Publ. Math. Debrecen
76/4 (2010), 453–464

Fuzzy approximation based on statistical rates

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Abstract. In this paper, we obtain some fuzzy Korovkin-type results based on statistical rates. Our results cover not only the fuzzy Korovkin theory but also the statistical fuzzy Korovkin theory. Important applications and remarks are also presented.

1. Introduction

Fuzzy analogue of the classical Korovkin theory was first introduced by Anastassiou [2] (see also [1], [3], [4], [10]). Recently, some statistical fuzzy approximation theorems have been obtained by using the concept of statistical convergence that is a weaker method than the ordinary convergence (see, e.g., [5], [7]).

The main goal of this paper is to construct some fuzzy Korovkin-type approximation results based on statistical rates instead of statistical converge or ordinary convergence. Our approach covers both the algebraic case and the trigonometric case in the fuzzy sense. We show that our new results are more general than the classical fuzzy Korovkin results in [2] and the statistical fuzzy approximation theorems in [5], [7]. Especially, we display a non-trivial application which verifies our generalization.

This paper is organized as follows. In the first section, we recall some basic concepts from fuzzy theory and summability theory. In the second section, we give fuzzy Korovkin-type results based on the statistical rates in algebraic case,

Mathematics Subject Classification: 26E50, 41A25, 41A36, 40G15.
Key words and phrases: fuzzy positive linear operators, fuzzy Korovkin theory, statistical convergence, statistical rates.
This research was supported by the Scientific and Technological Research Council of Turkey (TUBITAK); Project No: 108T229.

while, in the third section, we get ones in trigonometric case. The last section of the paper is devoted to some important applications and remarks.

A fuzzy number is a function $\mu : \mathbb{R} \to [0,1]$, which is normal, convex, upper semi-continuous and the closure of the set $\text{supp}(\mu)$ is compact, where $\text{supp}(\mu) := \{x \in \mathbb{R} : \mu(x) > 0\}$. The set of all fuzzy numbers are denoted by $\mathbb{R}_{\mathcal{F}}$. Let

$$[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) > 0\}} \text{ and } [\mu]^r := \{x \in \mathbb{R} : \mu(x) \geq r\},\ (0 < r \leq 1).$$

Then, it is well-known [11] that, for each $r \in [0,1]$, the set $[\mu]^r$ is a closed and bounded interval of $\mathbb{R}$. For any $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, it is possible to define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ as follows:

$$[u \oplus v]^r = [u]^r + [v]^r \text{ and } [\lambda \odot u]^r = \lambda [u]^r, \quad (0 \leq r \leq 1).$$

Now denote the interval $[u]^r$ by $[u_-^{(r)}, u_+^{(r)}]$, where $u_-^{(r)} \leq u_+^{(r)}$ and $u_-^{(r)}, u_+^{(r)} \in \mathbb{R}$ for $r \in [0,1]$. Then, for $u, v \in \mathbb{R}_{\mathcal{F}}$, define

$$u \preceq v \Leftrightarrow u_-^{(r)} \leq v_-^{(r)} \text{ and } u_+^{(r)} \leq v_+^{(r)} \quad \text{for all } 0 \leq r \leq 1.$$

Define also the following metric $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+$ by

$$D(u,v) = \sup_{r \in [0,1]} \max \left\{ |u_-^{(r)} - v_-^{(r)}|, |u_+^{(r)} - v_+^{(r)}| \right\}.$$

In this case, $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space (see [17]).

Now let $A = [a_{jn}]$, $j, n \in \mathbb{N}$, be an infinite summability matrix. For a given sequence $x := (x_n)_{n \in \mathbb{N}}$, the A-transform of x, denoted by $Ax := \{(Ax)_j\}_{j \in \mathbb{N}}$, is given by $(Ax)_j = \sum_{n=1}^{\infty} a_{jn} x_n$ provided the series converges for each j. We say that A is regular if $\lim_j (Ax)_j = L$ whenever $\lim_n x_n = L$ (see, for instance, [12]). Assume that $A = [a_{jn}]$ is a non-negative regular summability matrix. Then, for a given sequence $(\mu_n)_{n \in \mathbb{N}}$ of fuzzy numbers, we say that $(\mu_n)_{n \in \mathbb{N}}$ is A-statistically convergent to a fuzzy number $\mu \in \mathbb{R}_{\mathcal{F}}$, which is denoted by $st_A - \lim_n D(\mu_n, \mu) = 0$, if, for every $\varepsilon > 0$,

$$\lim_j \sum_{n : D(\mu_n, \mu) \geq \varepsilon} a_{jn} = 0$$

holds (see [5], [7]). Recall that the scalar version of the above definition was first introduced by Freedman and Sember [9]. Observe now that if we take $A = C_1 = (c_{jn})$, the Cesáro matrix of order one, then C_1-statistical convergence coincides with the notion of statistical convergence of fuzzy valued sequences given by Nuray and Savaş [16]. Actually, the idea of statistical converge of number sequences was noticed by Fast [8]. Furthermore, if A is replaced by the identity matrix, then we have the fuzzy convergence given by Matloka [14].

In order to obtain fuzzy approximation theorems based on statistical rates, we will use the following statistical rates. Let $(\mu_n)_{n\in\mathbb{N}}$ be a fuzzy valued sequence, and let $A = [a_{jn}]$ be a non-negative regular summability matrix. Assume that $(p_n)_{n\in\mathbb{N}}$ is a non-increasing sequence of positive real numbers. Then

- $(\mu_n)_{n\in\mathbb{N}}$ is said to be A-statistically convergent to a fuzzy number μ with the rate of $o(p_n)$ if, for every $\varepsilon > 0$,

$$\lim_j \frac{1}{p_j} \sum_{n:D(\mu_n,\mu)\geq\varepsilon} a_{jn} = 0.$$

In this case we write $\mu_n - \mu = st_A - o(p_n)$ as $n \to \infty$;

- $(\mu_n)_{n\in\mathbb{N}}$ is said to be is A-statistically convergent to μ with the rate of $\tilde{o}(p_n)$, denoted by $D(\mu_n,\mu) = st_A - \tilde{o}(p_n)$, as $n \to \infty$, if, for every $\varepsilon > 0$,

$$\lim_j \sum_{n:D(\mu_n,\mu)\geq p_n\varepsilon} a_{jn} = 0.$$

Note that the rate of convergence given by "o" is more controlled by the entries of the summability method rather than the terms of the sequence $(\mu_n)_{n\in\mathbb{N}}$. However, according to the statistical rate "$\tilde{o}$", the rate is mainly controlled by the terms of the fuzzy sequence $(\mu_n)_{n\in\mathbb{N}}$.

Furthermore, we can give the corresponding statistical rates of a real sequence $(x_n)_{n\in\mathbb{N}}$. In this case, we may write that

$$x_n - L = st_A - o(p_n) \text{ as } n \to \infty \text{ if } \lim_j \frac{1}{p_j} \sum_{n:|x_n-L|\geq\varepsilon} a_{jn} = 0 \quad \text{for every } \varepsilon > 0$$

and

$$x_n - L = st_A - \tilde{o}(p_n) \text{ as } n \to \infty \text{ if } \lim_j \sum_{n:|x_n-L|\geq\varepsilon p_n} a_{jn} = 0 \quad \text{for every } \varepsilon > 0.$$

2. Fuzzy Korovkin theory in algebraic case

Let $m \in \mathbb{N}$ and $\mathbf{\Gamma}_m := [a_1,b_1]\times\cdots\times[a_m,b_m]$. As usual, let $C(\mathbf{\Gamma}_m)$ denote the space of all real-valued continuous functions on $\mathbf{\Gamma}_m$ endowed with the usual supremum norm $\|\cdot\|$. Assume that $f : \mathbf{\Gamma}_m \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy number valued function.

Then f is said to be fuzzy continuous at $\mathbf{x}^0 := (x_1^0, \ldots, x_m^0) \in \mathbf{\Gamma}_m$ provided that whenever $\mathbf{x} := (\mathbf{x}_n)_{n\in\mathbb{N}} = (x_{n,1}, \ldots, x_{n,m})_{n\in\mathbb{N}} \to \mathbf{x}_0$, then $D(f(\mathbf{x}_n), f(\mathbf{x})) \to 0$ as $n \to \infty$. Also, we say that f is fuzzy continuous on $\mathbf{\Gamma}_m$ if it is fuzzy continuous at every point $\mathbf{x} \in \mathbf{\Gamma}_m$. The set of all fuzzy continuous functions on the set $\mathbf{\Gamma}_m$ is denoted by $C_{\mathcal{F}}(\mathbf{\Gamma}_m)$ (see, for instance, [2]). Notice that $C_{\mathcal{F}}(\mathbf{\Gamma}_m)$ is only a cone not a vector space. Now let $L : C_{\mathcal{F}}(\mathbf{\Gamma}_m) \to C_{\mathcal{F}}(\mathbf{\Gamma}_m)$ be an operator. Then L is said to be fuzzy linear if, for every $\lambda_1, \lambda_2 \in \mathbb{R}$ having the same sign and for every $f_1, f_2 \in C_{\mathcal{F}}(\mathbf{\Gamma}_m)$, and $\mathbf{x} = (x_1, \ldots, x_m) \in \mathbf{\Gamma}_m$,

$$L\left(\lambda_1 \odot f_1 \oplus \lambda_2 \odot f_2; \mathbf{x}\right) = \lambda_1 \odot L(f_1; \mathbf{x}) \oplus \lambda_2 \odot L(f_2; \mathbf{x})$$

holds. Also L is called fuzzy positive linear operator if it is fuzzy linear and the condition $L(f; \mathbf{x}) \preceq L(g; \mathbf{x})$ is satisfied for any $f, g \in C_{\mathcal{F}}(\mathbf{\Gamma}_m)$ and all $\mathbf{x} \in \mathbf{\Gamma}_m$ with $f(\mathbf{x}) \preceq g(\mathbf{x})$. Also, if $f, g : \mathbf{\Gamma}_m \to \mathbb{R}_{\mathcal{F}}$ are fuzzy number valued functions, then the distance between f and g is given by

$$D^*(f, g) = \sup_{\mathbf{x}\in\mathbf{\Gamma}_m} \sup_{r\in[0,1]} \max\left\{|f_-^{(r)} - g_-^{(r)}|, |f_+^{(r)} - g_+^{(r)}|\right\}.$$

In this section, for $\mathbf{x} = (x_1, \ldots, x_m) \in \mathbf{\Gamma}_m$, we use the following test functions

$$e_0(\mathbf{x}) := 1,\ e_j(\mathbf{x}) := x_j,\ e_{m+j}(\mathbf{x}) := x_j^2 \quad (j = 1, 2, \ldots, m).$$

Theorem 2.1. *Let $A = [a_{jn}]$ be a non-negative regular summability matrix, and let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of fuzzy positive linear operators from $C_{\mathcal{F}}(\mathbf{\Gamma}_m)$ into itself. Assume that there exists a corresponding sequence $\{\tilde{L}_n\}_{n\in\mathbb{N}}$ of positive linear operators from $C(\mathbf{\Gamma}_m)$ into itself with the property*

$$\left\{L_n(f; \mathbf{x})\right\}_{\pm}^{(r)} = \tilde{L}_n\left(f_{\pm}^{(r)}; \mathbf{x}\right) \quad \text{for } \mathbf{x} \in \mathbf{\Gamma}_m \text{ and } n \in \mathbb{N}. \tag{2.1}$$

Assume further that $(p_{k,n})_{n\in\mathbb{N}}$, $k = 0, 1, \ldots, 2m$, are non-increasing sequences of positive real numbers. If, for each $k = 0, 1, \ldots, 2m$,

$$\left\|\tilde{L}_n(e_k) - e_k\right\| = st_A - o(p_{k,n}) \quad \text{as } n \to \infty, \tag{2.2}$$

then, for all $f \in C_{\mathcal{F}}(\mathbf{\Gamma}_m)$, we have

$$D^*\left(L_n(f), f\right) = st_A - o(q_n) \quad \text{as } n \to \infty. \tag{2.3}$$

where $(q_n)_{n\in\mathbb{N}}$ is the sequence whose terms are defined by $q_n := \max_{0\leq k\leq 2m}\{p_{k,n}\}$ for every $n \in \mathbb{N}$.

Proof. Let $\mathbf{x} = (x_1, \dots, x_m) \in \mathbf{\Gamma}_m$ be fixed, and let $f \in C_{\mathcal{F}}(\mathbf{\Gamma}_m)$. Then, we may write that, for every $\varepsilon > 0$, there exist a $\delta > 0$ such that $D(f(\mathbf{y}), f(\mathbf{x})) < \varepsilon$ holds for every $\mathbf{y} = (y_1, \dots, y_m) \in \mathbf{\Gamma}_m$ satisfying $|\mathbf{y} - \mathbf{x}| := \sqrt{\sum_{i=1}^m (y_i - x_i)^2} < \delta$. So, the inequality $|f_{\pm}^{(r)}(\mathbf{y}) - f_{\pm}^{(r)}(\mathbf{x})| < \varepsilon$ holds true for every $r \in [0,1]$ and $\mathbf{y} \in \mathbf{\Gamma}_m$ satisfying $|\mathbf{y} - \mathbf{x}| < \delta$. Now, using (2.1) and also considering the multivariate case of the proof of Theorem 2.1 in [5], we immediately get

$$D^* \left(L_n(f), f\right) \leq \varepsilon + K(\varepsilon) \sum_{k=0}^{2m} \left\| \tilde{L}_n \left(e_k\right) - e_k \right\|, \tag{2.4}$$

where $K(\varepsilon)$ is a certain positive constant depending on ε. Then, for a given $\varepsilon' > 0$, chose $\varepsilon > 0$ such that $0 < \varepsilon < \varepsilon'$, and also define the following sets:

$$U := \{n \in \mathbb{N} : D^*(L_n(f), f) \geq \varepsilon'\},$$

$$U_k := \left\{ n \in \mathbb{N} : \left\| \tilde{L}_n(e_k) - e_k \right\| \geq \frac{\varepsilon' - \varepsilon}{(2m+1)K(\varepsilon)} \right\}, \ k = 0, 1, \dots, 2m.$$

Then inequality (2.4) gives

$$U \subseteq \bigcup_{k=0}^{2m} U_k,$$

which guarantees that, for each $j \in \mathbb{N}$,

$$\sum_{n \in U} a_{jn} \leq \sum_{k=0}^{2m} \Bigg(\sum_{n \in U_k} a_{jn} \Bigg).$$

Also, by the definition of $(q_n)_{n \in \mathbb{N}}$, we have

$$\frac{1}{q_j} \sum_{n \in U} a_{jn} \leq \sum_{k=0}^{2m} \Bigg(\frac{1}{p_{k,j}} \sum_{n \in U_k} a_{jn} \Bigg). \tag{2.5}$$

If we take limit as $j \to \infty$ in the both sides of inequality (2.5) and use the hypothesis (2.2), we see that

$$\lim_j \sum_{n \in U} a_{jn} = 0,$$

which gives (2.3). So, the proof is completed. □

We also get the next result.

Theorem 2.2. *Let* $A = [a_{jn}]$, $(p_{k,n})_{n\in\mathbb{N}}$ $(k = 0, 1, \ldots, 2m)$, $(q_n)_{n\in\mathbb{N}}$, $\{L_n\}_{n\in\mathbb{N}}$ *and* $\{\tilde{L}_n\}_{n\in\mathbb{N}}$ *be the same as in Theorem* 2.1 *with the property* (2.1). *If, for each* $k = 0, 1, \ldots, 2m$,

$$\left\|\tilde{L}_n(e_k) - e_k\right\| = st_A - \tilde{o}(p_{k,n}) \quad \text{as } n \to \infty, \tag{2.6}$$

then, for all $f \in C_{\mathcal{F}}(\mathbf{\Gamma}_m)$, *we have*

$$D^*\left(L_n(f), f\right) = st_A - \tilde{o}(q_n) \quad \text{as } n \to \infty. \tag{2.7}$$

Proof. By (2.4), it is clear that, for any $\varepsilon > 0$,

$$D^*\left(L_n(f), f\right) \le \varepsilon q_n + C'(\varepsilon) \sum_{k=0}^{2m} \left\|\tilde{L}_n(e_k) - e_k\right\| \tag{2.8}$$

holds for some $C'(\varepsilon) > 0$. Now, as in the proof of Theorem 2.1, for a given $\varepsilon' > 0$, choosing $\varepsilon > 0$ such that $\varepsilon < \varepsilon'$, we now consider the following sets:

$$E := \left\{n \in \mathbb{N} : D^*\left(L_n(f), f\right) \ge \varepsilon' q_n\right\},$$

$$E_k := \left\{n \in \mathbb{N} : \left\|\tilde{L}_n(e_0) - e_0\right\| \ge \left(\frac{\varepsilon' - \varepsilon}{(2m+1)C'(\varepsilon)}\right) p_{k,n}\right\}, \quad k = 0, 1, \ldots, 2m.$$

In this case, we claim that

$$E \subseteq \bigcup_{k=0}^{2m} E_k. \tag{2.9}$$

Indeed, otherwise, there would be an element $n \in E$ but $n \notin \bigcup_{k=0}^{2m} E_k$. So, we get

$$n \notin E_0 \Rightarrow \left\|\tilde{L}_n(e_0) - e_0\right\| < \left(\frac{\varepsilon' - \varepsilon}{(2m+1)C'(\varepsilon)}\right) p_{0,n},$$

$$n \notin E_1 \Rightarrow \left\|\tilde{L}_n(e_1) - e_1\right\| < \left(\frac{\varepsilon' - \varepsilon}{(2m+1)C'(\varepsilon)}\right) p_{1,n},$$

$$\ldots$$

$$n \notin E_{2m} \Rightarrow \left\|\tilde{L}_n(e_{2m}) - e_{2m}\right\| < \left(\frac{\varepsilon' - \varepsilon}{(2m+1)C'(\varepsilon)}\right) p_{2m,n}.$$

By the definition of $(q_n)_{n\in\mathbb{N}}$, we immediately see that

$$C'(\varepsilon) \sum_{k=0}^{2m} \left\|\tilde{L}_k(e_k) - e_k\right\| < (\varepsilon' - \varepsilon) q_n \tag{2.10}$$

Since $n \in E$, we have $D^*(L_n(f), f) \geq \varepsilon' q_n$, and hence, by (2.8),

$$C'(\varepsilon) \sum_{k=0}^{2m} \left\| \tilde{L}_k(e_k) - e_k \right\| \geq (\varepsilon' - \varepsilon) q_n,$$

which contradicts with (2.10). So, our claim (2.9) holds true. Now, it follows from (2.9) that

$$\sum_{n \in E} a_{jn} \leq \sum_{k=0}^{2m} \left(\sum_{n \in E_k} a_{jn} \right). \tag{2.11}$$

Letting $j \to \infty$ in (2.11) and also using (2.6), we observe that

$$\lim_j \sum_{n \in E} a_{jn} = 0,$$

which means (2.7). The proof is completed. □

Remark 2.1.

- If $m = 1$ and $p_{k,n} \equiv 1$ for each $k = 0, 1, 2$, then our Theorem 2.1 (or, Theorem 2.2) reduces to Theorem 2.1 of [5].
- If $A = I$, the identity matrix, $m = 1$ and $p_{k,n} \equiv 1$ for each $k = 0, 1, 2$, then we obtain the classical fuzzy Korovkin theorem in algebraic case (see [2]).

3. Fuzzy Korovkin theory in trigonometric case

Let $C_{2\pi}(\mathbb{R}^m)$ denote the space of all real-valued continuous and 2π-periodic functions on $\mathbb{R}^m$ ($m \in \mathbb{N}$). We recall that if a function f in $\mathbb{R}^m$ has period 2π, then, for every $\mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$,

$$\begin{aligned} f(x_1, \ldots, x_m) &= f(x_1 + 2k\pi, x_2, \ldots, x_m) \\ &= f(x_1, x_2 + 2k\pi, \ldots, x_m) \\ &\cdots \\ &= f(x_1, x_2, \ldots, x_m + 2k\pi) \end{aligned}$$

holds for $k = 0, \pm 1, \pm 2, \ldots$. (see, for instance, [15, p. 126]). Then, $C^*(\mathbb{R}^m)$ is a Banach space with the norm $\|\cdot\|_{2\pi}$ defined by

$$\|f\|_{2\pi} := \sup_{\mathbf{x} \in \mathbb{R}^m} |f(\mathbf{x})|, \quad f \in C_{2\pi}(\mathbb{R}^m).$$

In this section, for any $\mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$, we use the following test functions

$$f_0(\mathbf{x}) = 1,\ f_j(\mathbf{x}) = \cos x_j,\ f_{m+j}(\mathbf{x}) = \sin x_j \quad (j = 1, 2, \ldots, m).$$

Now Let $f : \mathbb{R}^m \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy number valued functions. Then, by $C_{\mathcal{F}}(\mathbb{R}^m)$ we denote the set of all fuzzy continuous functions on $\mathbb{R}^m$. Notice that $C_{\mathcal{F}}(\mathbb{R}^m)$ is only a cone not a vector space. By $C_{2\pi}^{(\mathcal{F})}(\mathbb{R}^m)$ we mean the space of all fuzzy continuous and 2π-periodic functions on $\mathbb{R}^m$ (see [6]).

Then we have the following result.

Theorem 3.1. *Let $A = [a_{jn}]$ be a non-negative regular summability matrix, and let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of fuzzy positive linear operators from $C_{2\pi}^{(\mathcal{F})}(\mathbb{R}^m)$ into itself. Assume that there exists a corresponding sequence $\{\tilde{L}_n\}_{n\in\mathbb{N}}$ of positive linear operators from $C_{2\pi}(\mathbb{R}^m)$ into itself with the property*

$$\{L_n(f;\mathbf{x})\}_{\pm}^{(r)} = \tilde{L}_n\big(f_{\pm}^{(r)};\mathbf{x}\big) \quad \text{for } \mathbf{x} \in \mathbb{R}^m \text{ and } n \in \mathbb{N}. \tag{3.1}$$

Assume further that $(p_{k,n})_{n\in\mathbb{N}}$, $k = 0, 1, \ldots, 2m$, are non-increasing sequences of positive real numbers. If, for each $k = 0, 1, \ldots, 2m$,

$$\big\|\tilde{L}_n(f_k) - f_k\big\|_{2\pi} = st_A - o(p_{k,n}) \quad \text{as } n \to \infty,$$

then, for all $f \in C_{2\pi}^{(\mathcal{F})}(\mathbb{R}^m)$, we have

$$D^*(L_n(f), f) = st_A - o(q_n) \quad \text{as } n \to \infty.$$

where $(q_n)_{n\in\mathbb{N}}$ is the sequence whose terms are defined by $q_n := \max_{0\le k\le 2m}\{p_{k,n}\}$ for every $n \in \mathbb{N}$.

Proof. By using a similar idea as in the proof of Theorem 2.1 in [7], it is not hard to see that, for every $f \in C_{2\pi}^{(\mathcal{F})}(\mathbb{R}^m)$,

$$D^*(L_n(f), f) \le \varepsilon + D(\varepsilon)\sum_{k=0}^{2m} \|L_n(f_k) - f_k\|_{2\pi}$$

holds for every $\varepsilon > 0$ and for some positive constant $D(\varepsilon)$. Now the remain part of the proof is completely similar to the proof of our Theorem 2.1. □

The above proof can easily be modified to prove the following analog.

Theorem 3.2. *Let* $A = [a_{jn}]$, $(p_{k,n})_{n\in\mathbb{N}}$ $(k = 0, 1, \ldots, 2m)$, $(q_n)_{n\in\mathbb{N}}$, $\{L_n\}_{n\in\mathbb{N}}$ *and* $\{\tilde{L}_n\}_{n\in\mathbb{N}}$ *be the same as in Theorem* 3.1 *with the property* (3.1)*. If, for each* $k = 0, 1, \ldots, 2m$,

$$\left\|\tilde{L}_n(f_k) - f_k\right\|_{2\pi} = st_A - \tilde{o}(p_{k,n}) \quad \text{as } n \to \infty,$$

then, for all $f \in C_{2\pi}^{(\mathcal{F})}(\mathbb{R}^m)$*, we have*

$$D^*\left(L_n(f), f\right) = st_A - \tilde{o}(q_n) \quad \text{as } n \to \infty.$$

Remark 3.1.

- If $m = 1$ and $p_{k,n} \equiv 1$ for each $k = 0, 1, 2$, then our Theorem 3.1 (or, Theorem 3.2) reduces to Theorem 2.1 of [7].
- If $A = I$, the identity matrix, $m = 1$ and $p_{k,n} \equiv 1$ for each $k = 0, 1, 2$, then we obtain the fuzzy Korovkin theorem in trigonometric case (see [6]).

4. Concluding remarks

In this section, we display an example that corrects our results. Take $A = C_1 = [c_{jn}]$, the Cesáro matrix of order, defined by

$$c_{jn} := \begin{cases} \dfrac{1}{j}, & \text{if } 1 \leq n \leq j \\ 0, & \text{otherwise.} \end{cases}$$

Consider the sequences (u_n) and (v_n) defined respectively by

$$u_n = \begin{cases} 0, & \text{if } n = i^2, \quad (i = 1, 2, \ldots), \\ 1, & \text{otherwise.} \end{cases}$$

and

$$v_n = \begin{cases} 0, & \text{if } n = i^3, \quad (i = 1, 2, \ldots), \\ 1, & \text{otherwise.} \end{cases}$$

Then, since, for every $\varepsilon > 0$,

$$\sqrt[3]{j} \sum_{n \leq j : |u_n - 1| \geq \varepsilon} \frac{1}{j} \leq \frac{\sqrt[3]{j}\sqrt{j}}{j} \to 0 \qquad \text{as } j \to \infty,$$

$$\sqrt{j} \sum_{n \leq j : |v_n - 1| \geq \varepsilon} \frac{1}{j} \leq \frac{\sqrt{j}\sqrt[3]{j}}{j} \to 0 \qquad \text{as} j \to \infty,$$

$$\sqrt[4]{j} \sum_{n \leq j : |u_n v_n - 1| \geq \varepsilon} \frac{1}{j} \leq \frac{\sqrt[4]{j}(\sqrt[3]{j} + \sqrt{j})}{j} \to 0 \quad \text{as } j \to \infty,$$

we may write that

$$1 - u_n = st_{C_1} - o\left(\frac{1}{\sqrt[3]{n}}\right) \quad \text{as } n \to \infty,$$

$$1 - v_n = st_{C_1} - o\left(\frac{1}{\sqrt{n}}\right) \quad \text{as } n \to \infty,$$

$$1 - u_n v_n = st_{C_1} - o\left(\frac{1}{\sqrt[4]{n}}\right) \quad \text{as } n \to \infty.$$

Now taking $m = 2$, we define the fuzzy positive linear operators as follows:

$$L_n(f; \mathbf{x}) = v_n \odot \bigoplus_{k,l=0}^{n} \binom{n}{k} \binom{n}{l} u_n^{k+l} x_1^k (1 - u_n x_1)^{n-k} x_2^l (1 - u_n x_2)^{n-l} \odot f\left(\frac{k}{n}, \frac{l}{n}\right), \tag{4.1}$$

where $f \in C_{\mathcal{F}}([0,1] \times [0,1])$, $\mathbf{x} = (x_1, x_2) \in [0,1] \times [0,1]$ and $n \in \mathbb{N}$. In this case, the corresponding positive linear operators as in Theorem 2.1 are given by

$$\{L_n(f; \mathbf{x})\}_{\pm}^{(r)} = \tilde{L}_n\left(f_{\pm}^{(r)}; \mathbf{x}\right)$$

$$= v_n \sum_{k,l=0}^{n} \binom{n}{k} \binom{n}{l} u_n^{k+l} x_1^k (1 - u_n x_1)^{n-k} x_2^l (1 - u_n x_2)^{n-l} f_{\pm}^{(r)}\left(\frac{k}{n}, \frac{l}{n}\right),$$

where $f_{\pm}^{(r)} \in C([0,1] \times [0,1])$ and $r \in [0,1]$. After some simple calculations, we get

$$\tilde{L}_n(e_0; \mathbf{x}) = v_n e_0(\mathbf{x}),$$

$$\tilde{L}_n(e_1; \mathbf{x}) = u_n v_n e_1(\mathbf{x}),$$

$$\tilde{L}_n(e_2; \mathbf{x}) = u_n v_n e_2(\mathbf{x})$$

$$\tilde{L}_n(e_3; \mathbf{x}) = u_n^2 v_n e_3(\mathbf{x}) - \frac{u_n^2 v_n x_1^2}{n} + \frac{u_n v_n x_1}{n},$$

$$\tilde{L}_n(e_4; \mathbf{x}) = u_n^2 v_n e_4(\mathbf{x}) - \frac{u_n^2 v_n x_2^2}{n} + \frac{u_n v_n x_2}{n}$$

Then, we obtain that

$$\|\tilde{L}_n(e_0) - e_0\| = 1 - v_n,$$

$$\|\tilde{L}_n(e_1) - e_1\| = 1 - u_n v_n,$$

$$\|\tilde{L}_n(e_2) - e_2\| = 1 - u_n v_n,$$

$$\|\tilde{L}_n(e_3) - e_3| \leq 1 - u_n^2 v_n + \frac{2}{n},$$

$$\|\tilde{L}_n(e_4) - e_4\| \leq 1 - u_n^2 v_n + \frac{2}{n}.$$

By the definition of $(u_n)_{n\in\mathbb{N}}$, it is not hard to see that $u_n^2 = u_n$ for each $n \in \mathbb{N}$. Hence, we immediately get

$$\|\tilde{L}_n(e_0) - e_0\| = st_{C_1} - o\left(\frac{1}{\sqrt{n}}\right) \text{as } n \to \infty,$$

$$\|\tilde{L}_n(e_1) - e_1\| = st_{C_1} - o\left(\frac{1}{\sqrt[4]{n}}\right) \text{as } n \to \infty,$$

$$\|\tilde{L}_n(e_2) - e_2\| = st_{C_1} - o\left(\frac{1}{\sqrt[4]{n}}\right) \text{as } n \to \infty,$$

$$\|\tilde{L}_n(e_3) - e_3\| = st_{C_1} - o\left(\frac{1}{\sqrt[4]{n}}\right) \text{as } n \to \infty,$$

$$\|\tilde{L}_n(e_4) - e_4\| = st_{C_1} - o\left(\frac{1}{\sqrt[4]{n}}\right) \text{as } n \to \infty.$$

Then, by Theorem 2.1, we obtain, for all $f \in C_{\mathcal{F}}([0,1] \times [0,1])$, that

$$D^*\left(L_n(f), f\right) = st_{C_1} - o\left(\frac{1}{\sqrt[4]{n}}\right) \quad \text{as } n \to \infty,$$

where $\{L_n\}_{n\in\mathbb{N}}$ is given by (4.1). However, since the sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ is non-convergent (in the usual sense), the sequence $\{L_n(f)\}_{n\in\mathbb{N}}$ is not fuzzy convergent to f. So, this example clearly shows that our fuzzy approximations based on statistical rates are more applicable than both fuzzy approximation in [2], [6] and statistical fuzzy approximation in [5], [7].

Acknowledgement. The author would like to thank the referee for carefully reading the manuscript.

References

[1] G. A. Anastassiou, Fuzzy random Korovkin theory and inequalities, *Math. Inequal. Appl.* **10** (2007), 63–94.

[2] G. A. Anastassiou, On basic fuzzy Korovkin theory, *Studia Univ. Babeş-Bolyai Math.* **50** (2005), 3–10.

[3] G. A. Anastassiou, Fuzzy approximation by fuzzy convolution type operators, *Comput. Math. Appl.* **48** (2004), 1369–1386.

[4] G. A. Anastassiou, High-order fuzzy approximation by fuzzy wavelet type and neural network operators, *Comput. Math. Appl.* **48** (2004), 1387–1401.

[5] G. A. Anastassiou and O. Duman, Statistical fuzzy approximation by fuzzy positive linear operators, *Comput. Math. Appl.* **55** (2008), 573–580.

[6] G. A. Anastassiou and S. G. Gal, On fuzzy trigonometric Korovkin theory, *Nonlinear Funct. Anal. Appl.* **11** (2006), 385–395.

[7] O. Duman and G. A. Anastassiou, On statistical fuzzy trigonometric Korovkin theory, *J. Comput. Anal. Appl.* **10** (2008), 333–344.

[8] H. Fast, Sur la convergence statistique, *Colloq. Math.* **2** (1951), 241–244.

[9] A. R. Freedman and J. J. Sember, Densities and summability, *Pacific J. Math.* **95** (1981), 293–305.

[10] S. G. Gal, Approximation Theory in Fuzzy Setting, Handbook of Analytic-Computational Methods in Applied Mathematics, *Chapman* & *Hall/CRC*, *Boca Raton, FL*, 2000, 617–666.

[11] R. J. Goetschel and W. Voxman, Elementary fuzzy calculus, *Fuzzy Sets* & *Systems* **18** (1986), 31–43.

[12] G. H. Hardy, Divergent Series, *Oxford Univ. Press*, *London*, 1949.

[13] P. Liu, Analysis of approximation of continuous fuzzy functions by multivariate fuzzy polynomials, *Fuzzy Sets* & *Systems* **127** (2002), 299–313.

[14] M. Matloka, Sequences of fuzzy numbers, *BUSEFAL* **28** (1986), 28–37.

[15] T. Myint-U and L. Debnath, Partial Differential Equations for Scientists and Engineers, *Prentice Hall*, *USA*, 1987.

[16] F. Nuray and E. Savaş, Statistical convergence of sequences of fuzzy numbers, *Math. Slovaca* **45** (1995), 269–273.

[17] C. X. Wu and M. Ma, Embedding problem of fuzzy number space I, *Fuzzy Sets* & *Systems* **44** (1991), 33–38.

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(Received February 16, 2009; revised October 13, 2009)