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On the distribution mod 1 of $\alpha\sigma(n)$

By IMRE KÁTAI (Budapest)

Abstract. The sequence $x_n = F(n)+\alpha\sigma(n) \pmod 1$ is investigated, where $\sigma(n) =$ sum of divisors of n, F is an additive arithmetical function. In an earlier paper De Koninck and the author proved that $x_n \mod 1$ is uniformly distributed if the approximation type of α is finite, and formulated the conjecture that it holds for every irrational α.

In this paper it is proved that the conjecture is not true in general, and it is true if $\alpha \in \mathcal{K}^*$. $\mathcal{K}^*$ is defined as follows. Let $M_x = \prod_p p^{r_p}$, p runs over the primes and r_p is the integer part of the number stated in the right hand side of (2.7). Let $\mathcal{K} = \mathcal{K}_x$ be the set of those irrational α, for which $\min_{H|M_x} \|H\alpha\| x > 1$ holds for every large x, $\mathcal{K}^* = \{\alpha \mid j\alpha \in \mathcal{K}\}$ for every $j = 1, 2, \dots$.

§ 1. Introduction

According to a reformulated version of a well known theorem of H. Daboussi (see Daboussi and Delange [1], [2]), for every additive arithmetical function $F(n)$ and any irrational α, the sequence $l_n := F(n) + \alpha n$ is uniformly distributed modulo 1. This famous theorem has a plenty of generalizations. It was proved in [3] that the same assertion holds for $l_n = F(n) + Q(n)$, where $Q(x) = \alpha_0 + \alpha_1 x + \dots + \alpha_k x^k \in \mathbb{R}[x]$, and at least one of $\alpha_1, \dots, \alpha_k$ is irrational.

Let $\|x\|$ stand for the distance between x and the closest integer. In [4] we proved the following result.

Theorem A. *Let α be a positive irrational number such that for each real*

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number $\kappa > 1$ there exists a positive constant $c = c(\kappa, \alpha)$ for which the inequality

$$\|\alpha q\| > \frac{c}{q^\kappa} \tag{1.1}$$

holds for every positive integer q. Then $l_n = \alpha\sigma(n) + F(n)$ is uniformly distributed modulo 1 for every additive arithmetical function $F(n)$. Here $\sigma(n)$ is the sum of divisors of n.

We mentioned that similar assertion can be proved for the integer valued multiplicative function h instead of σ, where $h(p) = Q(p)$ for every prime p and $h(p^a) = \mathcal{O}(p^{ad})$ for some fixed number d for every prime p and every integer $a \geq 2$, where

$$Q(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0 \in \mathbb{Z}[x],$$

$k \geq 1, a_k > 0$.

Especially, it is true for $h(n) = \varphi(n)$, where $\varphi(n)$ is Euler's totient function.

We formulated the conjecture that the assertion of Theorem A is true for every irrational α. We shall improve our theorem, and show that our conjecture is not true in general.

In [5] the function

$$\Delta(\alpha, x) = \frac{1}{\pi_2(x)} \max_{|X_p| \leq 1} \left| \sum_{\substack{p_1 p_2 \leq x \\ p_1 < p_2}} X_{p_1} X_{p_2} e(\alpha p_1 p_2) \right|$$

has been considered, where

$$\pi_2(x) = \sum_{\substack{p_1 p_2 \leq x \\ p_1 < p_2}} 1.$$

It was proved that $\Delta(\alpha, x) \to 0$ if α is not too well approximable by rationals, and the conjecture was formulated that this is true for every irrational α. Huixue Lao [6] proved that $\Delta(\alpha, x) \to 0$ if α was of finite type, in the sense that there exists a positive number σ such that $\|n\alpha\| > n^{-\sigma}$ for all sufficiently large n.

Finally Professor Glyn Harman [7] proved the conjecture. The main novelty of his method was that he could handle the case by the so called "major arc" estimate when α was very well approximable by rationals. Combining our method with his method we are able to prove the following theorems.

§ 2. Formulation of the results

2.1. Let $\mathcal{P}$ be the whole set of primes, p, q with or without indices denote prime numbers.

It is known that

$$\sum_{\substack{p\equiv -1 \pmod k\\ p<x}} \frac{1}{p} \le c_1 \frac{\log\log x}{\varphi(k)} \quad (k\in\mathbb{N}) \tag{2.1}$$

if $x>10$ (say), where c_1 is an absolute constant. (For a proof see [11]).

Let $c>0$ be a constant, $x>10$. Assume that S is such an integer depending on x for which

$$\#\{n\le x \mid \sigma(n)\equiv 0 \pmod S\} > cx. \tag{2.2}$$

Let us write every n as Km, where K is the square full part and m is the square free part of n. It is clear that

$$\#\{n\le x \mid K>Y\} \le \sum_{K>Y} \frac{x}{K} \le \frac{c_2 x}{\sqrt{Y}}. \tag{2.3}$$

Assume that $Y=\left(\frac{2c_2}{c}\right)^2$.

Then

$$\#\{n\le x \mid n=Km,\ K<Y,\ \sigma(K)\sigma(m)\equiv 0 \pmod S\} > \frac{c}{2}x. \tag{2.4}$$

Let us write

$$S=\prod p^{\gamma_p}. \tag{2.5}$$

Assume that $p>c_3\log\log x$. If $\gamma_p\ge 1$, then counting the integers $m\le x$ in (2.4) satisfying $p\mid\sigma(m)$ is bounded by (see (2.1))

$$\sum_{\substack{q\equiv -1 (p)\\ q\le x}} \frac{x}{q} < \frac{c_1 x\log\log x}{p-1}$$

and the right hand side should be larger than $\frac{c}{2}x$. Thus $p-1<\frac{2c_1}{c}\log\log x$. We proved that $\gamma_p=0$ if $p\ge\frac{2c_1}{c}\log\log x+1$.

Let now $Y<p\le\frac{2c_1}{c}\log\log x+1$. We count those integers n for which $p^{\gamma_p}\mid\sigma(n)$, i.e. $p^{\gamma_p}\mid\sigma(m)$. This can be overestimated by the sum

$$x\sum_{r=1}^{\gamma_p}\sum_{\alpha_1+\dots+\alpha_r=\gamma_p}\sum_{\substack{q_1<\dots<q_r<x\\ p^{\alpha_j}\mid q_j-1}}\frac{1}{q_1\dots q_r}\le x\sum_{r=1}^{\gamma_p}\frac{1}{r!}\sum_{\alpha_1+\dots+\alpha_r=\gamma_p}\frac{(c_1\log\log x)^r}{\varphi(p^{\alpha_1})\dots\varphi(p^{\alpha_r})}$$

$$\le \frac{2x}{\gamma_p!}\left(\frac{c_1\log\log x}{p-1}\right)^{\gamma_p}.$$

This number should be larger than cx, consequently

$$\frac{2}{\gamma_p!}\left(\frac{c_1\log\log x}{p-1}\right)^{\gamma_p} > c,$$

i.e.

$$\gamma_p \le \frac{c_1 e\log\log x}{p-1} \quad \text{if } x \text{ is large enough.} \tag{2.6}$$

Assume that $p < Y$. If $p^{\gamma_p} \mid \sigma(n)$, then $p^{\gamma_p-\Lambda} \mid \sigma(m)$, where $p^{\Lambda} \mid \sigma(K) \ll K\log\log K \ll Y\log\log Y$, whence we obtain that $\Lambda < c_4$. Furthermore as above we obtain that $\gamma_p - \Lambda \le \frac{c_1 e\log\log x}{p-1}$ and so we have

$$\gamma_p \le \begin{cases} \dfrac{c_1 e\log\log x}{p-1} & \text{if } Y < p \le \frac{2c_1}{c}\log\log x + 1 \\ \dfrac{c_1 e\log\log x}{p-1} + c_4 & \text{if } p \le Y \\ = 0 & \text{if } p > \dfrac{2c_1}{c}\log\log x + 1. \end{cases} \tag{2.7}$$

We proved

Lemma 1. *Assume that $c > 0$ is a constant, and with some integer S*

$$\frac{1}{x}\#\{n \le x \mid \sigma(n) \equiv 0 \pmod{S}\} > c.$$

Let $S = \prod p^{\gamma_p}$. Assume that x is large enough, $x > y_0$. Then for the exponents γ_p (2.7) hold true. Consequently $S \ll \exp(c_1 e(\log\log x)[(\log\log\log x) + c_5])$ with suitable constants c_1, c_5.

Remark. For some m let $\alpha_p(m)$ be that exponent for which $p^{\alpha_p(m)} \| \sigma(m)$. One can prove that

$$\alpha_p(m) > \frac{\log\log x}{(p-1)\,p} - \left(\frac{\log\log x}{p}\right)^{\frac{3}{4}} \tag{2.8}$$

holds for $2 \le p \le \sqrt{\log\log x}$ for all but $o(x)$ integers $m \le x$. Let $l_p :=$ integer part of the right hand side of (2.8).

Let $T = T_x = \prod_{p \le \sqrt{x_2}} p^{l_p}$. Then

$$\frac{1}{x}\#\{n \le x \mid \sigma(n) \equiv 0 \pmod{T}\} = (1 + o_x(1))\,.$$

As a consequence we have

Lemma 2. *Let α be such an irrational number for which there is a sequence $x_1 < x_2 < \dots$ tending to infinity and there is a sequence of integers $\mathcal{D}_\nu$ dividing T_ν such that $\|\alpha\mathcal{D}_\nu\| x_\nu \to 0$ ($\nu \to \infty$). Then*

$$\frac{1}{x_\nu} \sum_{n \le x_\nu} e\left(\alpha\sigma(n)\right) \to 1 \quad (\nu \to \infty)$$

and for all $\varepsilon > 0$:

$$\frac{1}{x_\nu} \#\{n \le x_\nu \mid \|\alpha\sigma(n)\| > \varepsilon\} \to 0 \quad (\nu \to \infty)\,. \tag{2.9}$$

Remark. Let $\alpha = \frac{1}{2^{t_1}} + \frac{1}{2^{t_2}} + \dots$, where $t_{k+1} = 2^{3^{t_k}}$. Assume that $t_1, \dots, t_k$, $x_1, \dots, x_{k-1}$ are chosen. Let $x_k = \exp(\exp(t_k + 1))$, $t_{k+1} > 3\exp(t_k + 1)$. Then

$$\alpha\sigma(n) = \left(\frac{1}{2^{t_1}} + \dots + \frac{1}{2^{t_k}}\right)\sigma(n) + \left(\frac{1}{2^{t_{k+1}}} + \dots\right)\sigma(n) = u(n) + v(n),$$

and for $n \le x_k$, $u(n)$ is integer for all but $o(x_k)$ integers, and from the known inequality $\sigma(n) \le n \log\log n$ we obtain that

$$v(n) \le \frac{2x_k \log\log x_k}{2^{t_{k+1}}} = \frac{2x_k(t_k + 1)}{2^{t_{k+1}}} = \eta_k,$$
$$\log \eta_k = \exp(t_k + 1) + \log(t_k + 1) - t_{k+1}\log 2 + \log 2,$$

and since $t_{k+1}\log 2 > 3(\log 2)\exp(t_k + 1)$, we have

$$\log \eta_k \le \left(\log \frac{e}{8}\right)\exp(t_k + 1) + \log(t_k + 1) \to -\infty,$$

and so $\eta_k \to 0$.

We clearly have (2.9).

2.2. Let

$$M_x = \prod_p p^{r_p},$$

where r_p is defined by the integer parts of the numbers stated on the right hand side of (2.7).

Let $\mathcal{K}$ be the set of those irrational α, for which

$$\min_{H | M_x} \|H\alpha\| x > 1 \tag{2.10}$$

holds for every large x. Let $\mathcal{K}^*$ be the set of those α for which $j\alpha \in \mathcal{K}$ for every $j \in \mathbb{N}$.

Theorem 1. *Let $\alpha \in \mathcal{K}^*$. For some additive arithmetical function F let $l_n = F(n) + \alpha\sigma(n)$. Then the sequence l_n $(n \in \mathbb{N})$ is uniformly distributed* mod 1, *and the discrepancy can be overestimated by a sequence of real numbers ϱ_x, which does not depend on F, and $\varrho_x \to 0$.*

Another formulation of Theorem 1 is

Theorem A'. *Let $\alpha \in \mathcal{K}^*$, $\tilde{\mathcal{M}}$ = set of complex valued multiplicative functions satisfying $|f(n)| \le 1$. Then*

$$\sup_{f\in\tilde{\mathcal{M}}} \frac{1}{x}\left|\sum_{n\le x} f(n)e\left(\sigma(n)\alpha\right)\right| \to 0 \quad (as \quad x\to\infty)\,.$$

Remark. The above theorems remain valid if we change $\sigma(n)$ by $\varphi(n)$, where $\varphi(n)$ is Euler's totient function.

2.3. Let $Q \ge 3$ be an integer $(1 \le)l_1 < \cdots < l_h(< Q)$ be coprime to Q, $h < \varphi(Q)$. Let $\mathcal{B}$ be the semigroup generated by the prime numbers p belonging to the arithmetical progressions $\equiv l_j \pmod{Q}$ $(j = 1, \dots, h)$. Let $N_{\mathcal{B}}(x)$ be the number of elements of $\mathcal{B}$ less than or equal to x.

Theorem 2. *Let α be an irrational number. Then*

$$\lim_{x\to\infty} \frac{1}{N_{\mathcal{B}}(x)} \sup_{f\in\tilde{\mathcal{M}}} \left|\sum_{\substack{n\le x\\ n\in\mathcal{B}}} f(n)e(n\alpha)\right| = 0$$

§ 3. Proof of Theorem 1.

It is enough to prove that if $\alpha \in \mathcal{K}^*$, then

$$\frac{1}{x}\sum_{n\le x} e\left(\alpha\sigma(n)\right) \to 0 \quad (x\to\infty)\,. \tag{3.1}$$

The further part of the proof is the same as in [4].

Let $P(n)$ be the largest prime factor of n. Let $\varepsilon > 0$ be fixed. Writing each integer $n \le x$ as $n = pm$, where $P(n) = p$, we have that if $\mathcal{N}_1 = \mathcal{N}_1(x) := \{n \le x : P(n) \le x^\varepsilon\}$, then

$$\lim_{\varepsilon\to 0}\lim_{x\to\infty} \frac{1}{x}\#\mathcal{N}_1 = 0.$$

On the other hand the contribution of those integers n for which $P(n)^2 \mid n$ is negligible.

Let

$$\mathcal{N}_2 = \mathcal{N}_2(x) := \{n \le x : P(n) > x^\varepsilon,\ P(n)^2 \nmid n\}.$$

Let

$$\sum_{n\in\mathcal{N}_2} e\left(\alpha\sigma(n)\right) = \sum_{m\le x^{1-\varepsilon}} \Sigma_m,$$

where

$$\begin{aligned}\Sigma_m &= e\left(\alpha\sigma(m)\right)\left(\sum_{p<\frac{x}{m}} e(\alpha\sigma(m)p) - \sum_{p<P(m)} e(\alpha\sigma(m)p)\right)\\ &= e(\alpha\sigma(m))\big(\Sigma_m^{(1)} - \Sigma_m^{(2)}\big).\end{aligned}$$

In [4] we proved that $\sum \Sigma_m^{(2)} \ll \frac{1}{\varepsilon^2}\frac{x}{(\log x)^2} + \frac{1}{\varepsilon}\frac{x}{\log x} = o(x)$. Let $\tau = \frac{x}{(\log x)^{30}}$. In order to estimate $\Sigma_m^{(1)}$, we shall approximate $\alpha\sigma(m)$ by rational number $\frac{a_m}{q_m}$ satisfying

$$\left|\alpha\sigma(m) - \frac{a_m}{q_m}\right| < \frac{1}{q_m\tau}, \quad 1 \le q_m < \tau.$$

In [4] we deduced from a theorem of I. M. Vinogradov that

$$\Sigma_m^{(1)} \ll \frac{\frac{x}{m}}{\log^2\frac{x}{m}}, \quad \text{if } q_m > (\log x)^4 \tag{3.2}$$

(see Lemma 1 in [4]).

Assume now that $q_m \le (\log x)^4$. Let $\gamma = \alpha\sigma(m)$. Then $\left|\gamma - \frac{a_m}{q_m}\right| < \frac{1}{q_m\tau}$, $\Sigma_m = \sum_{p\le\frac{x}{m}} e(\gamma p)$.

By using Lemma 3.1 in Vaughan [8], after partial summation we obtain that

$$\Sigma_m^{(1)} \ll \frac{x}{q_m m(\log\frac{x}{m})}. \tag{3.3}$$

The sum of $|\Sigma_m^{(1)}|$ with $q_m > \left(\frac{1}{\varepsilon}\right)^2$ is smaller than $\ll \varepsilon\frac{x}{\log x}\sum\frac{1}{m} \le \varepsilon c x$ with an absolute constant c. To prove (3.1) it is enough to prove that for every fixed $l \le \left(\frac{1}{\varepsilon}\right)^2$, the number of those $m \le x^{1-\varepsilon}$ for which $q_m = l$ is $o(x^{1-\varepsilon})$, as $x \to \infty$.

It is enough to prove that for every $U \in [x^\varepsilon, x^{1-\varepsilon}]$ and for every $l \le \left(\frac{1}{\varepsilon}\right)^2$ the number of $m \in [U, 2U]$ satisfying $q_m = l$ is $o_x(1)U$.

Let $(U \le)m_1, \dots, m_R(< 2U)$ be those numbers for which $\|l\alpha\sigma(m_j)\| < \frac{1}{\tau}$. Let $\beta = l\alpha$. Since $\alpha \in \mathcal{K}^*$, therefore $\beta \in \mathcal{K}^*$, and so $\beta \in \mathcal{K}$. Let R_{m_j} be the integer closest to $\beta\sigma(m_j)$. Then $\left|\beta - \frac{R_{m_j}}{\sigma(m_j)}\right| < \frac{1}{\sigma(m_j)\tau}$. If $\frac{R_{m_i}}{\sigma(m_i)} \ne \frac{R_{m_j}}{\sigma(m_j)}$, $(i \ne j)$ then

$$\frac{1}{\sigma(m_1)\sigma(m_2)} \le \left|\frac{R_{m_1}}{\sigma(m_1)} - \frac{R_{m_2}}{\sigma(m_2)}\right| < \frac{1}{\tau}\left(\frac{1}{\sigma(m_1)} + \frac{1}{\sigma(m_2)}\right)$$

which is impossible. Thus $\frac{R_{m_j}}{\sigma(m_j)} = \frac{R}{S}$ $(j = 1, \dots, T)$, $(R, S) = 1$. Thus $R\sigma(m_j) \equiv 0 \mod S$, and so $\sigma(m_j) \equiv 0 \mod S$ $(j = 1, \dots, T)$. Let us assume that $T > cU$ with a positive constant. Then S should be very special: $S \mid M_U$. But this is impossible since $\beta \in \mathcal{K}$. The proof is completed.

§ 4. Proof of Theorem 2

The proof depends on the following

Lemma 3. *Let α be an irrational number. Then*

$$\frac{1}{N_{\mathcal{B}}(x)} \sum_{\substack{n \le x \\ n \in \mathcal{B}}} e(\alpha n) \to 0 \quad (x \to \infty), \tag{4.1}$$

consequently the sequence $n\alpha$ $(n \in \mathcal{B})$ is uniformly distributed $\mod 1$.

First we deduce Theorem 2 from Lemma 3.

Let

$$S(x) = \sum_{\substack{n \le x \\ n \in \mathcal{B}}} f(n) e(n\alpha). \tag{4.2}$$

Let K be a positive number, $p_1 < \cdots < p_T$ be such primes $p_j \in \mathcal{B}$ for which

$$A_K := \sum_{j=1}^{T} \frac{1}{p_j} > K,$$

and $p_1 > K^2$.

Let $\mathcal{P}_K = \{p_1, \dots, p_T\}$,

$$\omega_{\mathcal{P}_K}(n) = \sum_{\substack{p \mid n \\ p \in \mathcal{P}_K}} 1. \tag{4.3}$$

One can prove the analogue of the Turán–Kubilius inequality, namely that

$$\frac{1}{N_{\mathcal{B}}(x)} \sum_{\substack{n \le x \\ n \in \mathcal{B}}} \left(\omega_{\mathcal{P}_K}(n) - A_K\right)^2 \le C A_K, \tag{4.4}$$

whence we obtain that

$$\frac{1}{N_{\mathcal{B}}(x)} \sum_{\substack{n \le x \\ n \in \mathcal{B}}} |\omega_{\mathcal{P}_K}(n) - A_K| \le \sqrt{C A_K}. \tag{4.5}$$

Let

$$S_1(x) = \sum_{\substack{n \le x \\ n \in \mathcal{B}}} f(n) e(n\alpha) \omega_{\mathcal{P}_K}(n). \tag{4.6}$$

We have

$$|A_K S(x)| \le |S_1(x)| + \sqrt{CA_K} N_{\mathcal{B}}(x). \tag{4.7}$$

We have

$$S_1(x) = \sum_{\substack{pm \le x \\ pm \in \mathcal{B} \\ p \in \mathcal{P}_K}} f(pm)\, e(pm\alpha). \tag{4.8}$$

Let

$$S_2(x) = \sum_{\substack{pm \le x \\ pm \in \mathcal{B} \\ p \in \mathcal{P}_K}} f(p) f(m) e(pm\alpha). \tag{4.9}$$

We have

$$|S_1(x) - S_2(x)| \le 2 \sum_{\substack{p^2\nu \le x \\ p^2\nu \in \mathcal{B}}} 1 \le c N_{\mathcal{B}}(x) \sum_{j=1}^{T} \frac{1}{p_j^2} \le \frac{c}{K} N_{\mathcal{B}}(x). \tag{4.10}$$

Furthermore

$$S_2(x) = \sum_{m \le \frac{x}{p_1}} f(m) \Sigma_m, \qquad \Sigma_m = \sum_{p_j \le \frac{x}{m}} f(p_j)\, e(\alpha p_j m). \tag{4.11}$$

Thus

$$|S_2(x)|^2 \le \sum_{\substack{m \le \frac{x}{p_1} \\ m \in \mathcal{B}}} |f(m)|^2 \sum_{\substack{m \le \frac{x}{p_1} \\ m \in \mathcal{B}}} |\Sigma_m|^2 = S \cdot H. \tag{4.12}$$

It is clear that $S \ll N_{\mathcal{B}}\big(\frac{x}{p_1}\big) \ll \frac{1}{p_1} N_{\mathcal{B}}(x)$.

$$H = \sum_{m \in \mathcal{B}} \sum_{\substack{p_i, p_j \le \frac{x}{m} \\ p_i, p_j \in \mathcal{B}}} f(p_i) \overline{f(p_j)} e\left((p_i - p_j)\alpha m\right) = H_1 + H_2.$$

In H_1 we sum over those m, p_i, p_j for which $i = j$, consequently

$$H_1 \ll \sum_{j=1}^{T} N_{\mathcal{B}}\left(\frac{x}{p_j}\right) \le c N_{\mathcal{B}}(x) \sum_{j=1}^{T} \frac{1}{p_j},$$

i.e.

$$H_1 \le c_3 A_K N_{\mathcal{B}}(x),$$

where c_3 is an absolute constant.

In H_2 we sum over those (m, p_i, p_j) for which $i \ne j$. Changing the order of summation, for fixed i, j

$$\sum_{\substack{m \le \min\left(\frac{x}{p_i}, \frac{x}{p_j}\right) \\ m \in N_{\mathcal{B}}(x)}} e\left(\alpha(p_i - p_j)m\right)$$

is $o(N_{\mathcal{B}}(x))$ due to Lemma 3, since $\alpha(p_i - p_j)$ is an irrational number also.

Hence we obtain that, for every large x, with a constant C,

$$\left| \frac{A_K S(x)}{N_{\mathcal{B}}(x)} \right| \le \sqrt{C A_K} + 2\frac{A_K}{K^2} + \frac{2}{K}\sqrt{A_K}, \tag{4.13}$$

consequently

$$\limsup_{x \to \infty} \frac{|S(x)|}{N_{\mathcal{B}}(x)} \le \frac{\sqrt{C}}{\sqrt{A_K}} + \frac{2}{K^2} + \frac{2}{K\sqrt{A_K}}. \tag{4.14}$$

Since K is arbitrarily large, $A_K > K$, we obtain that the left hand side of (4.14) is 0.

Finally we prove Lemma 3.

Let $\varepsilon > 0$ be fixed. Let us write each $n \in \mathcal{B}$ as $n = pm$, where $P(n) = p$ is the largest prime factor of n. Let $\mathcal{N}_1 = \mathcal{N}_1(x) := \{n \le x,\ n \in \mathcal{B},\ P(n) \le x^{\varepsilon}\}$. On can prove easily that

$$\#\mathcal{N}_1(x) \ll \varepsilon N_{\mathcal{B}}(x).$$

According to a well kown theorem due to Wirsing [10]

$$N_{\mathcal{B}}(x) = c\frac{x}{(\log x)^E}\left(1 + o_x(1)\right)$$

$$E = 1 - \frac{h}{\varphi(Q)},$$

we obtain that $N_{\mathcal{B}}\left(\frac{x}{u}\right) \le \frac{c}{u} N_{\mathcal{B}}(x)$ if $1 \le u \le x^{\lambda}$, where $(0 \le)\lambda < 1$, $c = c(\lambda)$. Let $\mathcal{N}_2 = \mathcal{B} \setminus \mathcal{N}_1$. Then

$$\sum_{\substack{n \in \mathcal{N}_2 \\ n \le x}} e(\alpha n) = \sum_{\substack{m \le x^{1-\varepsilon} \\ m \in \mathcal{B}}} \sum_{\substack{P(m) < p < \frac{x}{m} \\ p \in \mathcal{B}}} e(\alpha p m) = \sum_m \Sigma_m.$$

Further write

$$\Sigma_m = \sum_{\substack{p<\frac{x}{m}\\ p\in\mathcal{B}}} e(\alpha mp) - \sum_{\substack{p<P(m)\\ p\in\mathcal{B}}} e(\alpha mp) = \Sigma_m^{(1)} - \Sigma_m^{(2)}.$$

Assume that $1 \le m \le x^{1-\varepsilon}$. Let $\tau_m = \frac{x}{m}(\log x)^{-40}$,

$$\left|\alpha m - \frac{a_m}{q_m}\right| \le \frac{1}{q_m\tau_m}, \quad 1 \le q_m \le \tau_m,\ (a_m, q_m) = 1. \tag{4.15}$$

According to a result of A. Balog and A Perelli [9], if $\left|\alpha - \frac{a}{q}\right| \le \frac{2}{N}$, $(a,q)=1$, $h=(q,d)$, then

$$\sum_{\substack{n\le N\\ n\equiv f \pmod d}} \Lambda(n)e(n\alpha) \ll \left(\frac{hN}{d\sqrt{q}} + \frac{\sqrt{qN}}{\sqrt{h}} + \left(\frac{N}{d}\right)^{\frac{4}{5}}\right)(\log N)^3. \tag{4.16}$$

By using partial integration, the inequality remains true, if the left hand side is changed to

$$\sum_{\substack{p\le N\\ p\equiv f \pmod d}} e(p\alpha). \tag{4.17}$$

Let us apply the inequality (4.16), (4.17) for αm instead of α, and for $d=Q$, $f=l_j$, in the case if $\tau_m \ge (\log x)^{40}$.

Since l_j, Q, d are bounded as $x\to\infty$, we obtain that

$$\Sigma_m^{(1)} \ll \left\{\frac{x}{m\sqrt{q_m}} + \left(\frac{x}{m}q_m\right)^{\frac{1}{2}} + \left(\frac{x}{m}\right)^{\frac{4}{5}}\right\}(\log x)^3 \ll \frac{x}{m(\log x)^{17}} + \frac{x^{\frac{4}{5}}}{m^{\frac{4}{5}}},$$

and so

$$\sum_{q_m>(\log x)^{40}} |\Sigma_m^{(1)}| \ll \frac{x}{(\log x)^{16}}.$$

It is clear that

$$\frac{1}{\varphi(Q)}\sum_{a=0}^{Q-1} e\left(\frac{(p-l)a}{Q}\right) = \begin{cases} 1, & \text{if } p\equiv l \pmod Q,\\ 0, & \text{otherwise.}\end{cases}$$

We have

$$\Sigma_m^{(1)} = \sum_{j=1}^{h}\frac{1}{\varphi(Q)}\sum_{a=0}^{Q-1} e\left(\frac{-al_j}{Q}\right)\sum_{p<\frac{x}{m}} e\left(\left(\alpha m + \frac{a}{Q}\right)p\right)$$

whence

$$|\Sigma_m^{(1)}| \le \sum_{a=0}^{Q-1} \left| \sum_{p<\frac{x}{m}} e\left(\left(\alpha m + \frac{a}{Q}\right)p\right)\right| = \sum_{a=0}^{Q-1} |\Sigma_{m,a}^{(1)}|.$$

If (4.15) holds, then

$$\left|\alpha m + \frac{a}{Q} - \frac{E_m}{L_m}\right| < \frac{1}{q_m \tau_m},$$

where $\frac{E_m}{L_m} = \frac{a_m}{q_m} + \frac{a}{q}$, $(E_m, L_m) = 1$. It is clear that $\frac{q_m}{Q} \le L_m \le q_m Q$ if $q_m > Q$.

Let us assume that $q_m \le (\log x)^{40}$. By using Lemma 3.1 in Vaughan [8], after partial summation, we obtain that

$$\Sigma_{m,a}^{(1)} \ll \frac{x}{L_m m \log \frac{x}{m}}.$$

Hence we obtain that

$$\sum_{L_m > R} \Sigma_{m,a}^{(1)} \ll \frac{x}{R} \sum_{m \le x^{1-\varepsilon}} \frac{1}{m \log \frac{x}{m}} \ll \frac{N_{\mathcal{B}}(x)}{R}.$$

Let a be any of $a = 0, 1, \dots, Q-1$.
Let l be a fixed integer, and consider those m for which $q_m = l$. Let $x^\varepsilon \le U \le x^{1-\varepsilon}$, and consider the set of integers $m \in \mathcal{B}$ in $[U, 2U]$ for which

$$\left|m\alpha - \frac{a_m}{l}\right| < \frac{1}{l\tau_m}.$$

Assume that these numbers are $m_1, \dots, m_T$. Then $\left|l\alpha - \frac{a_m}{m}\right| < \frac{1}{m_\tau}$. If $\left|l\alpha - \frac{a_{m_u}}{m_u}\right| < \frac{1}{m_u \tau_m}$ holds for $u = i, j$, then

$$\left|\frac{a_{m_j}}{m_j} - \frac{a_{m_i}}{m_i}\right| < \frac{2}{\tau_{m_i}}\left(\frac{1}{m_i} + \frac{1}{m_j}\right)$$

which implies that $\frac{a_{m_u}}{m_u} = \frac{R}{S}$ $(u = 1, \dots, T)$, $(R, S) = 1$. Thus $R_{m_u} \equiv 0 \pmod S$ $(u = 1, \dots, T)$. S cannot be bounded as $x \to \infty$. Hence we obtain that

$$T \le \#\{m \in [U, 2U],\ m \in \mathcal{B},\ m \equiv 0 \pmod S\} \le \frac{N_{\mathcal{B}}(2U)}{S}.$$

Since α is irrational, therefore $S \to \infty$ as $x \to \infty$ uniformly as U varies in $[x^\varepsilon, x^{1-\varepsilon}]$. Thus we proved that

$$\sum_{\substack{m \\ m \le x^{1-\varepsilon} \\ m \in \mathcal{B}}} |\Sigma_m^{(1)}| = o_x(1) N_{\mathcal{B}}(x).$$

In order to estimate $\Sigma_m^{(2)}$, observe that $|\Sigma_m^{(2)}| \le c\frac{P(m)}{\log 2P(m)}$, and so

$$\sum_m \left|\Sigma_m^{(2)}\right| \le c \sum_{\substack{p\le x^{1-\varepsilon}\\ p\in\mathcal{B}}} \frac{p}{\log p} \sum_{\substack{pr\le x^{1-\varepsilon}\\ P(r)\le p\\ p^2 r\le x\\ r\in\overline{\mathcal{B}}}} 1 = \Sigma_A + \Sigma_B.$$

In Σ_A we sum over $p \le x^\varepsilon$, and in Σ_B over $p > x^\varepsilon$. Then

$$\Sigma_A \ll \sum_{\substack{p\le x^\varepsilon\\ p\in\mathcal{B}}} \frac{p}{\log p}\cdot N_{\mathcal{B}}\left(\frac{x^{1-\varepsilon}}{p}\right) \ll N_{\mathcal{B}}(x^{1-\varepsilon}) \sum_{p\le x^\varepsilon} \frac{1}{\log p}$$
$$\ll \frac{x^\varepsilon}{\varepsilon\log x} N_{\mathcal{B}}(x^{1-\varepsilon}) \ll \frac{N_{\mathcal{B}}(x)}{\varepsilon\log x} = o_x(1)N_{\mathcal{B}}(x),$$

Furthermore,

$$\Sigma_B \ll \sum_{x^\varepsilon<p\le x} \frac{p}{\log p} \sum_{\substack{r\le \frac{x}{p^2}\\ P(r)\le p\\ r\in\mathcal{B}}} 1 \ll \sum_{x^\varepsilon<p\le x} \frac{p}{\log p}\cdot\frac{x}{p^2} \ll \frac{1}{\varepsilon}\frac{x}{\log x} = o_x(1)N_{\mathcal{B}}(x).$$

The proof is completed.

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IMRE KÁTAI
DEPARTMENT OF COMPUTER ALGEBRA
EÖTVÖS LORÁND UNIVERSITY
PÁZMÁNY PÉTER SÉTÁNY 1/C
H-1117 BUDAPEST
HUNGARY

E-mail: katai@compalg.inf.elte.hu

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