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The influence of complemented minimal subgroups on the structure of finite groups

By JIAKUAN LU (Shanghai), SHIRONG LI (Nanning) and XIUYUN GUO (Shanghai)

Abstract. A subgroup H of a finite group G is said to be complemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K = 1$. In this paper the following theorem is proved: Let G be a finite group and let p be the smallest prime dividing the order of G. Then G is p-nilpotent if and only if every minimal subgroup of $P \cap G^{\mathcal{N}}$ is complemented in $N_G(P)$, where P is a Sylow p-subgroup of G and $G^{\mathcal{N}}$ is the nilpotent residual of G. As some applications, some interesting results related with complemented minimal subgroups are obtained.

1. Introduction

All groups considered in this paper are finite. A subgroup H of a group G is said to be complemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K = 1$. In this case we call the above subgroup K of G a complement of H in G. It is quite clear that the existence of complements for some families of subgroups of a group give a lot of information about its structure. For instance, Hall proved that a group G is solvable if and only if every Sylow subgroup is complemented [10]. In particular, Hall in 1937 proved that a group G is supersolvable with elementary abelian Sylow subgroups if and only if every subgroup of G is complemented in G [9]. We call such groups $\mathcal{H}$-groups. In 1994, Guo showed that a group G is an $\mathcal{H}$-group if and only if all normalizers of Sylow

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subgroups of G are $\mathcal{H}$-groups [8]. A minimal subgroup of a group G is a subgroup of G of prime order. Later, Ballester-Bolinches and Guo showed that a group G is an $\mathcal{H}$-group if and only if every minimal subgroup of G is complemented in G [1].

In recent years, there has been a lot of interest in investigating the influence of minimal subgroups on the structure of groups (for example [2], [4], [5], [11], [12]). In this paper, we shall continue the investigation on the influence of complemented minimal subgroups on the structure of finite groups. First, we give some new generalizations of the Hall theorem (Theorem 1.1). Second, we drop the assumption that every minimal subgroup is complemented. Our aim is to minimize the number of complemented minimal subgroups to determine the structure of a group. At the same time, we only assume that minimal subgroups are complemented in a subgroup of the group.

Recall that a class $\mathcal{F}$ of groups is called a formation provided that (i) $G \in \mathcal{F}$ and $N \trianglelefteq G$ imply $G/N \in \mathcal{F}$ and (ii) if both G/N and G/M are in $\mathcal{F}$, then $G/(N \cap M) \in \mathcal{F}$. If, in addition, $G/\Phi(G) \in \mathcal{F}$ implies $G \in \mathcal{F}$, then we say that $\mathcal{F}$ is saturated. Let $G^{\mathcal{F}}$ be the intersection of all normal subgroups N of G satisfying $G/N \in \mathcal{F}$. The subgroup $G^{\mathcal{F}}$ is called $\mathcal{F}$-residual of G. It is clear that a group $G \in \mathcal{F}$ if and only if $G^{\mathcal{F}} = 1$ (see [3, Chapter II]). Let $\mathcal{N}$ be the class of the nilpotent groups. It is well-known that

$$G^{\mathcal{N}} = \bigcap_{p} O^p(G),$$

where $O^p(G)$ is the subgroup generated by all p'-elements of G. Let $\mathcal{U}$ denote the saturated formation of all supersolvable groups. The following formations are also considered:

$\mathcal{H}$: the class of the supersolvable groups with elementary abelian Sylow subgroups (not saturated);

$\mathcal{H}^*$: the class of the supersolvable groups with abelian Sylow subgroups (not saturated).

The main results of this paper are as follows:

Theorem 1.1. *Let G be a group. Any two of the following statements are equivalent:*

(1) *G is an $\mathcal{H}$-group.*

(2) (P. Hall) *Every subgroup of G is complemented.*

(3) (A. Ballester-Bolinches, X. Guo) *Every minimal subgroup of G is complemented.*

(4) (W. Guo) *For every Sylow subgroup P of G, $N_G(P)$ is an $\mathcal{H}$-group.*

(5) *For every Sylow subgroup P of G, every minimal subgroup of P is complemented in $N_G(P)$.*

(6) *There exists a normal subgroup H of G such that $G/H \in \mathcal{H}$, and for every Sylow subgroup P of H, every minimal subgroup of P is complemented in $N_G(P)$.*

Theorem 1.2. *Let p be the smallest prime dividing the order of a group G and let P be a Sylow p-subgroup of G. The following statements are equivalent:*

(1) *G is p-nilpotent.*

(2) *Every subgroup of $P \cap G^{\mathcal{N}}$ of order p is complemented in $N_G(P)$.*

As some applications of Theorem 1.2, we have the following three results.

Theorem 1.3. *Let $\mathcal{F}$ be a formation containing $\mathcal{U}$ (non necessarily saturated) and let G be a group. The following statements are equivalent:*

(1) *$G \in \mathcal{F}$.*

(2) *There exists a normal subgroup H of G such that $G/H \in \mathcal{F}$, and for every Sylow subgroup P of H, every minimal subgroup of $P \cap G^{\mathcal{N}}$ is complemented in $N_G(P)$.*

Theorem 1.4. *Let $\mathcal{F}$ be a formation containing $\mathcal{H}^*$(non necessarily saturated) and let G be a group. The following statements are equivalent:*

(1) *$G \in \mathcal{F}$.*

(2) *There exists a normal subgroup H of G such that $G/H \in \mathcal{F}$, and for every Sylow subgroup P of H, every minimal subgroup of $P \cap N_G(P)'$ is complemented in $N_G(P)$.*

Theorem 1.5. *Let G ba a group. Any two of the following statements are equivalent:*

(1) *G is a supersolvable group with abelian Sylow subgroups.*

(2) *For every Sylow subgroup P of G, $N_G(P)$ is an $\mathcal{H}^*$-group.*

(3) *For every Sylow subgroup P of $G^{\mathcal{H}^*}$, every minimal subgroup of $P \cap N_G(P)'$ is complemented in $N_G(P)$.*

Let $\mathcal{H}^{**}$ be the class of groups G with the property: every minimal subgroup of the commutator subgroup G' of G is complemented in G. By Theorem 1.4, the groups in $\mathcal{H}^{**}$ are supersolvable groups with abelian Sylow subgroups. In fact, $\mathcal{H}^{**}$ is a proper sub-class of $\mathcal{H}^*$.

Corollary 1.6. *Let G be a group. The following statements are equivalent:*

(1) *G is an $\mathcal{H}^{**}$-group.*

(2) *For every Sylow subgroup P of G, $N_G(P)$ is an $\mathcal{H}^{**}$-group.*

Theorem 1.7. *Let $\mathcal{F}$ be a formation containing $\mathcal{U}$ (non necessarily saturated) and let G be a group. The following statements are equivalent:*

(1) *$G \in \mathcal{F}$.*

(2) *There exists a normal subgroup H of G such that $G/H \in \mathcal{F}$, and for every Sylow subgroup P of $F^*(H)$, the generalized Fitting subgroup of H, every minimal subgroup of $P \cap G^{\mathcal{N}}$ is complemented in $N_G(P)$.*

2. Lemmas

Lemma 2.1. *Let G be a group, $H \leq K \leq G$ and $N \trianglelefteq G$. Suppose that H is complemented in G.*

(1) *H is complemented in K.*

(2) *If $(|H|, |N|) = 1$, then HN/N is complemented in G/N.*

(3) *There is an example to show that the condition of (2) is necessary.*

Proof. (1) and (2) are well-known. In order to prove (3), consider the group $M = A_4 \times \langle a \rangle$, where $a^2 = 1$ and A_4 is the symmetric group of degree 4. Let b be an element of order 2 of A_4, $H = \langle ab \rangle$ and $N = \langle a \rangle$. Clearly, A_4 is a complement of H in M, while HN/N is a subgroup of order 2 of M/N. As $M/N \cong A_4$ and A_4 has no any complemented subgroup of order 2, we can conclude that HN/N is not complemented in M/N. □

Lemma 2.2. *Let G be a group and let P be a normal elementary abelian p-subgroup of G for some prime p. If every subgroup of P of order p is complemented in G, then P is generated by normal subgroups of G of order p.*

Proof. Note that $P \cap \Phi(G) = 1$. By [6, Satz 4.5, p. 279], $P = R_1 \times R_2 \times \cdots \times R_n$, where R_i are all minimal normal subgroups of G. Let X be a subgroup of R_i of order p. By hypothesis, there exists a subgroup M of G such that $G = XM$ and $X \cap M = 1$. Then $R_i = X(R_i \cap M)$ and $R_i \cap M \trianglelefteq G$. It follows that $R_i \cap M = R_i$ or $R_i \cap M = 1$. As $R_i \cap M = R_i$ is impossible, we get that $R_i \cap M = 1$ and hence $R_i = X$. The proof is now complete. □

Lemma 2.3. *Let P be a p-subgroup of a group G for some prime p. Suppose that every subgroup of P of order p is complemented in G. Then P is an elementary abelian group.*

Proof. Clearly, we may assume that $P > 1$. If the Frattini subgroup $\Phi(P) > 1$, then there exists a subgroup X of $\Phi(P)$ of order p. By hypothesis, X is complemented in G and it follows from Lemma 2.1(1) that X is complemented in P, which is impossible. So $\Phi(P) = 1$ and the lemma follows. □

Lemma 2.4. *Let G be a group, $H \trianglelefteq G$ and $P \in Syl_p(H)$. Suppose that N is a normal p'-subgroup of G contained in H and $X \leq P$ is complemented in $N_G(P)$. Then NX/N is complemented in $N_{G/N}(PN/N)$.*

Proof. Since $(|P|, |N|) = 1$, we have that $N_{G/N}(PN/N) = N_G(P)N/N$. By hypothesis, there exists a subgroup K of $N_G(P)$ such that $N_G(P) = XK$ and $X \cap K = 1$. Then the subgroup KN is a complement of X in $N_G(P)N$. The lemma follows from Lemma 2.1(2). □

The generalized Fitting subgroup $F^*(G)$ of a group G is the product of all normal quasinilpotent subgroups of G. We gather the following well-known facts about this subgroup for later use (see [7, Chapter X]).

Lemma 2.5. *Let G be a group and M a subgroup of G.*

(1) *If M is normal in G, then $F^*(M) \leq F^*(G)$;*

(2) *If $G \neq 1$, then $F^*(G) \neq 1$;*

(3) *$F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is soluble, then $F^*(G) = F(G)$.*

Lemma 2.6 ([13]). *Let P be a normal p-subgroup of a group G where p is a prime. Then $F^*(G/\Phi(P)) = F^*(G)/\Phi(P)$.*

3. Proofs of main theorems

Proof of Theorem 1.1. (1) $\Longleftrightarrow$ (4): see [8]. It is easy to see that (1) implies (2), (2) implies(3), (3) implies (5) and (5) implies (6). Thus we only need to prove that (6) implies (1).

Firstly, we claim that H is a Sylow tower group. Fix p to be the smallest prime dividing $|H|$. By Lemma 2.3, P is elementary abelian. Thus P is generated by subgroups U_i of order p which are normal in $N_H(P)$ by Lemma 2.2. Then all $U_i \leq Z(N_H(P))$ and hence $P \leq Z(N_H(P))$. By a theorem of Burnside [6, 2.6 Hauptsatz, p. 279], H is p-nilpotent. By the same arguments and induction, we have H is a Sylow tower group. Let q be the largest prime dividing $|H|$ and let Q be a Sylow q-subgroup of H. Then Q is normal in G. Thus, by hypothesis, every minimal subgroup of Q is complemented in G.

Now, by Lemma 2.4, we see that G/Q satisfies hypothesis, so $G/Q \in \mathcal{H}$ by induction. By Lemma 2.2, we can choose a subgroup U of Q of order q which is normal in G. Then there exists a subgroup M of G such that $G = UM$ and $U \cap M = 1$. Since $G/Q \cong M/(Q \cap M)$ and $G/Q \in \mathcal{H}$, it follows that $M/Q \cap M \in \mathcal{H}$. By induction, $M \in \mathcal{H}$, namely M is supersolvable with elementary abelian subgroups. Now $G = [U]M$ and G is supersolvable. Also, let Q_1 be a Sylow q-subgroup of M, then UQ_1 is a Sylow q-subgroup of G and Q_1 is a complement of U in UQ_1, so U is of index q in UQ_1 and hence normal in UQ_1. Consequently, $UQ_1 = U \times Q_1$. As Q_1 is elementary abelian, it follows that UQ_1 is elementary abelian. Thus all Sylow subgroups of G are elementary abelian and hence $G \in \mathcal{H}$. The proof is now complete. □

Proof of Theorem 1.2. It is easy to see that (1) implies (2).

We now prove that (2) implies (1). Assume that G satisfies (2) but G is not p-nilpotent. Let us choose G of minimal order. Write $D = G^{\mathcal{N}}$ and set $P_0 = P \cap D$. Then P_0 is a Sylow p-subgroup of D. By Lemma 2.3, P_0 is elementary abelian and $P_0 \trianglelefteq N_G(P)$.

(1) If $P \leq K < G$, then K is p-nilpotent.

Clearly, the nilpotent residual $K^{\mathcal{N}}$ of K is contained in $D \cap K$ and $P \cap K^{\mathcal{N}} \leq P \cap D = P_0$. By hypothesis, every subgroup of P_0 of order p is complemented in $N_G(P)$, and hence, by Lemma 2.1(1), in $N_K(P)$. So K satisfies hypothesis. By the choice of G, K is p-nilpotent, as desired.

(2) $P_0 \leq Z(N_G(P))$.

Clearly we may assume that P_0 is non-trivial. By Lemma 2.2, P_0 is generated by subgroups of order p which are normal in $N_G(P)$. Let $X \leq P_0$ be such a subgroup of order p and set $N = N_G(P)$. Then $N/C_N(X)$ is a cyclic group of order dividing $p-1$. Since p is the smallest prime dividing $|G|$, we have $C_N(X) = N$, namely $X \leq Z(N_G(P))$ as desired.

(3) $O_{p'p}(G)$ is a p-group.

By Lemma 1.4, we see that the quotient group $G/O_{p'}(G)$ satisfies hypothesis. The choice of G forces $O_{p'}(G) = 1$ and so (3) holds.

(4) $P_0 \trianglelefteq G$.

Assume that (4) is not true. As P_0 is normal in $N_G(P)$, we have $N_G(P) \leq N_G(P_0) < G$. By (1), $N_G(P_0)$ is p-nilpotent. Hence $N_D(P_0)$ is p-nilpotent, which implies that $N_D(P_0) = P_0 \times L$ where L is a Hall p'-subgroup of $N_D(P_0)$. As P_0 is an elementary abelian Sylow p-subgroup of D, by a theorem of Burnside

[6, 2.6 Hauptsatz, p. 279], D is p-nilpotent. So $D = P_0$ by (3). Hence $P_0 \trianglelefteq G$, a contradiction.

(5) $P \trianglelefteq G$.

Consider the normal subgroup $C_G(P_0)$ of G. If $C_G(P_0) = G$, then $P_0 \leq Z(G)$. As P_0 is a Sylow p-subgroup of D, we know that D is p-nilpotent. Applying (3) we see that $D = P_0 \leq Z(G)$, and hence G is nilpotent, contrary to the choice of G. Assume that $C_G(P_0) < G$. Then, by (2), $N_G(P) \leq C_G(P_0)$. So $C_G(P_0)$ is p-nilpotent by (1). Thus $C_G(P_0) \leq F_p(G)$, it follows from (3) that $C_G(P_0)$ is a p-group. Now, as $P \leq C_G(P_0)$, we have $P = C_G(P_0) \trianglelefteq G$, as desired.

(6) The final contradiction.

It is clear that $(G/P_0)/(D/P_0) \cong G/D$ is nilpotent. Since P_0 is central in G, we have that G is p-nilpotent, which contradicts the choice of G. The proof of the theorem is now complete. □

Corollary 3.1. *If, for every Sylow subgroup P of the group G, every minimal subgroup of $P \cap G^{\mathcal{N}}$ is complemented in $N_G(P)$, then G is a Sylow tower group.*

Proof. This follows from Theorem 1.2 and induction. □

The next corollary is required in the proof of Theorem 1.4.

Corollary 3.2. *If, for every Sylow subgroup P of the group G,every minimal subgroup of $P \cap N_G(P)'$ is complemented in $N_G(P)$, then G is a Sylow tower group.*

Proof. Let M be a proper Hall subgroup of G and let P be any Sylow subgroup of M. Then $P \cap N_M(P)' \leq P \cap N_G(P)'$ and $N_M(P) \leq N_G(P)$. Thus every minimal subgroup X of $P \cap N_M(P)'$ is also a minimal subgroup of $P \cap N_G(P)'$. Hence X is complemented in $N_G(P)$ by hypothesis, and hence in $N_M(P)$ by Lemma 2.1(1). By induction, M is a Sylow tower group. Let p be the smallest prime dividing $|G|$ and let P be a Sylow p-subgroup of G. Since $P' \leq \Phi(P)$, it follows that $P' = 1$, i.e., P is abelian. On the other hand, by Theorem 1.2, we know that $N_G(P)$ is p-nilpotent. By a theorem of Burnside [6, 2.6 Hauptsatz, p. 279], G is p-nilpotent. So $O^p(G)$ is a Hall p'-subgroup. By the above arguments, $O^p(G)$ is a Sylow tower group and so is G. The proof is now complete. □

Proofs of Theorem 1.3 and 1.4. We only need to prove that (2) implies (1). We use induction on $|G|$. Consider two cases:

Case 1. G satisfies the condition (2) of Theorem 1.3.

Case 2. G satisfies the condition (2) of Theorem 1.4.

By Corollaries 3.1 and 3.2, H is a Sylow tower group in both cases. Let q be the largest prime dividing $|H|$ and let Q be a Sylow q-subgroup of H. Then Q is normal in G and $(G/Q)/(H/Q) \cong G/H \in \mathcal{F}$. By Lemma 2.4, G/Q satisfies the hypothesis of Case 1 and Case 2 respectively, so by induction $G/Q \in \mathcal{F}$ in both cases. In Case 1, $\mathcal{F}$ has been assumed to contain $\mathcal{U}$, we have $G/G^{\mathcal{N}} \in \mathcal{F}$. Let $Q_0 = Q \cap G^{\mathcal{N}}$. For Case 2, $\mathcal{F}$ contains $\mathcal{H}^*$, so $G/G' \in \mathcal{F}$. In this case, write $Q_0 = Q \cap G'$. In both cases $G/Q_0 \in \mathcal{F}$.

Clearly, we may assume that $Q_0 > 1$ and let U be a minimal normal subgroup of G contained in Q_0. Then U is elementary abelian. By hypothesis, every minimal subgroup of U is complemented in $N_G(Q)$ and hence in G. By Lemma 2.2, U is generated by some minimal subgroups which are normal in G. Choose such a subgroup X, namely $X \leq U$ is cyclic of order q and $X \trianglelefteq G$. Then there exists a subgroup M of G such that $G = XM$ and $X \cap M = 1$. Thus $M/M \cap Q_0 \cong G/Q_0 \in \mathcal{F}$ and every minimal subgroup of $M \cap Q_0$ is complemented in G. Therefore, the pair $(M, M \cap Q_0)$ satisfies the condition of Case 1 and Case 2 respectively. By induction, M is an $\mathcal{F}$-group. Hence $Q_0 = X$. Consequently, $G = [Q_0]M$ with $Q_0 \cap M = 1$ and $|Q_0| = q$.

Now, as Q_0 is a normal subgroup of G of order q, we know that $G/C_G(Q_0)$ is cyclic of order dividing $q - 1$. Of course, $G' \leq C_G(Q_0)$. Write $N = C_G(Q_0) \cap M$. Then N is normal in G and G/N is a split extension of the normal subgroup Q_0N/N by M/N, where $M/N \cong G/C_G(Q_0)$ is cyclic. So $G/N \in \mathcal{H}^*$ and hence $\mathcal{F}$. Thus $G = G/(Q_0 \cap N) \in \mathcal{F}$, as desired. The proof is now complete. □

Proof of Theorem 1.5. Clearly (1) and (2) are equivalent. It follows from Theorem 1.4 that (1) and (3) are equivalent. □

Proof of Theorem 1.6. It is clear that (1) implies (2). Conversely, by Theorem 1.4, G is supersolvable. In particular, G is a Sylow tower group. Let Q be a Sylow q-subgroup of G for the largest prime q dividing $|G|$. Then Q is normal in G and so $N_G(Q) = G$. By hypothesis, $N_G(Q)$ is an $\mathcal{H}^{**}$-group and hence G is an $\mathcal{H}^{**}$-group. □

Proof of Theorem 1.7. It is clear that (1) implies (2). Conversely, assume that the result is not true and let G be a minimal counterexample. Then $H > 1$. By Lemma 2.5(2), $F^*(H) > 1$. Clearly, $F^*(H)$ satisfies the hypothesis of Corollary 3.1, so $F^*(H)$ is a Sylow tower group and hence is solvable. By Lemma 2.5(3), we get that $F^*(H) = F(H)$. Thus every Sylow subgroup of $F^*(H)$ is normal in G. We claim

(1) Every minimal subgroup of $F(H) \cap G^{\mathcal{N}}$ is complemented in G.

In fact, for each minimal subgroup X of $F(H) \cap G^{\mathcal{N}}$, let $|X| = p$ for some prime p. Then there exists a Sylow p-subgroup P of $F(H)$ such that $X \leq P$, so $X \leq P \cap G^{\mathcal{N}}$. By hypothesis, X is complemented in $N_G(P) = G$.

(2) $\Phi(F(H)) = 1$.

Let P be an arbitrary Sylow p-subgroup of $F(H)$ and $B = \Phi(P)$. Assume that $B \neq 1$. Consider the pair $(G/B, H/B)$. We have

(i) $(G/B)/(H/B) \in \mathcal{F}$ as $G/H \in \mathcal{F}$ and $(G/B)/(H/B) \cong G/H$.

(ii) $B \cap G^{\mathcal{N}} = 1$. If not, some minimal subgroup of B would be complemented in G contradicting the fact that B is a subgroup of $\Phi(G)$.

(iii) Every minimal subgroup of $F^*(H/B) \cap (G/B)^{\mathcal{N}}$ is complemented in G/B.

By Lemma 2.6, $F^*(H/B) = F^*(H)/B$. We hence have $F^*(H/B) \cap (G/B)^{\mathcal{N}} = F(H)/B \cap G^{\mathcal{N}}B/B = (F(H) \cap G^{\mathcal{N}})B/B$. Thus every minimal subgroup of $F^*(H/B) \cap (G/B)^{\mathcal{N}}$ has the form XB/B, where X is a minimal subgroup of $F(H) \cap G^{\mathcal{N}}$. By (1), there exists a subgroup M of G such that $G = XM$ and $X \cap M = 1$. Clearly, $B \leq M$. Thus M/B is a complement of XB/B in G/B and (iii) follows.

(iv) By (iii), G/B satisfies hypothesis and therefore G/B belongs to $\mathcal{F}$ by the choice of G. Since $\mathcal{F}$ contains $\mathcal{U}$, we have $G/G^{\mathcal{N}} \in \mathcal{F}$. Therefore $G = G/B \cap G^{\mathcal{N}} \in \mathcal{F}$, contrary to the choice of G.

Consequently $B = 1$ and $\Phi(F(H)) = 1$.

(3) $F(H)$ is a direct products of normal subgroups of G of prime order.

As $\Phi(F(H)) = 1$ by (2), $F(H) = R_1 \times R_2 \times \cdots \times R_n$, where each R_i is a minimal normal subgroup of G ([6, Satz 4.5, p. 279]). We claim that if $R_i \not\subseteq G^{\mathcal{N}}$, then R_i is of prime order. In fact, $R_i \cap G^{\mathcal{N}} = 1$, so $R_i G^{\mathcal{N}}/G^{\mathcal{N}} \cong R_i$ is a G-chief factor. As $G/G^{\mathcal{N}}$ is nilpotent, $R_i G^{\mathcal{N}}/G^{\mathcal{N}}$ must be prime order and R_i is of prime order. Now we suppose that there is a R_i such that $R_i \leq G^{\mathcal{N}}$. Then we can find a minimal subgroup X of $R_i \cap G^{\mathcal{N}} \leq F(H) \cap G^{\mathcal{N}}$. By (1), X is complemented in G, and hence $G = XM$ with $M \leq G$ and $X \cap M = 1$. Then $R_i = X(R_i \cap M)$. Thus $R_i \cap M$ is normal in G, which gives $R_i \cap M = 1$, namely $R_i = X$ is of prime order. Summing up, all R_i are of prime order and so (3) holds.

(4) Final contradiction.

Now let

$$F = \bigcap_{i=1}^{n} C_G(R_i) = C_G(F(H)).$$

Then G/F is supersolvable as $G/C_G(R_i)$ is cyclic for all R_i. Moreover, $F \cap H = C_G(F(H)) \cap H \leq F(H)$ by [6, III, 4.2 Satz, p. 277]. By hypothesis, $G/H \in \mathcal{F}$ so that $G/F \cap H \in \mathcal{F}$. Thus $G/F(H) \in \mathcal{F}$. Consequently, $G/F(H) \cap G^{\mathcal{N}} \in \mathcal{F}$ and every minimal subgroup of $F(H) \cap G^{\mathcal{N}}$ is complemented in G. Now, an application of Theorem 1.3 fields $G \in \mathcal{F}$. The proof of the theorem is now complete □

4. Examples

Example 4.1 (Theorem 1.4 is false if $\mathcal{H}^*$ is replaced by $\mathcal{H}$). Let $\mathcal{F} = \mathcal{H}$ and let $G = \langle a, b : a^5 = b^4 = 1, b^{-1}ab = a^2 \rangle$. This group satisfies the condition: For every Sylow subgroup P of G, every minimal subgroup of $P \cap G'$ is complemented in G. However, G is not in $\mathcal{H}$.

Example 4.2 (Theorem 1.3 is false if $\mathcal{U}$ is replaced by $\mathcal{H}^*$). Let $\mathcal{F} = \mathcal{H}^*$, and let G be a non-abelian nilpotent group. Then $G^{\mathcal{N}} = 1$. So G satisfies the condition: For every Sylow subgroup P of G, every minimal subgroup of $P \cap G^{\mathcal{N}}$ is complemented in G. But G is not in $\mathcal{H}^*$.

Example 4.3 (Comparison of Theorem 1.3 and Theorem 1.4). Let $G = S_4$, the symmetric group of degree 4. Then G satisfies the following condition: For every Sylow subgroup P of G, every minimal subgroup of $P \cap N_G(P)^{\mathcal{N}}$ is complemented in $N_G(P)$. However, G is non-supersolvable.

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JIAKUAN LU
DEPARTMENT OF MATHEMATICS
SHANGHAI UNIVERSITY
SHANGHAI 200444
P.R. CHINA

E-mail: jklu@shu.edu.cn

SHIRONG LI
DEPARTMENT OF MATHEMATICS
GUANGXI UNIVERSITY
NANNING, GUANGXI 530004
P.R. CHINA

E-mail: shirong@gxu.edu.cn

XIUYUN GUO
DEPARTMENT OF MATHEMATICS
SHANGHAI UNIVERSITY
SHANGHAI 200444
P.R. CHINA

E-mail: xyguo@staff.shu.edu.cn

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