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L_1-factorization of Pietsch integral operators

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Abstract. Given a compact Hausdorff space K and a regular positive finite, Borel measure μ on K, we characterize the operators on $C(K)$ admitting a factorization through the natural inclusion of $C(K)$ into $L_1(K,\mu)$. We also characterize the operators on $L_\infty(\Omega,\nu)$, with ν a positive finite measure, that factor through the natural inclusion of $L_\infty(\Omega,\nu)$ into $L_1(\Omega,\nu)$.

Throughout, E and F will denote Banach spaces, B_{E^*} will stand for the closed unit ball of the dual space E^* endowed with the weak-star topology, and $\mathcal{L}(E,F)$ will be the space of all (linear bounded) operators from E into F endowed with the supremum norm. The symbol k_F will be used for the canonical isometric embedding of F into F^{**}.

An operator $T \in \mathcal{L}(E,F)$ is *Pietsch integral* [DU, Definition VI.3.8] if there exists a countably additive F-valued, Borel measure $\mathcal{G}$ of bounded variation on B_{E^*} such that

$$T(x) = \int_{B_{E^*}} x^*(x)\, d\mathcal{G}(x^*) \quad (x \in E).$$

The *Pietsch integral norm* of T is given by

$$\|T\|_{\mathrm{PI}} := \inf|\mathcal{G}|(B_{E^*}),$$

where $|\mathcal{G}|$ denotes the variation of $\mathcal{G}$ and the infimum is taken over all vector measures $\mathcal{G}$ satisfying the definition.

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In the theory of Pietsch integral operators and its development, a fundamental rôle is played by the following well-known factorization theorem.

Theorem 1 ([DU, Theorem VI.3.11]). *An operator $T \in \mathcal{L}(E, F)$ is Pietsch integral if and only if there are a compact Hausdorff space K, a regular Borel measure μ on K, and operators $S \in \mathcal{L}(L_1(K,\mu), F)$ and $R \in \mathcal{L}(E, C(K))$ giving rise to the commutative diagram*

$$\begin{array}{ccc} E & \xrightarrow{T} & F \\ {\scriptstyle R}\big\downarrow & & \big\uparrow{\scriptstyle S} \\ C(K) & \xrightarrow[J]{} & L_1(K,\mu) \end{array}$$

where J denotes the natural inclusion of $C(K)$ into $L_1(K,\mu)$.

It is well-known that K may be chosen to be any weak-star compact norming subset of B_{E^*}, denoting by R the natural isometric embedding of E into $C(K)$ [DJT, Theorem 5.6].

Villanueva [V, pages 58–59] noticed that the result remains true if K is any compact space (not necessarily contained in B_{E^*}) such that there is an isomorphic embedding of E into $C(K)$. So we can state:

Theorem 2. *Let $T \in \mathcal{L}(E, F)$ be an operator, and let K be a compact Hausdorff space such that there is an isomorphic embedding $R \in \mathcal{L}(E, C(K))$. Then T is Pietsch integral if and only if there exist a regular Borel measure μ on K and an operator $S \in \mathcal{L}(L_1(K,\mu), F)$ such that the following diagram is commutative*

$$\begin{array}{ccc} E & \xrightarrow{T} & F \\ {\scriptstyle R}\big\downarrow & & \big\uparrow{\scriptstyle S} \\ C(K) & \xrightarrow[J]{} & L_1(K,\mu) \end{array}$$

where J denotes the natural inclusion of $C(K)$ into $L_1(K,\mu)$.

Therefore, we can select the compact space K and the operator $R \in \mathcal{L}(E, C(K))$, provided that it be an isomorphism. Then it seems natural to ask if we can also select the regular Borel measure μ on K.

It is easy to see that the answer is negative. Moreover, we give characterizations of the operators on $C(K)$ spaces that factor through the natural inclusion of $C(K)$ into $L_1(K,\mu)$ when μ is a given regular positive finite, Borel measure on K. We also characterize the operators on $L_\infty(\Omega,\nu)$, with ν a positive finite measure, that factor through the natural inclusion of $L_\infty(\Omega,\nu)$ into $L_1(\Omega,\nu)$.

The following simple example shows that there exists a functional on $C[0,1]$ (trivially, Pietsch integral) that cannot be factored through the natural inclusion of $C[0,1]$ into $L_1([0,1],\mu)$, where μ is Lebesgue measure.

Example 3. Consider $\delta_{1/2} \in C[0,1]^*$. Assume that it factors through the natural inclusion $J : C[0,1] \to L_1([0,1],\mu)$. Then $\delta_{1/2} = \xi \circ J$, where $\xi \in L_\infty([0,1],\mu)$. For each natural number n, let $f_n \in C[0,1]$ be a function such that

$$f_n(t) = \begin{cases} 0, & \text{if } t \leq \dfrac{1}{2} - \dfrac{1}{n} \\ 1, & \text{if } t = \dfrac{1}{2} \\ 0, & \text{if } t \geq \dfrac{1}{2} + \dfrac{1}{n}. \end{cases}$$

Then

$$1 = f_n\left(\frac{1}{2}\right) = \left|\delta_{1/2}(f_n)\right| = |\xi \circ J(f_n)| \leq \|\xi\| \, \|J(f_n)\|_{L_1} \xrightarrow[n\to\infty]{} 0,$$

a contradiction.

In fact, this example can be deduced easily from Theorem 5 below.

Given a compact Hausdorff space K, let Σ be the σ-algebra of the Borel subsets of K. Denote by $B(\Sigma)$ the space of all uniform limits of simple measurable functions on K, endowed with the supremum norm.

Let

$$\lambda : B(\Sigma) \longrightarrow C(K)^{**}$$

be the isometric embedding given by

$$\langle \lambda(f), \nu \rangle := \int_K f \, d\nu \quad (\nu \in C(K)^*, \ f \in B(\Sigma)),$$

where the integral is defined as in [DU, Definition I.1.12], that is, first on the simple measurable functions and then extended to $B(\Sigma)$.

Denote by

$$i : C(K) \longrightarrow B(\Sigma)$$

the natural embedding [D, Corollary §14.5.1].

Given an operator $T \in \mathcal{L}(C(K), F)$, the representing measure $m_T : \Sigma \to F^{**}$ associated with T is defined by

$$m_T(A) := T^{**}(\lambda(\chi_A)) \quad (A \in \Sigma)$$

(see the proof of [DU, Theorem VI.2.1]).

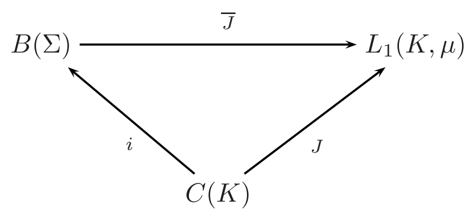
If μ is a finite positive measure on K, denote by

$$J : C(K) \longrightarrow L_1(K, \mu)$$

and

$$\overline{J} : B(\Sigma) \longrightarrow L_1(K, \mu)$$

the natural inclusions. Note that $\overline{J} \circ i = J$ (see the diagram below).



Indeed, given $\omega \in K$ and $f \in C(K)$, since J, $\overline{J}$, and i are natural inclusions, we have

$$J(f)(\omega) = f(\omega) = i(f)(\omega) = \overline{J}(i(f))(\omega) = \left(\overline{J} \circ i(f)\right)(\omega),$$

so $\overline{J} \circ i = J$.

The following lemma is well-known (see the proof of [DU, Example VI.3.6]), but we have not found its proof in the literature. We include it for completeness.

Lemma 4. *Let K be a compact Hausdorff space and let μ be a regular finite positive, Borel measure on K. Then the representing measure m_J of $J : C(K) \to L_1(K, \mu)$ is given by*

$$m_J(A) := \chi_A \qquad (A \in \Sigma).$$

Proof. Define a vector measure $\mathcal{G} : \Sigma \to L_1(K, \mu)$ by

$$\mathcal{G}(A) := \chi_A \qquad (A \in \Sigma).$$

Using the regularity of μ, and choosing $\xi \in L_\infty(K, \mu)$, it is easy to show that $\xi \circ \mathcal{G}$ is a regular set function. It is also easy to see that it is countably additive and has bounded variation.

By [DS, Theorem VI.7.3], there is a (weakly compact) operator

$$T : C(K) \longrightarrow L_1(K, \mu)$$

given by

$$T(f) := \int_K f \, d\mathcal{G} \quad (f \in C(K)),$$

whose representing measure is $\mathcal{G}$.

By [DU, Theorem I.1.13], there is an operator

$$U : B(\Sigma) \longrightarrow L_1(K, \mu)$$

given by

$$U(f) := \int_K f \, d\mathcal{G} \quad (f \in B(\Sigma)),$$

with representing measure $\mathcal{G}$.

Using the definitions of U and T, we obtain $U \circ i = T$.

For $A \in \Sigma$, we have

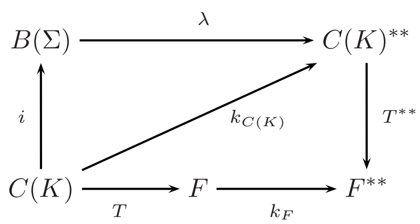
$$U(\chi_A) = \int_K \chi_A \, d\mathcal{G} = \int_A d\mathcal{G} = \mathcal{G}(A) = \chi_A = \overline{J}(\chi_A).$$

Given $f \in C(K)$, there is a sequence (f_n) of simple measurable functions such that $f_n \to i(f)$ in $B(\Sigma)$, under the supremum norm [D, Corollary §14.5.1]. Therefore,

$$T(f) = U \circ i(f) = \lim_n U(f_n) = \lim_n \overline{J}(f_n) = \overline{J} \circ i(f) = J(f),$$

so $T = J$ and $m_J = \mathcal{G}$. □

Given an operator $T \in \mathcal{L}(C(K), F)$, the following diagram is commutative:



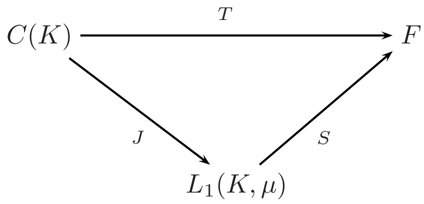
Indeed, it is enough to show that $\lambda \circ i = k_{C(K)}$. For $f \in C(K)$ and $\nu \in C(K)^*$, we have

$$\langle \lambda \circ i(f), \nu \rangle = \int_K i(f) \, d\nu = \int_K f \, d\nu = \langle \nu, f \rangle = \langle k_{C(K)}(f), \nu \rangle,$$

and the claim is proved.

Theorem 5. *Let K be a compact Hausdorff space, let μ be a regular positive finite, Borel measure on K, and let $T \in \mathcal{L}(C(K), F)$ be an operator. Then the following assertions are equivalent:*

(a) *There is an operator $S \in \mathcal{L}(L_1(K, \mu), F)$ such that $T = S \circ J$ (see the diagram below).*



(b) *There exists a constant $C > 0$ such that*

$$\|m_T(A)\| \leq C\mu(A) \quad (A \in \Sigma).$$

(c) *There exists a constant $C > 0$ such that*

$$|m_T|(A) \leq C\mu(A) \quad (A \in \Sigma).$$

(d) *There exists a constant $C > 0$ such that*

$$\|T(f)\| \leq C \|J(f)\|_{L_1} \quad (f \in C(K)).$$

Moreover, if (b), (c) *or* (d) *holds, then T is Pietsch integral, and*

$$\|T\|_{PI} \leq C\mu(K).$$

Proof. (a) $\Rightarrow$ (b). For $A \in \Sigma$, we have

$$\begin{aligned} \|m_T(A)\| &= \|T^{**}(\lambda(\chi_A))\| = \|S^{**} \circ J^{**}(\lambda(\chi_A))\| \\ &\leq \|S\| \|m_J(A)\| = \|S\| \|\chi_A\|_{L_1} \qquad \text{(by Lemma 4)} \\ &= \|S\| \mu(A). \end{aligned}$$

(b) $\Rightarrow$ (c). By the definition of variation, given $A \in \Sigma$, we have

$$|m_T|(A) = \sup_\pi \sum_{N \in \pi} \|m_T(N)\|,$$

where the supremum is taken over all partitions π of A into a finite number of pairwise disjoint members of Σ. Then,

$$\sup_\pi \sum_{N \in \pi} \|m_T(N)\| \leq C \sup_\pi \sum_{N \in \pi} \mu(N) = C\mu(A).$$

(c) $\Rightarrow$ (d). Let

$$g := \sum_{i=1}^{n} a_i \chi_{A_i}$$

be a simple measurable function on K, with $(A_i)_{i=1}^n$ a disjoint family of members of Σ. Then,

$$\begin{aligned}\|T^{**}(\lambda(g))\| &= \left\|\sum_{i=1}^{n} a_i T^{**}(\lambda(\chi_{A_i}))\right\| = \left\|\sum_{i=1}^{n} a_i m_T(A_i)\right\| \\ &\le \sum_{i=1}^{n} |a_i| \, \|m_T(A_i)\| \le \sum_{i=1}^{n} |a_i| \, |m_T|(A_i) \\ &\le C \sum_{i=1}^{n} |a_i| \, \mu(A_i) = C \int_K \left|\sum_{i=1}^{n} a_i \chi_{A_i}\right| d\mu = C \left\|\overline{J}(g)\right\|_{L_1}.\end{aligned}$$

Given $f \in C(K)$, let (f_n) be a sequence of simple measurable functions on K such that $i(f) = \lim_n f_n$ in $B(\Sigma)$. Then,

$$\begin{aligned}\|T(f)\| &= \|k_F \circ T(f)\| = \|T^{**}(\lambda(i(f)))\| \\ &= \lim_n \|T^{**}(\lambda(f_n))\| \le C \lim_n \left\|\overline{J}(f_n)\right\| \\ &= C \left\|\overline{J}(i(f))\right\| = C \, \|J(f)\|_{L_1}.\end{aligned}$$

(d) $\Rightarrow$ (a). Define an operator $S : J(C(K)) \to F$ by $S(J(f)) := T(f)$, for all $f \in C(K)$. We have

$$\|S(J(f))\| = \|T(f)\| \le C \, \|J(f)\|_{L_1} \quad (f \in C(K)),$$

so S is continuous on $J(C(K))$ endowed with the L_1-norm. Since $J(C(K))$ is dense in $L_1(K, \mu)$ [DS, Lemma IV.8.19], S has an extension, denoted also by S, to $L_1(K, \mu)$ such that $S \circ J = T$ and $\|S\| \le C$.

Suppose now that T satisfies (b), (c) or (d), for a constant $C > 0$. Then T satisfies also (a), with $\|S\| \le C$. Since J is Pietsch integral, so is $T = S \circ J$, and

$$\|T\|_{\mathrm{PI}} \le \|S\| \, \|J\|_{\mathrm{PI}} \le C\mu(K),$$

where we have used [DU, Example VI.3.10]. □

We now study the operators on $L_\infty(\Omega, \nu)$. We first give an example of a functional on $L_\infty([0,1], \mu)$, where μ is Lebesgue measure, that cannot be factored through the canonical inclusion into $L_1([0,1], \mu)$.

Example 6. For every $n \in \mathbb{N}$, let

$$g_n := n(n+1)\chi_{[\frac{1}{n+1},\frac{1}{n}]}.$$

Obviously, $g_n \in L_1([0,1],\mu)$ and $\|g_n\|_{L_1([0,1],\mu)} = 1$. Define a linear form

$$H : L_\infty([0,1],\mu) \longrightarrow \mathbb{R}$$

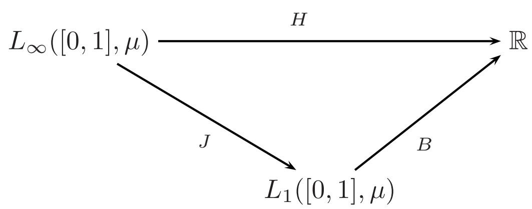
by

$$H(f) := \sum_{n=1}^{+\infty} \frac{1}{n^{3/2}} \int_{[0,1]} f g_n \, d\mu \quad \text{for} \quad f \in L_\infty([0,1],\mu).$$

Since

$$|H(f)| \leq \sum_{n=1}^{+\infty} \frac{1}{n^{3/2}} \|f\|_{L_\infty([0,1],\mu)} ,$$

H is continuous. Suppose that there exists $B \in L_1([0,1],\mu)^*$ such that the following diagram is commutative



where J is the natural inclusion of $L_\infty([0,1],\mu)$ into $L_1([0,1],\mu)$. For each $m \in \mathbb{N}$, consider the function

$$f_m := \chi_{[\frac{1}{m+1},\frac{1}{m}]}.$$

We have

$$B \circ J(f_m) = H(f_m) = \sum_{n=1}^{+\infty} \frac{1}{n^{3/2}} \int_{[0,1]} \chi_{[\frac{1}{m+1},\frac{1}{m}]} n(n+1)\chi_{[\frac{1}{n+1},\frac{1}{n}]} \, d\mu = \frac{1}{m^{3/2}}.$$

Therefore

$$\frac{1}{m^{3/2}} = |B \circ J(f_m)| \leq \|B\| \, \|f_m\|_{L_1([0,1],\mu)} = \|B\| \, \frac{1}{m(m+1)},$$

so

$$\frac{m(m+1)}{m^{3/2}} \leq \|B\| \quad \text{for all} \quad m \in \mathbb{N},$$

a contradiction.

The following result is mentioned in [LM, Exercise 10.10]. We give a proof for completeness.

Lemma 7. *Let (Ω, Σ, μ) be a measure space. Then the space of simple measurable functions is dense in $L_\infty(\Omega, \mu)$.*

Proof. Let $f \in L_\infty(\Omega, \mu)$. Suppose first that $f \geq 0$. There is a subset $M \subset \Omega$ with $\mu(M) = 0$ such that f is bounded on $\Omega \backslash M$. By [R, Theorem 1.17], there is a nondecreasing sequence (f_n) of simple measurable functions such that

$$\lim_n f_n(\omega) = f(\omega) \quad (\omega \in \Omega \backslash M).$$

Since f is bounded, the convergence is uniform on $\Omega \backslash M$ (see the comment after [R, Theorem 1.17]). Therefore, the L_∞-norm of $(f_n - f)$ tends to zero.

If f is arbitrary, we decompose $f = f^+ - f^-$ as usual. There are sequences (g_n) and (h_n) of simple measurable functions converging respectively to f^+ and f^-. Then, the sequence of simple measurable functions $(g_n - h_n)$ converges to f in $L_\infty(\Omega, \mu)$. □

The following result is contained in [DJT, Examples 2.9 and Corollary 5.22].

Lemma 8. *Let (Ω, Σ, μ) be a finite measure space. Then the natural inclusion*

$$J : L_\infty(\Omega, \mu) \longrightarrow L_1(\Omega, \mu)$$

is Pietsch integral with norm $\|J\|_{PI} = \mu(\Omega)$.

Given an operator $T \in \mathcal{L}(L_\infty(\Omega, \mu), F)$, its representing measure $m_T : \Sigma \to F$ is defined by

$$m_T(A) := T(\chi_A) \quad (A \in \Sigma)$$

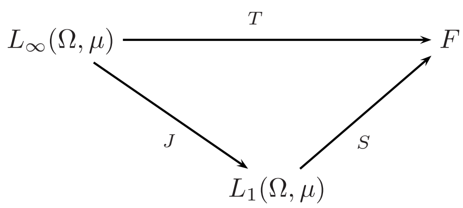
[DU, page 148].

Theorem 9. *Let (Ω, Σ, μ) be a positive finite measure space and let*

$$T \in \mathcal{L}(L_\infty(\Omega, \mu), F)$$

be an operator. Then the following assertions are equivalent:

(a) *There is an operator $S \in \mathcal{L}(L_1(\Omega, \mu), F)$ such that $T = S \circ J$ (see the diagram below).*



(b) *There exists a constant* $C > 0$ *such that*

$$\|m_T(A)\| \leq C\mu(A) \quad (A \in \Sigma).$$

(c) *There exists a constant* $C > 0$ *such that*

$$|m_T|(A) \leq C\mu(A) \quad (A \in \Sigma).$$

(d) *There exists a constant* $C > 0$ *such that*

$$\|T(f)\| \leq C \|J(f)\|_{L_1} \quad (f \in L_\infty(\Omega, \mu)).$$

Moreover, if (b), (c) or (d) holds, then T *is Pietsch integral, and*

$$\|T\|_{PI} \leq C\mu(\Omega).$$

PROOF. (a) ⇒ (b). For $A \in \Sigma$, we have

$$\|m_T(A)\| = \|T(\chi_A)\| = \|S \circ J(\chi_A)\| \leq \|S\| \|\chi_A\|_{L_1} = \|S\| \mu(A).$$

(b) ⇒ (c) as in Theorem 5.

(c) ⇒ (d). Let

$$g := \sum_{i=1}^{n} a_i \chi_{A_i}$$

be a simple measurable function on Ω, with $(A_i)_{i=1}^n$ a disjoint family of members of Σ. Then,

$$\begin{aligned} \|T(g)\| &= \left\| \sum_{i=1}^{n} a_i T(\chi_{A_i}) \right\| = \left\| \sum_{i=1}^{n} a_i m_T(A_i) \right\| \\ &\leq C \int_\Omega \left| \sum_{i=1}^{n} a_i \chi_{A_i} \right| d\mu \qquad \text{(as in the proof of Theorem 5)} \\ &= C \|J(g)\|_{L_1} . \end{aligned}$$

Let $f \in L_\infty(\Omega, \mu)$. By Lemma 7, there is a sequence (f_n) of simple measurable functions on Ω such that $\lim_n f_n = f$ in $L_\infty(\Omega, \mu)$. Then,

$$\|T(f)\| = \lim_n \|T(f_n)\| \leq C \lim_n \|J(f_n)\|_{L_1} = C \|J(f)\|_{L_1} .$$

(d) ⇒ (a) as in the proof of Theorem 5, bearing in mind that $J(L_\infty(\Omega, \mu))$ is dense in $L_1(\Omega, \mu)$ [R, Theorem 3.13].

Suppose now that T satisfies (b), (c) or (d), for a constant $C > 0$. Then T satisfies also (a), with $\|S\| \leq C$. Since J is Pietsch integral, so is $T = S \circ J$, and

$$\|T\|_{\mathrm{PI}} \leq \|S\| \|J\|_{\mathrm{PI}} \leq C\mu(\Omega),$$

where we have used Lemma 8. □

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