Publ. Math. Debrecen
76/4 (2010), 495–508

Divisor functions arising from q-series

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Abstract. We consider some of the properties of divisor functions arising from q-series and theta functions. Using these we obtain several new identities involving divisor functions. On the other hand we prove a conjecture of Z. H. Sun concerning representations by the ternary quadratic form $x^2 + y^2 + 3z^2$, and also get an analogous result for $x^2 + y^2 + 2z^2$.

1. Introduction

Throughout this paper we use the standard notation

$$(a;q)_\infty := \prod_{n\geq 0}(1-aq^n).$$

If there is no confusion, we write $(a)_\infty$ for $(a;q)_\infty$. In general, q denotes a fixed complex number of absolute value less than 1, so we may write $q = e^{\pi it}$ with Im $t > 0$.

For $N, m, r, s, k, l \in \mathbb{Z}$ with $N, m, k, l > 0$ satisfying $k \equiv l \bmod 2$, $k \geq 3$ and $l \leq k-2$, we define some necessary divisor functions for later use, which appear in many areas of number theory:

$$E_r(N;m) = \sum_{\substack{d|N \\ d\equiv r \bmod m}} 1 - \sum_{\substack{d|N \\ d\equiv -r \bmod m}} 1,$$

Mathematics Subject Classification: 11P83, 05A17.
Key words and phrases: basic hypergeometric series.
The second author is partially supported by NAP of Korea Research Council of Fundamental and the third partially supported by the SCR Program of NRF of Korea funded by MEST 2009-0063182.

$$\begin{aligned}
E_{r,\dots,s}(N;m) &= E_r(N;m) + \cdots + E_s(N;m), \\
\sigma_r(N;m) &= \sum_{\substack{d|N \\ d\equiv r \bmod m}} d, \\
\sigma_{r,\dots,s}(N;m) &= \sigma_r(N;m) + \cdots + \sigma_s(N;m), \\
\tau_{k,l}(N) &= \sum_{\substack{d|N \\ d\equiv (k-l)/2 \bmod k}} N/d + \sum_{\substack{d|N \\ d\equiv (k+l)/2 \bmod k}} N/d.
\end{aligned}$$

Let $r_Q(N)$ be the number of representations of a positive integer N by a positive definite quadratic form Q. Finding explicit formulas for $r_Q(N)$ is a classical problem in number theory (see for example [7]). In some special cases we can use divisor functions to evaluate $r_Q(N)$ rather than utilizing the standard approach via local densities. For example, Gauss (1801) showed that for $Q = x^2 + y^2$, $r_Q(N) = 4E_1(N;4)$. Moreover, by the theory of theta functions we can further show that this is equivalent to

$$\frac{(q^2;q^2)_\infty^{10}}{(q)_\infty^4 (q^4;q^4)_\infty^4} = 1 + 4\sum_{N=1}^{\infty} E_1(N;4) q^N.$$

On the other hand, Berndt found that many of the modular equations and theta function identities of Ramanujan yield elegant partition identities ([1], [2], [9], [10]).

In §2 we consider various identities, that is, those whose coefficients in q-series are given by divisor functions. As a special example of Theorem 4, we can derive the identity

$$\begin{aligned}
&\frac{(q^2;q^2)_\infty^2(-q;q^2)_\infty(-q;q^2)_\infty}{(-q^2;q^2)_\infty^2(q;q^2)_\infty(q;q^2)_\infty} + 2\frac{(q^4;q^4)_\infty^2(-q;q^4)_\infty(-q^3;q^4)_\infty}{(-q^4;q^4)_\infty^2(q;q^4)_\infty(q^3;q^4)_\infty} \\
&\qquad = 3 + 8\sum_{N=1}^{\infty} E_1(N;8) q^N.
\end{aligned}$$

Farkas showed in [4] and [5] that for $N, k, l \geq 1$ with $k \equiv l \bmod 2$, $k \geq 3$ and $l \leq k-2$,

$$\begin{aligned}
k \cdot \tau_{k,l}(N) &= 2 \cdot \sigma_{0,\frac{k-l}{2},\frac{k+l}{2}}(N;k) + l \cdot E_{\frac{k-l}{2}}(N;k) \\
&\quad + k \cdot \sum_{j=1}^{N-1} E_{\frac{k-l}{2}}(j;k) E_{\frac{k-l}{2}}(N-j;k). \qquad (1)
\end{aligned}$$

By applying Farkas' result to Theorem 4 we will obtain Theorem 5.

In §3 by means of divisor functions we prove Sun's conjecture ([11]) on the number of representations of the square of a prime by the ternary quadratic form $x^2 + y^2 + 3z^2$. An analogous result for $x^2 + y^2 + 2z^2$ is obtained by the same argument. We also obtain some new properties of divisor functions.

2. Special examples for divisor functions

From Fine's list of identities of the basic hypergeometric series ([6]), we list following ones for later use.

$$\frac{(q)_\infty^9}{(q^3;q^3)_\infty^3} = 1 - 9\sum_{N=1}^{\infty}\left(\sigma_1^{\{2\}}(N;3) - \sigma_2^{\{2\}}(N;3)\right)q^N. \qquad ([6], \text{p. } 85) \qquad (2)$$

$$\begin{aligned}\frac{(q)_\infty^3}{(q^3;q^3)_\infty} &= 1 - 3\sum_{N=0}^{\infty}\{E_1(N;3) - 3E_1(N/3;3)\}q^N \\ &= 1 - 3\sum_{n=0}^{\infty}E_1(3n+1;3)q^{3n+1} + 6\sum_{n=1}^{\infty}E_1(3n;3)q^{3n} \\ &= 1 - 3\sum_{n=0}^{\infty}E_1(3n+1;3)q^{3n+1} + 6\sum_{n=1}^{\infty}E_1(n;3)q^{3n}. \qquad ([6], \text{p. } 79) \qquad (3)\end{aligned}$$

$$\frac{(q^3;q^3)_\infty^3}{(q)_\infty} = \sum_{n=0}^{\infty}E_1(3n+1;3)q^n. \qquad ([6], \text{p. } 79) \qquad (4)$$

$$\frac{(q^2;q^2)_\infty^{10}}{(q)_\infty^4(q^4;q^4)_\infty^4} = 1 + 4\sum_{N=1}^{\infty}E_1(N;4)q^N. \qquad ([6], \text{p. } 78) \qquad (5)$$

$$\frac{(q^2;q^2)_\infty^{20}}{(q)_\infty^8(q^4;q^4)_\infty^8} = 1 + 8\sum_{N=1}^{\infty}q^N(2+(-1)^N)\sum_{\substack{\omega|N \\ omega\,\text{odd}}}\omega. \qquad ([6], \text{p. } 78) \qquad (6)$$

$$\frac{q(q^4;q^4)_\infty^8}{(q^2;q^2)_\infty^4} = \sum_{\substack{N=1 \\ N\text{odd}}}^{\infty}\sigma(N)q^N. \qquad ([6], \text{p. } 79) \qquad (7)$$

$$\frac{(q)_\infty^3(q^2;q^2)_\infty^3}{(q^3;q^3)_\infty(q^6;q^6)_\infty} = 1 - 3\sum_{N=1}^{\infty}\{\sigma_{1,-1}(N;6) - 2\sigma_3(N;6)\}q^N. \qquad ([6], \text{p. } 84) \qquad (8)$$

$$\frac{q(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q)_\infty(q^2;q^2)_\infty} = \sum_{N=1}^{\infty}\left\{\sigma_{1,-1}(N;6) + \frac{2}{3}\sigma_3(N;6)\right\}q^N. \qquad ([6], \text{p. } 86) \qquad (9)$$

$$\frac{(q)_\infty^4(q^3;q^3)_\infty^4}{(q^2;q^2)_\infty^2(q^6;q^6)_\infty^2}=1-4\sum_{N=1}^{\infty}\{\sigma_{1,-1}(N;6)-\sigma_{2,-2}(N;6)\}q^N. \quad \text{([6], p. 85)} \tag{10}$$

$$\frac{q(q^2;q^2)_\infty^4(q^6;q^6)_\infty^4}{(q)_\infty^2(q^3;q^3)_\infty^2}=\sum_{N=1}^{\infty}\left\{\sigma_{1,-1}(N;6)+\frac{1}{2}\sigma_{2,-2}(N;6)\right\}q^N. \quad \text{([6], p. 87)} \tag{11}$$

$$\frac{(q^2;q^2)_\infty^3(q^3;q^3)_\infty^2(q^6;q^6)_\infty}{(q)_\infty^2(q^4;q^4)_\infty(q^{12};q^{12})_\infty}=1+2\sum_{N=1}^{\infty}E_{1,5}(N;12)q^N. \quad \text{([6], p. 80)} \tag{12}$$

$$\frac{(q)_\infty(q^3;q^3)_\infty(q^4;q^4)_\infty^2(q^6;q^6)_\infty^2}{(q^{12};q^{12})_\infty^2}$$
$$=1-\sum_{N=1}^{\infty}\{\sigma_{1,-1}(N;12)-\sigma_{5,-5}(N;12)\}q^N. \quad \text{([6], p. 85)} \tag{13}$$

$$\frac{(q^p;q^p)_\infty^2(-q^r;q^p)_\infty(-q^{p-r};q^p)_\infty}{(-q^p;q^p)_\infty^2(q^r;q^p)_\infty(q^{p-r};q^p)_\infty}$$
$$=1+2\sum_{N=1}^{\infty}E_{r,p-r}(N;2p)q^N,\quad 0<r<p,\ (r,p)=1. \quad \text{([6], p. 72)} \tag{14}$$

Now, we deduce from (8) and (9) that

$$\frac{(q)_\infty^3(q^2;q^2)_\infty^3}{(q^3;q^3)_\infty(q^6;q^6)_\infty}+3\frac{q(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q)_\infty(q^2;q^2)_\infty}=1+8\sum_{N=1}^{\infty}\sigma_3(N;6)q^N, \tag{15}$$

and

$$\frac{(q)_\infty^3(q^2;q^2)_\infty^3}{(q^3;q^3)_\infty(q^6;q^6)_\infty}-9\frac{q(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q)_\infty(q^2;q^2)_\infty}=1-12\sum_{N=1}^{\infty}\sigma_{1,-1}(N;6)q^N. \tag{16}$$

In particular, if $N=2^A3^Bp_1^{e_1}\dots p_r^{e_r}q_1^{f_1}\dots q_s^{f_s}$, where $p_i\equiv 1 \bmod 6$ and $q_j\equiv -1 \bmod 6$, then

$$\frac{(q)_\infty^3(q^2;q^2)_\infty^3}{(q^3;q^3)_\infty(q^6;q^6)_\infty}+3\frac{q(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q)_\infty(q^2;q^2)_\infty}=1+8\sum_{N=1}^{\infty}\sigma_3(N;6)q^N$$
$$=1+8\sum_{N=1}^{\infty}q^N\frac{3^B-1}{3-1}\sigma(p_1^{e_1}\dots p_r^{e_r}q_1^{f_1}\dots q_s^{f_s}), \tag{17}$$

and

$$\frac{(q)_\infty^3(q^2;q^2)_\infty^3}{(q^3;q^3)_\infty(q^6;q^6)_\infty}-9\frac{q(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q)_\infty(q^2;q^2)_\infty}=1-12\sum_{N=1}^{\infty}\sigma_{1,-1}(N;6)q^N$$
$$=1-12\sum_{N=1}^{\infty}q^N\sigma(p_1^{e_1}\dots p_r^{e_r}q_1^{f_1}\dots q_s^{f_s}). \tag{18}$$

Similarly, by (10),(11),(13) and (16) we establish that

$$\frac{(q)^3_\infty(q^2;q^2)^3_\infty}{(q^3;q^3)_\infty(q^6;q^6)_\infty} - 9\frac{q(q^3;q^3)^3_\infty(q^6;q^6)^3_\infty}{(q)_\infty(q^2;q^2)_\infty} + 12\frac{(q)_\infty(q^3;q^3)_\infty(q^4;q^4)^2_\infty(q^6;q^6)^2_\infty}{(q^{12};q^{12})^2_\infty}$$
$$= 13 - 24\sum_{N=1}^{\infty}\sigma_{1,-1}(N;12)q^N,$$

$$\frac{(q)^3_\infty(q^2;q^2)^3_\infty}{(q^3;q^3)_\infty(q^6;q^6)_\infty} - 9\frac{q(q^3;q^3)^3_\infty(q^6;q^6)^3_\infty}{(q)_\infty(q^2;q^2)_\infty} - 12\frac{(q)_\infty(q^3;q^3)_\infty(q^4;q^4)^2_\infty(q^6;q^6)^2_\infty}{(q^{12};q^{12})^2_\infty}$$
$$= -11 - 24\sum_{N=1}^{\infty}\sigma_{5,-5}(N;12)q^N,$$

$$\frac{(q)^4_\infty(q^3;q^3)^4_\infty}{(q^2;q^2)^2_\infty(q^6;q^6)^2_\infty} - 8\frac{q(q^2;q^2)^2_\infty(q^6;q^6)^2_\infty}{(q)^2_\infty(q^3;q^3)^2_\infty} = 1 - 12\sum_{N=1}^{\infty}\sigma_{1,-1}(N;6)q^N,$$

and

$$\frac{(q)^3_\infty(q^2;q^2)^3_\infty}{(q^3;q^3)_\infty(q^6;q^6)_\infty} - 9\frac{q(q^3;q^3)^3_\infty(q^6;q^6)^3_\infty}{(q)_\infty(q^2;q^2)_\infty}$$
$$= \frac{(q)^4_\infty(q^3;q^3)^4_\infty}{(q^2;q^2)^2_\infty(q^6;q^6)^2_\infty} - 8\frac{q(q^2;q^2)^2_\infty(q^6;q^6)^2_\infty}{(q)^2_\infty(q^3;q^3)^2_\infty}.$$

It follows from (3) and (8) that

$$\begin{aligned}
&1 - 3\sum_{N\ge1}\{\sigma_{1,-1}(N;6) - 2\sigma_3(N;6)\}q^N \\
&= \frac{(q)^3_\infty(q^2;q^2)^3_\infty}{(q^3;q^3)_\infty(q^6;q^6)_\infty} = \frac{(q)^3_\infty}{(q^3;q^3)_\infty}\cdot\frac{(q^2;q^2)^3_\infty}{(q^6;q^6)_\infty} \\
&= \Big(1 - 3\sum_{n\ge0}E_1(3n+1;3)q^{3n+1} + 6\sum_{n\ge1}E_1(3n;3)q^{3n}\Big) \\
&\quad\times\Big(1 - 3\sum_{n\ge0}E_1(3n+1;3)q^{6n+2} + 6\sum_{n\ge1}E_1(3n;3)q^{6n}\Big) \\
&= 1 - 3\sum_{n\ge0}E_1(6n+1;3)q^{6n+1} - 3\sum_{n\ge0}E_1(6n+4;3)q^{6n+4} + 6\sum_{n\ge0}E_1(6n;3)q^{6n} \\
&\quad + 6\sum_{n\ge0}E_1(6n+3;3)q^{6n+3} + 9\sum_{n\ge0,\,N\ge0}E_1(6n+1;3)E_1(3N+1;3)q^{6n+6N+3} \\
&\quad + 9\sum_{n\ge0,\,N\ge0}E_1(6n+4;3)E_1(3N+1;3)q^{6n+6N+6}
\end{aligned}$$

$$- 18 \sum_{n\geq 1,\, N\geq 0} E_1(6n;3)E_1(3N+1;3)q^{6n+6N+2}$$
$$- 18 \sum_{n\geq 0,\, N\geq 0} E_1(6n+3;3)E_1(3N+1;3)q^{6n+6N+5}$$
$$+ 6 \sum_{n\geq 1} E_1(3n;3)q^{6n} - 3 \sum_{n\geq 0} E_1(3n+1;3)q^{6n+2}$$
$$- 18 \sum_{n\geq 0,\, N\geq 1} E_1(6n+1;3)E_1(3N;3)q^{6n+6N+1}$$
$$- 18 \sum_{n\geq 0,\, N\geq 1} E_1(6n+4;3)E_1(3N;3)q^{6n+6N+4}$$
$$+ 36 \sum_{n\geq 1,\, N\geq 1} E_1(6n;3)E_1(3N;3)q^{6n+6N} + 36 \sum_{n\geq 0,\, N\geq 1} E_1(6n;3)E_1(3N;3)q^{6n+6N+3}.$$

From these results we obtain the following new identities involving divisor functions.

Lemma 1. *For M a positive integer, we have*

(a) $\sigma_{1,5}(6M+1;6) = \sigma(6M+1) = E_1(6M+1;3) + 6 \sum_{n=0}^{M-1} E_1(6n+1;3)E_1(M-n;3).$

(b) $\sigma_{1,5}(6M+2;6) = E_1(3M+1;3) + 6 \sum_{n=0}^{M-1} E_1(3n+1;3)E_1(2(M-n);3).$

(c) $\sigma_{1,5}(6M+4;6) = E_1(6M+4;3) + 6 \sum_{n=0}^{M-1} E_1(6n+4;3)E_1(M-n;3).$

(d) $\sigma_{1,5}(6M+5;6) = \sigma(6M+5) = 6 \sum_{n=0}^{M} E_1(2n+1;3)E_1(M-n;3).$

(e) $2\sigma_3(6M+3;6) - \sigma_{1,5}(6M+3;6) = 2E_1(2M+1;3) + 3 \sum_{n=0}^{M} E_1(6n+1;3)E_1(3(M-n)+1;3) + 12 \sum_{n=0}^{M} E_1(2n+1;3)E_1(M-n;3).$

(f) $2\sigma_3(6M;6) - \sigma_{1,5}(6M;6) = 2E_1(2M;3) + 2E_1(M;3) + 3 \sum_{n=0}^{M-1} E_1(6n+4;3)E_1(3(M-n-1)+1;3) + 12 \sum_{n=1}^{M-1} E_1(2n;3)E_1(M-n;3).$

Next, we derive from (4) and (9) that

$$\frac{q(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q)_\infty(q^2;q^2)_\infty} = \sum_{N=1}^{\infty} q^N\left(\sigma_{1,5}(N;6) + \frac{2}{3}\sigma_3(N;6)\right)$$
$$= q\Big(\sum_{n=0}^{\infty} E_1(3n+1;3)q^n\Big)\Big(\sum_{m=0}^{\infty} E_1(3m+1;3)q^{2m}\Big)$$

$$= \sum_{N=0}^{\infty} q^{2N+1} \sum_{n=0}^{N} E_1(6n+1;3)E_1(3(N-n)+1;3)$$
$$+ \sum_{N=1}^{\infty} q^{2N} \sum_{n=0}^{N-1} E_1(6n+4;3)E_1(3(N-n-1)+1;3),$$

and this implies the following lemma.

Lemma 2. *Let M be a positive integer.*

(a) $\sigma_{1,5}(6M+1;6) = \sigma(6M+1) = \sum_{n=0}^{3M} E_1(6n+1;3)E_1(9M-3n+1;3)$.

(b) $\sigma_{1,5}(6M+2;6) = \sum_{n=0}^{3M} E_1(6n+4;3)E_1(9M-3n+1;3)$.

(c) $\sigma_{1,5}(6M+4;6) = \sum_{n=0}^{3M+1} E_1(6n+4;3)E_1(9M-3n+4;3)$.

(d) $\sigma_{1,5}(6M+5;6) = \sigma(6M+5) = \sum_{n=0}^{3M+2} E_1(6n+1;3)E_1(9M-3n+7;3)$.

(e) $\frac{2}{3}\sigma_3(6M+3;6) + \sigma_{1,5}(6M+3;6) = \sum_{n=0}^{3M+1} E_1(6n+1;3)E_1(9M-3n+4;3)$.

(f) $\frac{2}{3}\sigma_3(6M;6) + \sigma_{1,5}(6M;6) = \sum_{n=0}^{3M-1} E_1(6n+4;3)E_1(9M-3n+1;3)$.

(g) $\sigma_3(6M+3;6) = \frac{3}{8}\Big\{2E_1(2M+1;3) + 3\sum_{n=0}^{M} E_1(6n+1;3)E_1(3(M-n)+1;3)$
$+12\sum_{n=0}^{M} E_1(2n+1;3)E_1(M-n;3) + \sum_{n=0}^{3M+1} E_1(6n+1;3)E_1(9M-3n+4;3)\Big\}$.

(h) $\sigma_{1,5}(6M+3;6) = \frac{1}{4}\Big\{-2E_1(2M+1;3) - 3\sum_{n=0}^{M} E_1(6n+1;3)E_1(3(M-n)+1;3)$
$-12\sum_{n=0}^{M} E_1(2n+1;3)E_1(M-n;3) + 3\sum_{n=0}^{3M+1} E_1(6n+1;3)E_1(9M-3n+4;3)\Big\}$.

(i) $\sigma_3(6M;6) = \frac{3}{8}\Big\{2E_1(2M;3) + 2E_1(M;3) + 3\sum_{n=0}^{M-1} E_1(6n+4;3)$
$E_1(3(M-n-1)+1;3) + 12\sum_{n=1}^{M-1} E_1(2n;3)E_1(M-n;3)$
$+ \sum_{n=0}^{3M-1} E_1(6n+4;3)E_1(9M-3n+1;3)\Big\}$.

(j) *For $M > 1$, we have* $\sigma_{1,5}(6M;6) = \frac{1}{4}\Big\{ -2E_1(2M;3) - 2E_1(M;3)$

$$-3\sum_{n=0}^{M-1} E_1(6n+4;3)E_1(3(M-n-1)+1;3) - 12\sum_{n=1}^{M-1} E_1(2n;3)E_1(M-n;3)$$
$$+3\sum_{n=0}^{3M-1} E_1(6n+4;3)E_1(9M-3n+1;3)\Big\}.$$

Here, we observe that

$$\sum_{\substack{d|N \\ d\equiv 1 \bmod 6}} 1 - \sum_{\substack{d|N \\ d\equiv 5 \bmod 6}} 1 = \sum_{\substack{d|N \\ d\equiv 1 \bmod 12}} 1 + \sum_{\substack{d|N \\ d\equiv 7 \bmod 12}} 1 - \sum_{\substack{d|N \\ d\equiv 5 \bmod 12}} 1 - \sum_{\substack{d|N \\ d\equiv 11 \bmod 12}} 1.$$

Then we readily see that

$$\frac{(q^2;q^2)_\infty^6 (q^3;q^3)_\infty}{(q)_\infty^3 (q^6;q^6)_\infty^2} = 1 + 3\sum_{N=1}^{\infty} E_1(N;6)q^N = 1 + 3\sum_{N=1}^{\infty} E_{1,7}(N;12)q^N,$$

which gives the following lemma by (12).

Lemma 3.

(a) $$\frac{3}{5}\frac{(q^2;q^2)_\infty^3 (q^3;q^3)_\infty^2 (q^6;q^6)_\infty}{(q)_\infty^2 (q^4;q^4)_\infty (q^{12};q^{12})_\infty} + \frac{2}{5}\frac{(q^2;q^2)_\infty^6 (q^3;q^3)_\infty}{(q)_\infty^3 (q^6;q^6)_\infty^2}$$
$$= 1 + \frac{12}{5}\sum_{N=1}^{\infty} E_1(N;12)q^N.$$

(b) $$3\frac{(q^2;q^2)_\infty^3 (q^3;q^3)_\infty^2 (q^6;q^6)_\infty}{(q)_\infty^2 (q^4;q^4)_\infty (q^{12};q^{12})_\infty} - 2\frac{(q^2;q^2)_\infty^6 (q^3;q^3)_\infty}{(q)_\infty^3 (q^6;q^6)_\infty^2}$$
$$= 1 + 12\sum_{N=1}^{\infty} E_5(N;12)q^N.$$

More generally we can derive the following identities.

Theorem 4. *For $n \geq 1$ we have*

(a) $$\sum_{k=1}^{n} 2^{k-1}\frac{(q^{2^k};q^{2^k})_\infty^2 (-q;q^{2^k})_\infty (-q^{2^k-1};q^{2^k})_\infty}{(-q^{2^k};q^{2^k})_\infty^2 (q;q^{2^k})_\infty (q^{2^k-1};q^{2^k})_\infty}$$
$$= (2^n - 1) + 2^{n+1}\sum_{N=1}^{\infty} E_1(N;2^{n+1})q^N.$$

(b) $$2\frac{(q^2;q^2)_\infty^6 (q^3;q^3)_\infty}{(q;q)_\infty^3 (q^6;q^6)_\infty^2}$$
$$+ 3\sum_{k=1}^{n} 2^{k-1}\frac{(q^{3\cdot 2^k};q^{3\cdot 2^k})_\infty^2 (-q;q^{3\cdot 2^k})_\infty (-q^{3\cdot 2^k-1};q^{3\cdot 2^k})_\infty}{(-q^{3\cdot 2^k};q^{3\cdot 2^k})_\infty^2 (q;q^{3\cdot 2^k})_\infty (q^{3\cdot 2^k-1};q^{3\cdot 2^k})_\infty}$$
$$= (3\cdot 2^n - 1) + 3\cdot 2^{n+1}\sum_{N=1}^{\infty} E_1(N;3\cdot 2^{n+1})q^N.$$

Proof. Note that $E_r(N;p) = E_{r,p+r}(N;2p)$ and $E_{p+r}(N;2p) = -E_{p-r}(N;2p)$ imply $2E_r(N;2p) = E_{r,p-r}(N;2p) + E_r(N;p)$. Hence, we inductively get by the identity (14) that

$$(2^n-1) + 2^{n+1}\sum_{N=1}^{\infty} E_r(N;2^n p)q^N$$
$$= 2\sum_{N=1}^{\infty} E_r(N;p)q^N + \sum_{k=0}^{n-1} 2^k \frac{(q^{2^k p};q^{2^k p})_\infty^2(-q^r;q^{2^k p})_\infty(-q^{2^k p-r};q^{2^k p})_\infty}{(-q^{2^k p};q^{2^k p})_\infty^2(q^r;q^{2^k p})_\infty(q^{2^k p-r};q^{2^k p})_\infty}.$$

The first assertion follows from the above identity by observing $E_r(N;p) = 0$ for $p = 2$, $r = 1$.

As for the second, we know by [6, p. 80] that

$$\frac{(q^2;q^2)_\infty^6(q^3;q^3)_\infty}{(q;q)_\infty^3(q^6;q^6)_\infty^2} = 1 + 3\sum_{N=1}^{\infty} E_1(N;6)q^N.$$

Therefore, we have the assertion by the above identity with $p = 6, r = 1$. □

By applying the Farkas' identity (1) to Theorem 4 we also have the following theorem.

Theorem 5. *For $n \geq 1$ we have*

$$\sum_{1\leq k,l\leq n} 2^{k+l-2}$$
$$\frac{(q^{2^k};q^{2^k})_\infty^2(-q;q^{2^k})_\infty(-q^{2^k-1};q^{2^k})_\infty(q^{2^l};q^{2^l})_\infty^2(-q;q^{2^l})_\infty(-q^{2^l-1};q^{2^l})_\infty}{(-q^{2^k};q^{2^k})_\infty^2(q;q^{2^k})_\infty(q^{2^k-1};q^{2^k})_\infty(-q^{2^l};q^{2^l})_\infty^2(q;q^{2^l})_\infty(q^{2^l-1};q^{2^l})_\infty}$$
$$= (2^n-1)^2 + 2^{n+2}(2^n-1)q + 2^{n+2}\sum_{N=2}^{\infty}(2^n\tau_{2^{n+1},2^{n+1}-2}(N) - \sigma_{0,1,-1}(N;2^{n+1}))q^N.$$

Proof. The square of the right hand side of Theorem 4 (a) is equal to

$$(2^n-1)^2 + 2^{n+2}(2^n-1)\sum_{N=1}^{\infty} E_1(N;2^{n+1})q^N + 2^{2n+2}$$
$$\times \sum_{N=2}^{\infty}\sum_{j=1}^{N-1} E_1(j;2^{n+1})E_1(N-j;2^{n+1})q^N,$$

and the term $\sum_{j=1}^{N-1} E_1(j;2^{n+1})E_1(N-j;2^{n+1})$ can be replaced by using the identity (1). □

3. Divisor functions and Sun's conjecture

We first recall some well-known facts (for instance, see [6], pp. 72-77) about a quantitative problem of quadratic forms over $\mathbb{Z}$ in terms of divisor functions.

Lemma 6. *Let N be a positive integer.*

(1) *If $Q_1(x,y) = x^2 + 3y^2$, then* $r_{Q_1}(N) = \begin{cases} 2E_1(N;3) & \text{if } N \text{ is odd} \\ 6E_1(N;3) & \text{if } N \text{ is even.} \end{cases}$

(2) *If $Q_2(x,y,z,w) = x^2 + y^2 + 3z^2 + 3w^2$, then $r_{Q_2}(N) = (-1)^{N-1} \cdot 4 \cdot \sum_{d|N} d \cdot \chi(d)$, where $\chi(d) = 1, -1$, and 0 according as $d \equiv \pm 1 \bmod 6$, $d \equiv \pm 2 \bmod 6$, and $d \equiv 0 \bmod 3$, respectively.*

(3) *If $Q_3(x,y) = x^2 + y^2$, then $r_{Q_3}(N) = 4E_1(N;4)$.*

(4) *If $Q_1'(x,y) = x^2 + 2y^2$, then $r_{Q_1'}(N) = 2E_{1,3}(N;8)$.*

(5) *If $Q_2'(x,y,z,w) = x^2 + y^2 + 2z^2 + 2w^2$, then $r_{Q_2'}(N) = 4k(\alpha)\sum_{d|m} d$, where $N = 2^{\alpha} m$ with $(2,m) = 1$ and $k(0) = 1$, $k(1) = 2$, $k(\alpha) = 6$ for $\alpha \geq 2$.*

Here we note that $E_1(N;3) = \sum_{d|N} (\frac{d}{3})$ is a multiplicative function of N, that is, for coprime positive integers N and M, $E_1(NM;3) = E_1(N;3)E_1(M;3)$. Since $E_1(2^{2n};3) = 1$ and $E_1(2^{2n+1};3) = 0$ for any nonnegative integer n, we also get $E_1(4N;3) = E_1(N;3)$ by observing that E_1 is multiplicative. Now, we are ready to prove Sun's conjecture ([11], Conjecture 17) on the number of representations of the square of a prime by the ternary quadratic form $x^2 + y^2 + 3z^2$, and also to prove an analogous result for $x^2 + y^2 + 2z^2$. The method of our proof goes back to Hurwitz, and the work of Hurwitz can be found on [8, pp. 5–7] or [3, p. 311 of Volume 2].

Proposition 7. *Let $Q(x,y,z) = x^2+y^2+3z^2$, and $Q'(x,y,z) = x^2+y^2+2z^2$. Then we have*

(1) *$r_Q(p^2) = 4(p + 1 - (\frac{p}{3}))$ for any prime $p \geq 5$.*

(2) *$r_{Q'}(p^2) = 4(p + 1 - (\frac{-2}{p}))$ for any prime $p \geq 3$.*

Proof. (1) Since p is odd, $p^2 \equiv 1 \bmod 8$. Thus $p^2 = x^2 + y^2 + 3z^2$ implies that $(x,y,z) =$ (odd, even, even) or (even, odd, even). With the notation as in Lemma 6 we derive that

$$r_Q(p^2) = 4 \sum_{\substack{1 \leq i \leq p \\ i = \text{odd}}} r_{Q_1}(p^2 - i^2) = 4\left(\sum_{\substack{1 \leq i < p \\ i = \text{ odd}}} r_{Q_1}(p^2 - i^2) + 1 \right)$$

$$= 4\left(6\sum_{\substack{1\le i<p\\ i=\text{odd}}} E_1(p^2-i^2;3)+1\right) \quad \text{by Lemma 6-(1)}$$

$$= 4\left(6\sum_{\substack{1\le i<p\\ i=\text{odd}}} E_1(\frac{p-i}{2}\cdot\frac{p+i}{2};3)+1\right), \quad \text{because } E_1(4N;3)=E_1(N;3)$$

$$= 4\left(6\sum_{\substack{1\le i<p\\ i=\text{odd}}} E_1\left(\frac{p-i}{2};3\right)E_1\left(\frac{p+i}{2};3\right)+1\right), \quad \text{since } \left(\frac{p-i}{2},\frac{p+i}{2}\right)=1$$

$$= 4\left(6\sum_{j=1}^{\frac{p-1}{2}} E_1(j;3)E_1(p-j;3)+1\right) = 4\left(3\sum_{j=1}^{p-1} E_1(j;3)E_1(p-j;3)+1\right)$$

$$= \sum_{j=1}^{p-1} 12E_1(j;3)E_1(p-j;3)+4 = \sum_{j=1}^{p-1} r_{Q_1}(j)r_{Q_1}(p-j)+4$$

$$= \sum_{j=0}^{p} r_{Q_1}(j)r_{Q_1}(p-j)+4-2r_{Q_1}(p) = r_{Q_2}(p)+4-2r_{Q_1}(p)$$

$$= 4\left(p+1-\left(\frac{p}{3}\right)\right) \quad \text{by Lemma 6-(1) and (2).}$$

(2) In a similar way we see that $E_{1,3}(N;8)=\sum_{0<d|N}(\frac{-2}{d})$ is a multiplicative function of N and $E_{1,3}(2^\alpha;8)=1$. Then we claim that

$$r_{Q'}(p^2) = 4\left(\sum_{\substack{1\le i<p\\ i=\text{odd}}} r_{Q_1'}(p^2-i^2)+1\right)$$

$$= 4\left(2\sum_{\substack{1\le i<p\\ i=\text{ odd}}} E_{1,3}(p^2-i^2;8)+1\right) \quad \text{by Lemma 6-(4)}$$

$$= 4\left(2\sum_{\substack{1\le i<p\\ i=\text{odd}}} E_{1,3}\left(\frac{p-i}{2}\cdot\frac{p+i}{2};8\right)+1\right), \quad \text{because } E_{1,3}(2^\alpha N;8)=E_{1,3}(N;8)$$

$$= 4\left(\sum_{j=1}^{p-1} E_{1,3}(j;8)E_{1,3}(p-j;8)+1\right)$$

$$= \sum_{j=0}^{p} r_{Q_1'}(j)r_{Q_1'}(p-j)+4-2r_{Q_1'}(p)$$

$$= 4\left(p+1-\left(\frac{-2}{p}\right)\right) \quad \text{by Lemma 6-(4) and (5).} \qquad \square$$

As an application we have the following corollary.

Corollary 8. *Let* $[r]$ *be the greatest integer less than or equal to* $r \in \mathbb{R}$.

(1) *For any prime* $p \geq 5$,

$$\sum_{k=1}^{\frac{p-1}{2}} \big(3E_1(p^2-(2k-1)^2;3)+E_1(p^2-4k^2;3)\big) = \begin{cases} p-2, & p \equiv 1 \bmod 3 \\ p+1, & p \equiv 2 \bmod 3, \end{cases}$$

$$\sum_{k=1}^{[\frac{p}{\sqrt{3}}]} E_1(p^2-3k^2;4) = \begin{cases} \dfrac{p-3}{2}, & p \equiv 1 \bmod 12 \\ \dfrac{p-1}{2}, & p \equiv \pm 5 \bmod 12 \\ \dfrac{p+1}{2}, & p \equiv 11 \bmod 12. \end{cases}$$

(2) *For any prime* $p \geq 3$,

$$\sum_{k=1}^{p-1} E_{1,3}(p^2-k^2;8) = \begin{cases} p-2, & p \equiv 1,3 \bmod 8 \\ p+1, & p \equiv 5,7 \bmod 8, \end{cases}$$

$$\sum_{k=1}^{[\frac{p}{\sqrt{2}}]} E_1(p^2-2k^2;4) = \begin{cases} \dfrac{p-3}{2}, & p \equiv 1 \bmod 8 \\ \dfrac{p-1}{2}, & p \equiv \pm 3 \bmod 8 \\ \dfrac{p+1}{2}, & p \equiv -1 \bmod 8. \end{cases}$$

Proof. (1) It follows from by Lemma 6-(1) that

$$\begin{aligned} r_Q(p^2) &= \sum_{i=-p}^{p} r_{Q_1}(p^2-i^2) = r_{Q_1}(p^2)+2\sum_{i=1}^{p-1} r_{Q_1}(p^2-i^2)+2r_{Q_1}(0) \\ &= 2E_1(p^2;3)+2+2\Bigg\{6\sum_{\substack{i=1\\ \text{odd}}}^{p-2} E_1(p^2-i^2;3)+2\sum_{\substack{i=1\\ \text{even}}}^{p-1} E_1(p^2-i^2;3)\Bigg\}. \end{aligned}$$

On the other hand, we know by Proposition 7-(1) that $r_Q(p^2) = 4(p+1-(\frac{p}{3}))$. Hence we have that

$$\begin{aligned} &\sum_{k=1}^{\frac{p-1}{2}} \{3E_1(p^2-(2k-1)^2;3)+E_1(p^2-4k^2;3)\} \\ &= \left(p+1-\left(\frac{p}{3}\right)\right)-\frac{1}{2}E_1(p^2;3)-\frac{1}{2} = p-\frac{1}{2}\left(1+3\left(\frac{p}{3}\right)\right), \end{aligned}$$

from which we derive the first statement.

Next, we obtain by Lemma 6-(3) and Proposition 7-(1) that

$$r_Q(p^2) = \sum_{i=-[\frac{p}{\sqrt{3}}]+1}^{[\frac{p}{\sqrt{3}}]} r_{Q_3}(p^2-3i^2) = 4E_1(p^2;4) + 8\sum_{i=1}^{[\frac{p}{\sqrt{3}}]} E_1(p^2-3i^2;4)$$
$$= 4\left(p+1-\left(\frac{p}{3}\right)\right).$$

Since $E_1(N;4) = \sum_{0<d|N}(\frac{-1}{d})$,

$$\sum_{k=1}^{[\frac{p}{\sqrt{3}}]} E_1(p^2-3k^2;4) = \frac{1}{2}\left(p+1-\left(\frac{p}{3}\right) - E_1(p^2;4)\right)$$
$$= \frac{1}{2}\left(p-1-\left(\frac{p}{3}\right) - \left(\frac{-1}{p}\right)\right).$$

This enables us to conclude the second statement.

(2) We first observe by Lemma 6-(4) and Proposition 7-(2) that

$$r_{Q'}(p^2) = \sum_{i=-p}^{p} r_{Q_1'}(p^2-i^2) = r_{Q_1'}(p^2) + 2\sum_{i=1}^{p-1} r_{Q_1'}(p^2-i^2) + 2r_{Q_1'}(0)$$
$$= 2E_{1,3}(p^2;8) + 2 + 4\sum_{i=1}^{p-1} E_{1,3}(p^2-i^2;8) = 4\left(p+1-\left(\frac{-2}{p}\right)\right),$$

and by Lemma 6-(3) and Proposition 7-(2) that

$$r_{Q'}(p^2) = \sum_{i=-[\frac{p}{\sqrt{2}}]+1}^{[\frac{p}{\sqrt{2}}]} r_{Q_3}(p^2-2i^2) = 4E_1(p^2;4) + 8\sum_{i=1}^{[\frac{p}{\sqrt{2}}]} E_1(p^2-2i^2;4)$$
$$= 4\left(p+1-\left(\frac{-2}{p}\right)\right).$$

Therefore, we deduce the conclusions by similar arguments as in (1). □

Acknowledgement. The authors would like to express hearty thanks to the referee for many valuable comments on an earlier version of this paper.

References

[1] N. D. Baruah and B. C. Berndt, Partition identites and Ramanujan's modular equations, *J. Combin.* Theory Ser A. **114** (2007), 1024–1045.

[2] B. C. Berndt, Ramanujan's Notebooks III, *Springer-Verlag*, 1991.

[3] L. E. Dickson, History of the Theory of Numbers, Vol II: Diophantine analysis, *Chelsea Publishing Co.*, *New York*, 1966.

[4] H. M. Farkas, On an arithmetical function, *Ramanujan J.* **8** (2004), 309–315.

[5] H. M. Farkas, On an arithmetical function II, *Contemp. Math.* **382** (2005), 121–130.

[6] N. J. Fine, Basic Hypergeometric Series and Applications, *American Mathematical Society, Providence, RI*, 1988.

[7] E. Grosswald, Representations of Integers as Sums of Squares, *Springer-Verlag*, 1985.

[8] A. Hurwitz, Mathematische Werke, Vol. 2, *Basel*, 1933.

[9] S. Ramanujan, Proof of certain identities in combinatory analysis, *Proc. Cambridge Philos. Soc.* **19** (1919), 214–216.

[10] L. J. Rogers, Second memoir on the expansion of certain infinite products, *Proc. London Math. Soc.* **25** (1894), 318–343.

[11] Z. H. Sun, My conjectures in number theory, http://www.hytc.edu.cn/xsjl/szh/mycon.pdf.

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(Received April 27, 2009; revised October 8, 2009)