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Discussion on "A fixed point theorem of Banach–Caccioppoli type on a class of generalized metric spaces" by A. Branciari

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Abstract. The aim of this discussion is to expose incorrect property of the generalized metric space introduced by A. Branciari in previous Publ. Math. Debrecen article.

1. Preliminaries

Recently Branciari [1] introduced the concept of a generalized metric space where the triangular inequality of a metric space has been replaced by a more general inequality involving four points instead of three. More precisely, Let X be a non-empty set and $d : X \times X \to [0, +\infty)$ be a mapping such that for all $x, y \in X$ and for all distinct point $\xi, \eta \in X$, each of them different from x and y, one has

(i) $d(x, y) = 0 \Leftrightarrow x = y$

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, y) \leq d(x, \xi) + d(\xi, \eta) + d(\eta, y)$.

Then, (X, d) is called a generalized metric space, (shortly a g.m.s.).

In [1], Branciari said that the fnuction d is continuous in each coordinates. More precisely, if x_n, a, b are distinct points in X ($n \in \mathbb{N}$) and if $\lim_{n\to+\infty} x_n = a$ then we have

$$|d(x_n, b) - d(a, b)| \to 0 \quad \text{as} \quad n \to +\infty. \tag{1}$$

The aim of this paper is to show that (1) is not true in general.A counter-example is given in the next section.

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2. Counter-example

Let $(x_n)_{n\in\mathbb{N}^*}$ be a sequence in $\mathbb{Q}$ and $a, b \in \mathbb{R}\backslash\mathbb{Q}$, $a \neq b$. We put the set $X = \{x_1, x_2, \cdots, x_n, \cdots\} \cup \{a, b\}$ and we consider $d : X \times X \to \mathbb{R}$ defined by

$$\begin{cases} d(x,x) = 0, & \forall\, x \in X, \\ d(x,y) = d(y,x), & \forall\, x, y \in X, \\ d(x_n, x_m) = 1, & \forall\, n, m \in \mathbb{N}^*,\ n \neq m, \\ d(x_n, b) = \dfrac{1}{n}, & \forall\, n \in \mathbb{N}^*, \\ d(x_n, a) = \dfrac{1}{n}, & \forall\, n \in \mathbb{N}^*, \\ d(a,b) = 1. \end{cases}$$

It is not difficult to show that (X, d) is a g.m.s. Here, we have $\lim_{n\to+\infty} x_n = a$ because $d(x_n, a) = \frac{1}{n} \to 0$ as $n \to +\infty$. But

$$|d(x_n, b) - d(a, b)| = 1 - \frac{1}{n} \to 1 \quad \text{as} \quad n \to +\infty.$$

Then, we show that (1) is false in this case. This error can be explained as follows. To obtain (1), the author used that if (x_n) is a convergent sequence in X then $d(x_n, x_m) \to 0$ as $n, m \to +\infty$. In the other manner, the author used that every convergent sequence is a Cauchy sequence. This result is true in the case of a metric space, but in the case of a g.m.s it is false in general as it is shown in the given counter-example.

References

[1] A. Branciari, A fixed point theorem of Banach–Caccioppoli type on a class of generalized metric spaces, *Publ. Math. Debrecen* **57** (2000), 31–37.

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