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Finite groups in which the degrees of non-linear constituents of some induced characters are distinct

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Berkovich, Chillag and Herzog [1] classified all finite groups G, in which the degrees of the non-linear irreducible characters are distinct. If G is such a non-abelian group, then one of the following assertions holds [1]:

(a) $G = \mathrm{ES}(m, 2)$, an extra-special group of order 2^{1+2m};

(b) $G = (C(p^m - 1), E(p^m))$, a Frobenius group with elementary abelian kernel $E(p^m)$ of order p^m (p is a prime), and a complementary cyclic factor $C(p^m - 1)$ of order $p^m - 1$;

(c) $G = (Q(8), E(9))$, a Frobenius group with the elementary abelian kernel $E(9)$ of order 9, and a complementary factor $Q(8)$, the ordinary quaternion group of order 8.

In this note we study a more general class of groups, which we call D-groups:

D : If $1 < N \leq G'$, N is normal in G, and $1_N \neq \lambda \in \mathrm{Irr}(N)$, then the degrees of the irreducible constituents of the induced character λ^G are distinct.

Let $\mathrm{Irr}_1(G)$ denote the set of all non-linear irreducible characters of G.

We denote by $\mathrm{Irr}(\chi)$ the set of all irreducible constituents of the character χ. Let $\mathrm{Irr}_1(\chi)$ denote the set of all non-linear irreducible constituents of the character χ, and $cd_1(\chi) = \{\varphi(1) \mid \varphi \in \mathrm{Irr}_1(\chi)\}$. A character χ is said to be a D-character if the sets $cd_1(\chi)$ and $\mathrm{Irr}_1(\chi)$ contain the same number of elements.

Lemma 1. *Suppose that H is a non-trivial normal subgroup of a non-abelian group G, $G/H \simeq C(m)$ ($C(m)$ is a cyclic group of order m). If,*

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for some non-principal $\lambda \in \mathrm{Irr}(H)$, *the character* λ^G *is a* D*-character, then* $|\mathrm{Irr}_1(\lambda^G)| \leq 1$.

Proof. Suppose that $\chi, \tau \in \mathrm{Irr}_1(\lambda^G)$. Then

$$\langle \chi_H, \lambda \rangle = 1 = \langle \tau_H, \lambda \rangle$$

by Clifford's theory. So by Clifford's theorem $\chi(1) = \tau(1)$ and $\chi = \tau$ since χ, τ are non-linear irreducible constituents of the same degree of the D-character λ^G. □

Corollary 1.1. *Suppose that* $1 < G' < G$, $G/G' \simeq C(m)$. *If, for every non-principal* $\lambda \in \mathrm{Irr}(G')$, *the character* λ^G *is a D-character, then* $G = (C(m), G')$, *a Frobenius group with a complementary factor* $C(m)$ *and the kernel* G'.

Proof. If $1_{G'} \neq \lambda \in \mathrm{Irr}(G')$ then $\mathrm{Irr}(\lambda^G) = \mathrm{Irr}_1(\lambda^G)$, so by Lemma 1, $\lambda^G \in \mathrm{Irr}(G)$. Now the result follows from [2, Corollary 2.5] (see also [5, corollary 37.5.4]).

Remark. If $G' \leq N \leq G$ and a non-principal $\lambda \in \mathrm{Irr}(N)$, then all irreducible constituents of the character λ^G have the same degree (see [4], Problem 6.2).

Lemma 2. *Suppose that* H *is a non-trivial normal subgroup of* G, $G/H \simeq Q(8)$ *and, for some non-principal* $\lambda \in \mathrm{Irr}(H)$ *the character* λ^G *is a* D*-character. Then* λ^G *has at most one non-linear irreducible constituent.*

Proof. Let

$$\lambda^G = e_1\chi^1 + \ldots + e_s\chi^s, \qquad \mathrm{Irr}(\lambda^G) = \{\chi^1, \ldots, \chi^s\}.$$

By Clifford's theory $e_1, \ldots, e_s$ are degrees of irreducible projective representations of the group $I_G(\lambda)/H$, where $I_G(\lambda)$ is the inertia group of λ in G. Since the Schur's multiplier of any subgroup of $Q(8)$ is trivial then in fact $e_1, \ldots, e_s$ are degrees of ordinary irreducible representations of $I_G(\lambda)/H$. Hence $e_i \leq 2$ for all i.

Suppose that distinct $\chi^1, \chi^2 \in \mathrm{Irr}_1(\lambda^G)$. By reciprocity and Clifford's theorem $e_1 \neq e_2$. Let $e_1 > e_2$. Then $e_1 = 2$, $e_2 = 1$. Since $e_1 = 2$ then $I_G(\lambda)/H \simeq Q(8)$, i.e., $I_G(\lambda) = G$ and λ is invariant under G. Then

$$\chi^1_H = 2\lambda, \quad \chi^2_H = \lambda,$$

and λ is non-linear (since χ^2 is non-linear). Hence $\mathrm{Irr}_1(\lambda^G) = \mathrm{Irr}(\lambda^G)$ by reciprocity. Therefore $s = 2$ and

$$|G : H|\lambda(1) = 8\lambda(1) = \lambda^G(1) = e_1\chi^1(1) + e_2\chi^2(1) = 5\lambda(1),$$

a contradiction. □

Corollary 2.1. *Suppose that H is a non-trivial normal subgroup of G, $G/H \simeq Q(8)$ and $H < G'$. If for every non-principal $\lambda \in \mathrm{Irr}(H)$ the character λ^G is a* D*-character, then $G = (Q(8), G')$.*

See the proof of Corollary 1.1.

Lemma 3. *Let p be a prime. Suppose that*

$$p^n = p^k + a_1 p^{2c(1)} + \ldots + a_s p^{2c(s)},$$

where $s, n, k, a_1, \ldots, a_s$, $c(1), \ldots, c(s)$ are positive integers. Then

(a) $k \geq 2c(1)$.

(b) *If $a_1 < p^2 - p$ then $s = 1$, $k = 2c(1)$, $n = 2c(1)+1$, $a_1 = p-1$.*

We omit an easy proof of this lemma.

Let G be a group, $cd(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$. Then

$$d(G) = (a_0 \cdot 1, a_1 \cdot d_1, \ldots, a_t \cdot d_t)$$

denotes that $|G : G'| = a_0$, and $\mathrm{Irr}(G)$ contains exactly a_i characters of degree d_i, $i \in \{1, \ldots, t\}$. Usually we assume that $1 < d_1 < \ldots < d_t$.

Lemma 4 (see [1])**.** *Suppose that G is a non-abelian p-group, $d(G) = (p^k \cdot 1, a_1 \cdot p^{c(1)}, \ldots, a_t \cdot p^{c(t)})$. If $a_1 < p^2 - p$ then $G \simeq ES(m, p)$.*

Proof. Let $|G| = p^n$. Then

$$p^n = p^k + a_1 p^{2c(1)} + \ldots a_t p^{2c(t)}.$$

By the condition $t \geq 1$. Hence (Lemma 3)

$$t = 1, \quad k = 2c(1) = n - 1, \quad a_1 = p - 1.$$

Therefore $|G'| = p$ and $c(1) = (n-1)/2$. If $\chi \in \mathrm{Irr}(G)$ is a non-linear character then

$$p^{n-1} = \chi(1)^2 \leq |G : Z(G)| \leq p^{n-1}.$$

Then $|Z(G)| = p = |G'| \implies G' = Z(G)$ and $G \simeq \mathrm{ES}(m, p)$. $\square$

Lemma 5. *Suppose that N is a non-trivial normal subgroup of G, $N \leq G'$, G/N is a p-group. If for some non-principal $\lambda \in \mathrm{Irr}(N)$ the character λ^G is a* D*-character, then $\lambda^G = e\chi$ with $\chi \in \mathrm{Irr}(G)$.*

Proof. Let

$$\lambda^G = e_1\chi^1 + \ldots + e_s\chi^s, \quad \mathrm{Irr}(\lambda^G) = \{\chi^1, \ldots, \chi^s\}.$$

Since $N \leq G'$ and $\lambda \neq 1_N$, then all χ^i are non-linear. Let

$$\chi^i_N = e_i(\lambda_1 + \ldots + \lambda_n)$$

be Clifford's decomposition, $\lambda_1 = \lambda$. Then $\chi^i(1) = e_i n\lambda(1)$,

$$|G:N|\lambda(1) = \lambda^G(1) = n\lambda(1)\ (e_1^2 + \ldots + e_s^2),$$
$$|G:N| = n(e_1^2 + \ldots + e_s^2).$$

Here $n = |G : I_G(\lambda)|$ is a power of p since $N \le I_G(\lambda)$. If $|I_G(\lambda) : N| = p^\alpha$, then

$$p^\alpha = e_1^2 + \ldots + e_s^2.$$

Since $I_G(\lambda)/N$ is a p-group then $e_1, \ldots, e_s$ are powers of p. If $i \ne j$ then $e_i n\lambda(1) = \chi^i(1) \ne \chi^j(1) = e_j n\lambda(1) \implies e_i \ne e_j$. Suppose that $e_1 < \ldots < e_s$, $e_i = p^{\beta(i)}$. If $s > 1$ then

$$p^{\alpha - 2\beta(1)} = 1 + p^{2(\beta(2)-\beta(1))} + \ldots + p^{2(\beta(s)-\beta(1))},$$

which is impossible. Hence $s = 1$ and $\lambda^G = e_1\chi^1$, $\chi^1 \in \mathrm{Irr}(G)$. □

Lemma 6. *Suppose that $G = (A, H)$ is a Frobenius group, $1 < N < H$ and N is a normal in G, H/N is a p-group. If a non-principal $\lambda \in \mathrm{Irr}(N)$ and the character λ^G is a* D*-character, then $\lambda^G = e\psi$ where $\psi \in \mathrm{Irr}(G)$.*

Proof. Suppose that

$$\lambda^G = e_1\chi^1 + \ldots + e_s\chi^s, \quad \mathrm{Irr}(\lambda^G) = \{\chi^1, \ldots, \chi^s\}.$$

Obviously $N < G'$, so that $\chi^1, \ldots, \chi^s$ are non-linear. Let

$$\chi^i_N = e_i(\lambda_1 + \ldots + \lambda_n)$$

be the Clifford's decomposition, $\lambda_1 = \lambda$. Then $\chi^i(1) = e_i n\lambda(1)$, $|G:N| = n(e_1^2 + \ldots = e_s^2)$. Since (A, N) is a Frobenius group then $I_G(\lambda) \le H$, so that $n = |A|n_0$. Therefore

$$|G:N| = |G:H|\,|H:N| = |A|\,|H:N| = |A|n_0(e_1^2 + \ldots + e_s^2),$$
$$|H:N|n_0^{-1} = e_1^2 + \ldots + e_s^2.$$

Since $N \le I_G(\lambda) \le H$ then n_0 is a power of p. As before $e_1, \ldots, e_s$ are distinct powers of p (as degrees of irreducible projective representations of a p-group $I_G(\lambda)/N$). As in Lemma 5 this implies $s = 1$. □

Lemma 7. *Suppose that H is a normal Hall subgroup of a group G. Then $H \cap \Phi(G) = \Phi(H)$; here $\Phi(G)$ is the Frattini subgroup of G.*

Proof. The inclusion $\Phi(H) \le \Phi(G)$ follows from the modular law. So we may assume without loss of generality that $\Phi(H) = 1$. Suppose that $D = H \cap \Phi(G) > 1$. Let A be the least subgroup of H such that $AD = H$ and D does not contained in A (A exists since $D > 1$ and

$\Phi(H) = 1$). Let $D_1 = A \cap D$. From the choice of A it follows easily that $D_1 \le \Phi(A)$. Since D is abelian, then $N_H(D_1) \ge \langle A, D\rangle = H$. Then as it is known $D_1 \le \Phi(H) = 1$. Thus $1 = D_1 = A \cap D$, and $H = AD$, a semi-direct product. Since D is abelian then by Gaschutz's theorem [3, §1.17] there exists a subgroup F in G such that $G = FD$ and $F \cap D = 1$. Since $1 < D \le \Phi(G)$ one obtains a contradiction. Thus $D = 1$ and $H \cap \Phi(G) = 1 = \Phi(H)$. □

Lemma 8 (see [5, Lemma 37.3.3])**.** *Suppose that P is a non-trivial minimal normal p-subgroup of a group $G = CP$, $C \cap P = 1$ and a cyclic subgroup C of order b acts on P faithfully. Let m be the order of $p \pmod b$. Then $|P| = p^m$.*

Proof (A. Mann). Put $E = \mathrm{End}_{GF(p)C}(P)$. Then E is a finite skew field (Schur's lemma). By Wedderburn's theorem E is commutative. Obviously $C \subset E$. So all E-subspaces of P are trivial. Therefore $\dim_E P = 1$. Let F be the least subfield of E containing C. As above $\dim_F P = 1$. Hence $|E| = |P| = |F|$, $F = E$. Put $|E| = p^n$. Then $|C|\,|(p^n - 1)$. Since C generates E as field then n is the least positive integer such that $p^n \equiv 1 \pmod b$. □

Main Theorem. *Suppose that G is a non-abelian solvable D-group. Then one and only one of the following assertions holds:*

(a) $G = \mathrm{ES}(m, p)$, an extra-special p-group of order p^{1+2m}.

(b) $G = (Q(8), E(p^n))$, $Q(8)$ acts on $E(p^n)$ irreducibly.

(c) $G = (C(s), E(p^n))$, $C(s)$ acts on $E(p^n)$ irreducibly (in particular n is the order of $p \pmod s$).

Proof. It is easy to see that groups (a)–(c) are in fact D-groups.

Suppose that R is a minimal normal subgroup of G such that $R \le G'$. Let $|R| = p^n$.

(i) *G/R is a D-group.*

This is obvious.

(ii) *If G is nilpotent then $G \simeq \mathrm{ES}(m, p)$.*

Proof. Suppose that $G = P \times Q$ where $P \in \mathrm{Syl}_p(G)$ is non-abelian, $Q > 1$ (so we may assume that $R < P$). Let $1_R \ne \lambda \in \mathrm{Irr}(R)$, $\chi \in \mathrm{Irr}(\lambda^G)$; χ is non-linear since $R \le G'$. Let μ be a non-principal linear character of Q. Then $\chi \times 1_Q$, $\chi \times \mu$ are distinct non-linear irreducible constituents of λ^G of the same degree, a contradiction. Thus G is a p-group.

Let a non-principal $\lambda \in \mathrm{Irr}(R)$. Then (Lemma 5) $\lambda^G = e\chi$, $\chi \in \mathrm{Irr}(G)$. Since λ is G-invariant then $e = \chi(1)$ (Clifford) and

$$\lambda^G = \chi(1)\chi,\ |G : R| = \lambda^G(1) = \chi(1)^2 \le |G : Z(G)| \implies R = Z(G).$$

If $|G| = p^n$ then $\mathrm{Irr}(G)$ contains exactly $p-1$ characters of degree $p^{(n-1)/2}$. Since $n-1$ is even then $n-2$ is odd. Hence G/R is abelian. Then $R = G' = Z(G)$ has order p, and $G = \mathrm{ES}(m,p)$. $\square$

In the sequel we suppose that G is non-nilpotent.

(iii) *If G/R is abelian then $G = (C(s), R)$.*

Proof. By the condition $R = G'$. Since G is non-nilpotent then R does not contained in $\Phi(G)$. So $G = AR$, $A \cap R = 1$; here A is a maximal subgroup of G. Obviously A is abelian, $Z(G) < A$ and $G/Z(G) = (C(s), E(p^n))$. In particular every non-principal character from $\mathrm{Irr}(R)$ belongs to the G-orbit of length s. If a non-principal $\lambda \in \mathrm{Irr}(R)$ then $I_G(\lambda) = RZ(G)$ and $cd(G) = \{1, s\}$. Since λ^G has no linear constituents then by the condition $\lambda^G = e\chi$ with $\chi \in \mathrm{Irr}(G)$, $\chi(1) = s$. So by the Clifford's theorem $e = 1$. Then $|G : R| = \lambda^G(1) = \chi(1) = s \Rightarrow Z(G) = 1$ and $G = (C(s), E(p^n))$. $\square$

(iv) *If G' is abelian (and G is non-nilpotent) then $G = (C(s), E(p^n)) = (C(s), R)$, i.e. $G' = R$ is a minimal normal subgroup of G.*

Proof. In view of (iii) we may assume that $R < G'$.

Let T be the greatest normal subgroup of G which is properly contained in G'. It has been proved in (iii) that

$$G/T \in \{\mathrm{ES}(m,q)\},\ (C(s), E(q^m));\text{ here } q \text{ is a prime.}$$

(1iv) $G/T \simeq \mathrm{ES}(m,q)$.

In view of (ii), q does not divide $|T|$ (we recall that T is abelian). Now $T > 1$ since G is non-nilpotent.

So $G = QT$ where $Q \in \mathrm{Syl}_q(G)$. If a non-principal $\lambda \in \mathrm{Irr}(T)$ then $\lambda^G = e\chi$, $\chi \in \mathrm{Irr}(G)$ (Lemma 5). In particular χ vanishes on $G - T$, and so also on $Q^\# = Q - \{1\}$. Hence $|Q| | \chi(1)$. Since $\lambda^G(1) = |Q|$ then $\lambda^G = \chi$ for any choice of non-principal $\lambda \in \mathrm{Irr}(T)$. Hence $G = (Q, T)$ (see [5, Corollary 37.5.4])]. Then $G' = (Q', T)$ is non-abelian, a contradiction.

(2iv) $G/T = (C(s), E(q^m))$, the subgroup $E(q^m)$ is a minimal normal subgroup of G/T.

Then $G/G' \simeq C(s)$, so $G = (C(s), G')$ (Corollary 1.1), $cd(G) = \{1, s\}$. Let a non-principal $\lambda \in \mathrm{Irr}(T)$. Since λ^G has no linear constituents then $\lambda^G = e\chi$, $\chi \in \mathrm{Irr}(G)$ (Lemma 6), $\chi(1) = s$. Since every non-principal irreducible character of T has exactly s conjugates under G, then $e = 1$ (Clifford) and $\lambda^G = \chi \in \mathrm{Irr}(G)$. In this case G is a Frobenius group with the kernel T [5, Corollary 37.5.4], a contradiction since a Frobenius group has only one Frobenius kernel. $\square$

(v) *If* $G/G' = C(s)$ *then* $G = (C(s), G')$, $G' \in \mathrm{Syl}_p(G)$, $\Phi(G') = \Phi(G)$.

PROOF. By Corollary 1.1, $G = (C(s), G')$. In particular G' is nilpotent (Thompson). Since G'/G'' is a minimal normal subgroup of G/G'' by (iv), then G'/G'' is primary $\implies$ G' is primary, say, G' is a p-group. In particular $G' \in \mathrm{Syl}_p(G)$. Now $\Phi(G') = \Phi(G)$ (Lemma 7). $\square$

(vi) *If* $G/R \simeq \mathrm{ES}(m, q)$ *then* $G = (Q(8), R)$.

PROOF. Recall that R is a minimal normal subgroup of order p^n in G and G is non-nilpotent. So $q \neq p$. Now $R < G'$ by the choice of R. In view of (iv) we may assume that G' is non-abelian. We have $|G'| = qp^n$. Since G' is non-abelian, it is non-nilpotent. Hence $Z(G') < R = 1 \implies G' = (Z(Q), R)$ where $Q \in \mathrm{Syl}_q(G)$, $Q \simeq \mathrm{ES}(m, q)$. If a non-principal $\lambda \in \mathrm{Irr}(R)$ then $\lambda^G = e\chi$, $\chi \in \mathrm{Irr}(G)$ (Lemma 5). In particular, χ vanishes on $G - R$, and, as in (2iv) one obtains $G = (Q, R)$. Hence $Q = Q(8)$, $G = (Q(8), R) = (Q(8), E(p^n))$. $\square$

(vii) *If* $G/G'' \simeq \mathrm{ES}(m, q)$ *and* $G'' > 1$ *then* $G = (Q(8), E(p^n))$ *and* $E(p^n)$ *is a minimal normal subgroup of* G.

PROOF. Let T be the greatest normal subgroup of G which is properly contained in G''. Then by (vi), $G/T \simeq (Q(8), E(p^n))$. In particular $G/G'' \simeq Q(8)$. Hence $G = (Q(8), G'')$ by Corollary 2.1. Then G'' is abelian (Burnside) and $cd(G) = \{1, 2, 8\}$. If a non-principal $\lambda \in \mathrm{Irr}(T)$ then $\lambda^G = e\chi$, $\chi \in \mathrm{Irr}(G)$ (Lemma 6). Now λ belongs to a G-orbit of length $8 = \chi(1)$. Hence $e = 1$ by the Clifford's theorem. Thus $\lambda^G = \chi$ and $\chi(1) = 8 = \lambda^G(1) = |G : T|$, a contradiction. Hence $T = 1$. $\square$

(viii) *If* $G/G'' = (C(s), E(p^m))$ *then* $G'' = 1$. *In particular* G' *is a minimal normal subgroup of* G.

PROOF. One has $G = (C(s), G')$ where $G' \in \mathrm{Syl}_p(G)$ by (v). If $G'' = 1$ then the result follows from (iv). Suppose that $G'' > 1$. Without loss of generality we may assume that G'' is a minimal normal subgroup of G. Since $G'' \leq Z(G')$ then G'' is a minimal normal subgroup of $C(s)G''$. So (Lemma 8) $|G''| = |G'/G''| = p^m$. Take an element x in $G' - G''$. Then the mapping $\varphi : G' \to G''$ which is defined by $\varphi(a) = [x, a]$ $(a \in G')$ is a homomorphism. Obviously the kernel of φ is equal to $C_{G'}(x)$. Hence $[x, G']$, the image of φ is a proper subgroup of G''. Take a non-principal $\lambda \in \mathrm{Irr}(G'')$ such that $[x, G'] \leq \ker\lambda$. Then $x\ker\lambda \leq Z(G'/\ker\lambda)$. Now $\lambda^G = e\chi$, $\chi \in \mathrm{Irr}(G)$ (Lemma 6). So [4, Theorem 6.11] $\lambda^{G'} = e\psi$, $\psi \in \mathrm{Irr}(G')$. Obviously

$$\ker\psi = \ker\lambda.$$

Since λ is G'-invariant then $e = \psi(1)$ (Clifford). Now

$$|G' : G''| = \lambda^{G'}(1) = \psi(1)^2 \leq |G'/\ker\psi : Z(G'/\ker\psi)|,$$

so that $|Z(G'/\ker\psi)| = p$. Hence $Z(G'/\ker\psi) = G''/\ker\psi$, a contradiction since $x\ker\lambda \in Z(G'/\ker\lambda) - G''/\ker\lambda$. Thus $G'' = 1$. The theorem is proved. □

Corollary [1]. *Suppose that the degrees of the non-linear irreducible characters of a non-abelian solvable group G are distinct. Then*

$$G \in \{ES(m, 2),\ (Q(8), E(9)),\ (C(p^n - 1),\ E(p^n))\}.$$

Proof. Obviously G is a D-group. Hence by the Main Theorem we have to consider the following three cases.

(i) $G \simeq \mathrm{ES}(m, p)$.

In this case $\mathrm{Irr}(G)$ contains exactly $p-1$ characters of degree p^m. Hence $p - 1 = 1$, $p = 2$, $G \simeq \mathrm{ES}(m, 2)$.

(ii) $G = (Q(8), E(p^n))$.

In this case $\mathrm{Irr}(G)$ contains exactly $(p^n - 1)/8$ characters of degree 8. Therefore $(p^n - 1)/8 = 1$, $p^n = 9$, $G = (Q(8), E(9))$.

(iii) $G = (C(s), E(p^n))$.

In this case $\mathrm{Irr}(G)$ contains exactly $(p^n - 1)/s$ characters of degree s. So $(p^n - 1)/s = 1$, $s = p^n - 1$, $G = (C(p^n - 1), E(p^n))$. □

Note that all non-abelian simple groups are D-groups.

Next we consider $\overline{D}$-groups, i.e. groups G, satisfying the following condition:

$\overline{D}$: *If $N > 1$ is any normal subgroup of G and $1_N \neq \lambda \in \mathrm{Irr}(N)$ then λ^G is a D–character.*

Obviously $\overline{D}$-groups are D-groups.

Theorem 9. *If G is a non-solvable $\overline{D}$-group then $G' = G$.*

Proof. Suppose that $G' < G$. Let H be the last term of the derived series of G. Then G/H is a non-identity $\overline{D}$-group. Since G/H is also a D-group, we have to consider the following four possibilities.

(i) $G/H \simeq \mathrm{ES}(m, p)$.

If $1_H \neq \lambda \in \mathrm{Irr}(H)$ then $\lambda^G = e\chi$, $\chi \in \mathrm{Irr}(G)$ (Lemma 5). If $\tau \in \mathrm{Irr}(G)$ and H does not contained in $\ker\tau$ then $\langle \tau_H, 1_H \rangle = 0$, so $\mu^G = f\tau$ for a certain non-linear $\mu \in \mathrm{Irr}(H)$. Then p divides $\chi(1)$ for all $\chi \in \mathrm{Irr}(G)$ such that H does not contained in $\ker\chi$. If $\varphi \in \mathrm{Irr}(G)$ is non-linear and $H \leq \ker\varphi$, then $\varphi \in \mathrm{Irr}(G/H)$ so that $\varphi(1) = p^m$. Thus p divides degrees of all non-linear irreducible characters of G. By Thompson's theorem

[4, corollary 12.2] the group G has a normal p-complement, and this p-complement coincides with H since $H' = H$. Let $P \in \mathrm{Syl}_p(G)$. Then for a non-principal $\lambda \in \mathrm{Irr}(H)$, the character $\lambda^G = e\chi$ (Lemma 5) vanishes on $P^\# \subseteq G - H$. So $|P| | \chi(1)$, $e = 1$ and $\lambda^G = \chi$ is irreducible. Hence $G = (P, H)$ [5, corollary 37.5.4], H is nilpotent by Thompson's theorem [5, theorem 37.3.3], G is solvable, a contradiction.

(ii) $G/H \simeq (Q(8), E(p^n))$.

Then $G/G'' \simeq Q(8)$ and $G = (Q(8), G'')$ by Corollary 2.1, G'' is abelian (Burnside), G is solvable, a contradiction.

(iii) $G/H \simeq (C(s), P/H)$, $P/H \in \mathrm{Syl}_p(G/H)$.

Then $P/H = G'/H$, $G/G' \simeq C(s)$ and $G = (C(s), G')$ (Corollary 1.1), G' is nilpotent (Thompson), G is solvable, a contradiction.

(iv) G/H is abelian.

Then $H = G'$. Let $n = \exp G/H$, and let $G' \leq T < G$ be such that $G/T \simeq C(n)$. Take a non-linear $\chi \in \mathrm{Irr}(G)$. Then by Clifford's theory

$$\chi_T = \lambda_1 + \ldots + \lambda_s,$$

where $\lambda_1, \ldots, \lambda_s \in \mathrm{Irr}(T)$ are pairwise distinct of the same degree, and

$$T \cap \ker \chi = \ker \chi_T = \bigcap_{i=1}^{s} \ker \lambda_i.$$

Since $G/\ker \chi$ is non-abelian then $G/(T \cap \ker \chi)$ is non-abelian. Hence all λ_i are non-linear since $T' = G'$. So $\lambda_i^G = \chi$ by reciprocity and Lemma 1. In particular $n \mid \chi(1)$ for all $\chi \in \mathrm{Irr}_1(G)$. Therefore for all prime divisors p of n the group G has a normal p-complement [4, corollary 12.2]. So $H = G'$ is a Hall subgroup of G.

Take a non-principal $\lambda \in \mathrm{Irr}(H)$. Since G is a D-group then $\lambda^G = e\chi$ by [4, problem 6.2]. Hence χ vanishes on $R^\#$ where R is a complement to H (R exists by Schur–Zassenhaus theorem). Then $|R| \mid \chi(1)$. Since R and H are Hall subgroups of G and

$$\lambda^G(1) = |G : H|\lambda(1) = |R|\lambda(1), \quad (|R|, \lambda(1)) = 1, \quad \lambda(1) \mid \chi(1)$$

then $\lambda^G(1) = \chi(1)$, $\lambda^G = \chi$ for all non-principal $\lambda \in \mathrm{Irr}(H)$. Therefore $G = (R, H)$ [5, corollary 37.5.4], H is nilpotent (Thompson [5, theorem 37.3.3]), a contradiction. □

Let a D_0-group be a group in which the degrees of the non-linear irreducible characters are distinct. Since all non-linear irreducible characters of D_0-group G are rational-valued, then $G' < G$[1] (it is a corollary of well-known Feit–Seitz theorem). Hence G is solvable by Theorem 9. We have obtained a new proof of the main theorem of note [1].

Remark. If for any $N \geq G'$ it follows from $1_N \neq \lambda \in \mathrm{Irr}(N)$ that the character λ^G is a D–character, then G is solvable or $G' = G$. My proof of this assertion uses the classification of finite simple groups.

Conjecture. Non-solvable $\overline{D}$–groups are simple.

A character χ of a group G is said to be a D_1–character if $|\mathrm{Irr}_1(\chi)| \geq |cd_1(\chi)| - 1$. A group G is said to be a D_1–group if for any non-identity normal subgroup N of G, $N \leq G'$, and for any non-principal $\lambda \in \mathrm{Irr}(N)$ the character λ^G is a D_1–character.

Question. Classify all D_1–groups.

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