F^2G_x)z + \\
&(-6F^3H_x^2KH - 4F^2H_xHK_xG^2 - 3FH_xKG^3G_x + 2K_xF^2KG^2G_x + FH_xG^4K_x - \\
&5FH_xKHF_xG^2 - K_xFKG^3F_x + 3F^2H_x^2KG^2 - 6K^2F^2GG_x^2 + 6F^2H_xKGG_xH + \\
&7K^2G^2F_xFG_x - 4K_xF^3KHG_x - 2K^2G^3F_x^2 + 2F^2H_xKH^2F_x + 2K_xF^2KHGF_x + \\
&2KG^4H_xF_x + 4F^3H_xH^2K_x - 2K^2F^2HF_xG_x + K^2FHF_x^2G)z^2.
\end{aligned}
$$

Therefore we will have to choose the functions F, G, H and K so that at least one of the coefficients of 1, z or z^2 in the last expression is non-zero in order for equation (3.6) not to be satisfied as well as to require that $2FH - G^2 \neq 0$. However, so far it is has not proved possible to obtain an explicit example. The difficulty lies in finding a solution of (3.3) that does not satisfy equation (3.6). These equations have a lot of obvious common "gauge" solutions, namely, an arbitrary quadratic function in z with function coefficients depending on x and y. It may well be the case that no other explicit solutions of (3.3) can be obtained. In a future work we hope to be able to give a more geometric interpretation of the conditions obtained here.

4. Higher dimensions

We shall briefly consider some calculations that are valid in arbitrary dimensions. They will also serve to put the two-dimensional case into a better perspective. As such we resume the discussion from equation 3.1 so that $L = L(x^i, u^i)$ is a Finsler function of degree two, that is,

$$u^k L_{,u^k} = 2L.$$

Next we use the fact that a Finsler function is first integral of its own geodesic equations: see [1]: thus

$$u^k L_{,x^k} + f^k L_{,u^k} = 0.$$

Now we differentiate the last condition with respect to u^i then u^j to obtain

$$2g_{ij,x^k} + f^l_{,u^i} g_{ij,u^l} + f^l_{,u^i u^k} g_{lj} + f^l_{,u^i u^j} g_{lk} + f^l_{,u^i u^j u^k} L_{,u^l} = 0. \tag{4.1}$$

Comparing equation 3.1 we have:

Proposition 4.1. *The Landsberg condition is equivalent to* $f^l_{,u^i u^j u^k} L_{,u^l} = 0$.

Again we obtain a well known result:

Corollary 4.2. *If the f^l's are quadratic and therefore homogeneous quadratic the Landsberg condition is satisfied.*

We now change point of view and suppose that f is a function such that $f = f(x^i, u^i)$ and put $z^A = \frac{u_A}{u_n}$ where $1 \leq A \leq n-1$ and $F = \frac{f}{u_n^2}$. Then we find the following:

$$\begin{cases} f_{,u^A u^B u^C} = \dfrac{1}{u_n} F_{,z^A z^B z^C} \\ f_{,u^A u^B u^n} = \dfrac{-z^C}{u_n} F_{,z^A z^B z^C} \\ f_{,u^A u^n u^n} = \dfrac{z^B z^C}{u_n} F_{,z^A z^B z^C} \\ f_{,u^n u^n u^n} = \dfrac{-z^A z^B z^C}{u_n} F_{,z^A z^B z^C}. \end{cases} \tag{4.2}$$

Coming back now to the Landsberg condition we put $L = u_n^2 M(x^i, z^A)$ and also write $F^i = \frac{f^i}{u_n^2}$ so that $F^i = F^i(x^i, z^A)$ where $1 \leq A \leq n-1$. Then the Landsberg condition becomes in the (x^i, z^A) coordinates

$$F^E_{,z^A z^B z^C} M_{,z^E} + F^n_{,z^A z^B z^C}(2M - z^E M_{,z^E}) = 0. \tag{4.3}$$

It should be apparent that the last condition generalizes equation 3.3 to higher dimensions. Likewise the generalization of 2.5 and 2.6 are given by

$$z^E M_{x^E} + M_{x^n} = F^n(z^E M_{z^E} - 2M) - F^E M_{z^E} \tag{4.4}$$

$$z^E M_{z^A x^E} + M_{x^n z^A} - M_{x^A} = F^n(z^E M_{z^A z^E} - M_{z^A}) - F^E M_{z^A z^E}. \tag{4.5}$$

Computing the z^A-derivative of 4.4 gives

$$\begin{aligned} z^E M_{z^A x^E} + M_{x^A} + M_{z^A x^n} = {} & F^n(z^E M_{z^A z^E} - M_{z^A}) + F^n_{z^A}(z^E M_{z^E} - 2M) \\ & - F^E M_{z^A z^E} - F^E_{z^A} M_{z^E} \end{aligned} \tag{4.6}$$

and hence subtracting 4.5 from 4.6

$$2M_{x^A} = F^n_{z^A}(z^E M_{z^E} - 2M) - F^E_{z^A} M_{z^E} \tag{4.7}$$

and hence from 4.4

$$2M_{x^n} = (2F^n - z^B F^n_{z^B})(z^E M_{z^E} - 2M) + M_{z^E}(z^B F^E_{z^B} - 2F^E). \tag{4.8}$$

Now compute the difference of the x^n-derivative of 4.7 and the x^A-derivative of 4.8: we find that

$$\lambda_A^D M_{z^D} - \mu_A M = 0$$

where

$$\begin{aligned}\lambda_A^D = z^D \Bigg(& 2F^n_{x^A} - z^B F^n_{z^B x^A} - z^B F^n_{z^B x^A} - F^n_{z^A x^n} + \frac{1}{2}(z^C F^E_{z^C} - 2F^E + 2z^E F^n \\ & - z^C z^E F^n_{z^C}) F^n_{z^A z^E} - \frac{z^E}{2} F^n_{z^A} F^n_{z^E} + \frac{z^B z^E}{2} F^n_{z^A} F^n_{z^B z^E} + \frac{1}{2} F^E_{z^A} F^n_{z^E} \\ & - \frac{z^B}{2} F^E_{z^A} F^n_{z^B z^E} + F^n F^n_{z^A} - \frac{z^E}{2} F^n_{z^A} F^n_{z^E} \Bigg) + F^D_{x^n z^A} + z^B F^D_{x^A z^B} - F^D_{x^A} \\ & - \frac{z^C}{2} F^n_{z^A} F^D_{z^C} + F^n_{z^A} F^D - z^D F^n_{z^A} F^n + \frac{z^C z^D}{2} F^n_{z^A} F^n_{z^C} - \frac{z^B z^C}{2} F^n_{z^A} F^D_{z^B z^C} \\ & + \frac{z^C}{2} F^n_{z^A} F^D_{z^C} + \frac{z^B}{2} F^C_{z^A} F^D_{z^B z^C} - \frac{1}{2} F^C_{z^A} F^D_{z^C} - \Bigg(\frac{z^C}{2} F^B_{z^C} - F^B + z^B F^n \\ & - \frac{z^C z^E}{2} F^n_{z^C} \Bigg) z^B F^D_{z^A} F^n_{z^B} - F^n F^D_{z^A} + z^B F^n_{z^A} F^D_{z^B} - 2F^D F^n_{z^A}\end{aligned}$$

and

$$\begin{aligned}\mu_A = {} & 4F^n_{x^A} - 2z^B F^n_{z^B x^A} - 2F^n_{z^A x^n} + \big(z^C F^E_{z^C} - 2F^E + 2z^E F^n - z^C z^E F^n_{z^C}\big) F^n_{z^A z^E} \\ & - z^E F^n_{z^A} F^n_{z^E} + z^B z^E F^n_{z^A} F^n_{z^B z^E} + F^E_{z^A} F^n_{z^E} - z^B F^E_{z^A} F^n_{z^B z^E} + 2F^n F^n_{z^A}.\end{aligned}$$

Suppose now in the generic case that the matrix λ_A^D is invertible. Then we can solve explicitly for M_{z^D} and obtain the one-form

$$\begin{aligned}\omega = \Bigg(& \frac{1}{2}\big(z^E F^n_{Z^A} - F^E_{Z^A}\big)(\lambda_B^E)^{-1}\mu_B, \frac{1}{2}\big(z^E(2F^n - z^C F^n_{z^C}) \\ & + z^C F^E_{z^C} - 2F^E\big)(\lambda_B^E)^{-1}\mu_B, (\lambda_E^A)^{-1}\mu_E \Bigg).\end{aligned}$$

Thus we may state the following Theorem which generalizes 2.1:

Theorem 4.1. *If the matrix λ_A^D is invertible then the necessary and sufficient condition for a Finsler function to be compatible with the system 2.1 is that the one-form ω should be exact, say, equal to dm. The only stipulation is that the Hessian should be regular, which is equivalent to saying that the matrix*

$$\begin{bmatrix} m_{z^A z^B} + m_{z^A} m_{z^B} & m_{z^A} - z^A(m_{z^A z^B} + m_{z^A} m_{z^B}) \\ m_{z^A} - z^A(m_{z^A z^B} + m_{z^A} m_{z^B}) & 2 - z^A m_{z^A} + z^A z^B(m_{z^A z^B} + m_{z^A} m_{z^B}) \end{bmatrix}$$

should be non-singular. In that case the Finsler function is given by $u_n^2 e^m$ and is unique up to scaling by a constant.

If the matrix λ_A^D is not invertible then it is not clear if a Finsler function exists. In view of the complexity of the expressions involved we shall not pursue the existence of Finsler functions in the non-generic cases here.

5. Prognosis of the Landsberg unicorns

Let us come back to equation 4.3 and the Landsberg problem. Take the derivative with respect to z^D, interchange the indices C and D and subtract and then use 4.3 to eliminate $F^n_{,z^Az^Bz^C}$. The result is

$$\begin{aligned}((z^PM_{z^P}-2M)M_{z^Dz^Q}+(M_{z^D}-z^PM_{z^Dz^P})M_{z^Q})F^Q_{z^Az^Bz^C}\\ -((z^PM_{z^P}-2M)M_{z^Cz^Q}+(M_{z^C}-z^PM_{z^Cz^P})M_{z^Q})F^Q_{z^Az^Bz^D}=0. \quad (5.1)\end{aligned}$$

One interpretation of 5.1 is that there are $\binom{n-1}{2}\binom{n}{2}$ linear conditions for the $(n-1)\binom{n+1}{3}$ unknowns $F^P_{z^Az^Bz^C}$. Note that these two numbers are equal for $n=8$ and for $n>8$ the former is larger than the latter. On the other hand the conditions are vacuous if for example M is homogeneous of degree two in z^A. In any case since we have argued that Landsberg metrics in arbitrary dimensions do exist we may not conclude unconditionally that the $F^P_{z^Az^Bz^C}$ and hence $F^n_{z^Az^Bz^C}$ are zero.

Let us take stock of the situation. In the strict sense the answer to the Landsberg problem is affirmative. In dimension two the Landsberg condition 4.3 reduces to a single condition and so by appealing to the Cauchy–Kowaleskaya theorem there are solutions in the analytic category. Furthermore Landsberg metrics of arbitrary dimensions can be obtained by adding "free particle" systems to the two-dimensional Landsberg system. In higher dimensions the Landsberg condition 4.3 is a genuine *system* of fourth order. Conditions 4.4 and 4.5 allow the functions F^A and F^n to be determined in terms of the unknown M and its derivatives *in principle*. Unfortunately it appears to be impossible to obtain F^A and F^n explicitly in arbitrary dimensions; thus it is not possible to see how to generalize equation 3.5. If such a generalization could be found it might be possible to attack the problem using Cartan-Kähler theory. However, we have argued above that conditions 5.1 must arise as implicit fifth order conditions in any such analysis.

Recently there have been several attempts to throw light on the Landsberg problem. Szabó claimed the non-existence of the so-called Landsberg "unicorns"; in a subsequent paper [9] he seems to say that his proof in [8] only works with additional assumptions. Sabau, Shibuya and Shimada [7] studied the Landsberg problem from the point of view of Cartan–Kähler theory. Apparently they accept Szabó's result as originally stated but do not claim the existence of Landsberg unicorns. Matveev wrote a short paper [5] which points out the gap in Szabó's proof. Finally Torrome [11] claims the non-existence of Landsberg unicorns. The consensus seems to be that Landsberg unicorns are indeed chimeras. We must point out that we have not studied any of these papers in detail. The approaches adopted appear to be completely different from the one adopted here. They involve holonomy groups of the Berwald and parallel transport of the Finsler metric. The discrepancy between the conclusions of [8], [9], [7], [5], [11] and the present paper probably resides in some implicit assumption of smoothness whereas we do not assume any and do not make the assumption that the Finsler metric is positive definite. We have not imposed smoothness restrictions that would insure for example that the holonomy group of the non-linear Berwald connection would be well defined. In particular we do not assume smoothness on the zero section. We do assume real analyticity of the Finsler metric in order to invoke the Cauchy-Kowaleskaya theorem. In any case perhaps the discrepancy between the different approaches is more apparent than real since it is unlikely that a concrete example can ever be found. It is hoped that the methods used here may complement other contributions and take the understanding of Landsberg metrics to a new level.

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