Publ. Math. Debrecen
77/1-2 (2010), 15–31

Functional equations and functional relations for the Euler double zeta-function and its generalization of Eisenstein type

By YASUSHI KOMORI (Nagoya), KOHJI MATSUMOTO (Nagoya)
and HIROFUMI TSUMURA (Tokyo)

Abstract. We consider certain double series in two variables such as the Euler double zeta-function and its generalization of Eisenstein type. In the former part, we give some functional equations among these series, which are Eisenstein type analogues of a previous result on double zeta-functions given by the second-named author. We point out that, on certain hyperplanes, we can show functional equations of traditional symmetric type for these double series. In the latter part, we give some functional relations for these series and double series of another type involving hyperbolic functions. As special cases, we can obtain the known value-relation formulas for these series given by the third-named author recently.

1. Introduction

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the ring of rational integers, $\mathbb{R}$ the field of real numbers, $\mathbb{C}$ the field of complex numbers and $i = \sqrt{-1}$.

For $\omega_1, \omega_2 \in \mathbb{C}$, we define

$$\zeta_2(s_1, s_2; \omega_1, \omega_2) = \sum_{m=1}^{\infty} \frac{1}{(m\omega_1)^{s_1}} \sum_{n=1}^{\infty} \frac{1}{(m\omega_1 + n\omega_2)^{s_2}} \tag{1}$$

for $s_1, s_2 \in \mathbb{C}$. Note that here, and throughout this paper, we interpret z^s as $e^{s \log z}$, where $\log z = \log|z| + i \arg z$ with $-\pi < \arg z \leq \pi$ unless otherwise indicated. To ensure the convergence of (1), we assume the following condition.

Mathematics Subject Classification: 11M41, 11M99.
Key words and phrases: double zeta-functions, functional equation, Eisenstein series, Riemann zeta-function.

Let ℓ be any line on the complex plane crossing the origin. Then ℓ divides the plane into two half-planes. Let $H(\ell)$ be one of those half-planes, not including ℓ itself. Using this notation, we assume that $\omega_1, \omega_2 \in H(\ell)$ for some ℓ. Then, under this assumption, we see that the series (1) is absolutely convergent in the region

$$\{(s_1, s_2) \in \mathbb{C}^2 \mid \Re(s_1 + s_2) > 2,\ \Re s_2 > 1\} \tag{2}$$

(see [12, Theorem 3]). The meromorphic continuation of (1) to the whole space $\mathbb{C}^2$ was first established by Atkinson [3] in the case $\omega_1 = \omega_2 = 1$ in his study on the mean square of the Riemann zeta-function $\zeta(s)$; for general case, see [13].

In the former part of this paper, we consider a certain "functional equation" for $\zeta_2(s_1, s_2; \omega_1, \omega_2)$. First, in Theorem 2.1, we give a generalization of the previous result given by the second-named author in [14], that is, a functional equation for the Euler double zeta-function which is written in terms of confluent hypergeometric functions.

Next, by restricting this result to certain hyperplanes, we give a symmetric form of functional equation for $\zeta_2(s_1, s_2; \omega_1, \omega_2)$ (see Theorem 2.2). More precisely, if we put

$$\begin{aligned} \xi(s_1, s_2; \omega_1, \omega_2) &= \left(\frac{2\pi i}{\omega_1\omega_2}\right)^{\frac{1-s_1-s_2}{2}} \Gamma(s_2) \\ \times \Bigg\{\zeta_2(s_1, s_2; \omega_1, \omega_2) &- \frac{\Gamma(1-s_1)}{\Gamma(s_2)}\Gamma(s_1+s_2-1)\zeta(s_1+s_2-1)\omega_1^{-1}\omega_2^{1-s_1-s_2}\Bigg\}, \end{aligned} \tag{3}$$

we can see that

$$\xi(s_1, s_2; \omega_1, \omega_2) = \xi(1-s_2, 1-s_1; \omega_1, \omega_2) \tag{4}$$

on certain hyperplanes. This can be regarded as a double analogue of the functional equation for $\zeta(s)$.

As a corollary, we can evaluate $\zeta_2(s_1, s_2; \omega_1, \omega_2)$ at certain negative integers in terms of Bernoulli numbers (see Corollary 2.4).

In the latter part of this paper, we consider the double zeta-function of Eisenstein type

$$\widetilde{\zeta}_2(s_1, s_2; \tau) = \sum_{\substack{m\in\mathbb{Z} \\ m\neq 0}} \sum_{n\in\mathbb{Z}} \frac{1}{m^{s_1}(m+n\tau)^{s_2}} \tag{5}$$

and the hyperbolic-sine double series of Eisenstein type

$$\begin{aligned} \mathcal{S}_2(s_1, s_2; \tau) &= \sum_{m\neq 0} \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{\sinh(m\pi i/\tau) m^{s_1}(m+n\tau)^{s_2}} \\ &= \sum_{m\neq 0} \sum_{n\in\mathbb{Z}} \frac{1}{\sinh((m+n\tau)\pi i/\tau) m^{s_1}(m+n\tau)^{s_2}} \end{aligned} \tag{6}$$

for $\tau \in \mathbb{C}$ with $\Im\tau > 0$, where we slightly modify the definition of $\arg m$ for $m < 0$ as follows. In the term $m^{s_1}(m + n\tau)^{s_2}$ of each summand for $m < 0$ on the right-hand sides of (5) and (6), we define $\arg m = \pi$ (resp. $\arg m = -\pi$) when $n \geq 0$ (resp. $n < 0$). This definition can be seen as a natural one because $\lim_{\tau\to 0} \arg(m + n\tau) = \arg m$ for any $m < 0$. Furthermore this definition makes the expression given in Lemma 3.1 simpler.

We will prove certain functional relations among (5) and (6), for example,

$$\mathcal{S}_2(s, 3; \tau) = -\frac{\pi i}{6\tau}\widetilde{\zeta}_2(s, 2; \tau) + \frac{\tau}{\pi i}\widetilde{\zeta}_2(s, 4; \tau) \quad (s \in \mathbb{C})$$

(see Theorem 3.2). Therefore, combining this result and Theorem 2.1, we can show a functional equation for $\mathcal{S}_2(s, k; \tau)$ ($k \in \mathbb{N}$; $k \geq 2$) in terms of $F_\pm$ (see Theorem 3.5).

Using the well-known formula of Hurwitz ((35) below) for Eisenstein series, we can confirm that our functional relations include value relations which we proved previously (see [21, Example 4.1]), for example,

$$\mathcal{S}_2(0, 3; i) = \sum_{m\neq 0}\sum_{n\in\mathbb{Z}} \frac{(-1)^n}{\sinh(m\pi)(m + ni)^3} = \frac{1}{15}\frac{\varpi^4}{\pi} - \frac{7}{90}\pi^3 + \frac{1}{6}\pi^2, \tag{7}$$

$$\mathcal{S}_2(0, 5; i) = \sum_{m\neq 0}\sum_{n\in\mathbb{Z}} \frac{(-1)^n}{\sinh(m\pi)(m + ni)^5} = -\frac{1}{90}\varpi^4\pi + \frac{31}{2520}\pi^5 - \frac{7}{360}\pi^4, \tag{8}$$

where ϖ is the lemniscate constant defined by

$$\varpi = 2\int_0^1 \frac{dx}{\sqrt{1 - x^4}} = 2.622057\ldots.$$

Note that these formulas can be regarded as certain double analogues of the following classical result on the hyperbolic-sine zeta-function, studied by Cauchy, Mellin, Ramanujan, Berndt and so on (see [4], [5], [6], [7], [16], [17]):

$$\sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^m}{\sinh(m\pi)m^{4k+3}} = (2\pi)^{4k+3}\sum_{j=0}^{2k+2}(-1)^{j+1}\frac{B_{2j}(1/2)}{(2j)!}\frac{B_{4k+4-2j}(1/2)}{(4k + 4 - 2j)!} \tag{9}$$

for $k \in \mathbb{N}_0$, where $B_n(x)$ is the n-th Bernoulli polynomial defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}. \tag{10}$$

2. Functional equations

In this section, we give functional equations for $\zeta_2(s_1, s_2; \omega_1, \omega_2)$.

First we recall the definition of the confluent hypergeometric function

$$\Psi(a, c; x) = \frac{1}{\Gamma(a)} \int_0^{e^{i\theta}\infty} e^{-xy} y^{a-1} (1+y)^{c-a-1} dy, \tag{11}$$

where $\Re a > 0$, $-\pi < \theta < \pi$, $|\theta + \arg x| < \pi/2$ (see Erdélyi et al. [8, formula 6.5 (3)]). Define

$$F_{\pm}(s_1, s_2; \tau) = \sum_{k=1}^{\infty} \sigma_{s_1+s_2-1}(k) \Psi(s_2, s_1 + s_2; \pm 2\pi i k \tau), \tag{12}$$

where $\sigma_{s_1+s_2-1}(k) = \sum_{d|k} d^{s_1+s_2-1}$, and

$$g_0(s_1, s_2; \omega_1, \omega_2) = \frac{\Gamma(1-s_1)}{\Gamma(s_2)} \Gamma(s_1 + s_2 - 1) \zeta(s_1 + s_2 - 1) \omega_1^{-1} \omega_2^{1-s_1-s_2}.$$

Then $F_{\pm}$ can be continued meromorphically to $\mathbb{C}^2$ as a function in (s_1, s_2), and we obtain the following theorem.

Theorem 2.1. *Let $\omega_1, \omega_2 \in \mathbb{C}$ with $\Re\omega_1 > 0$ and $\Re\omega_2 > 0$. Then the following "functional equation" holds:*

$$\begin{aligned} \zeta_2(s_1, s_2; \omega_1, \omega_2) &= g_0(s_1, s_2; \omega_1, \omega_2) + \Gamma(1-s_1) \omega_1^{-1} \omega_2^{1-s_1-s_2} \\ &\quad \times \left\{ F_+\left(1-s_2, 1-s_1; \frac{\omega_2}{\omega_1}\right) + F_-\left(1-s_2, 1-s_1; \frac{\omega_2}{\omega_1}\right) \right\}. \end{aligned} \tag{13}$$

We discuss here why we call (13) the "functional equation". It is classically known that the Hurwitz zeta-function $\zeta(s, \alpha) = \sum_{n=0}^{\infty} (n+\alpha)^{-s}$ $(\alpha > 0)$ satisfies the functional equation

$$\zeta(s, \alpha) = \frac{\Gamma(1-s)}{i(2\pi)^{1-s}} \left\{ e^{\pi i s/2} \phi(1-s, \alpha) - e^{-\pi i s/2} \phi(1-s, -\alpha) \right\}, \tag{14}$$

where $\phi(s, \alpha) = \sum_{n=1}^{\infty} e^{2\pi i n \alpha} n^{-s}$ (see Titchmarsh [20, (2.17.3)]). Formula (13) may be regarded as the double analogue of (14). In fact, the asymptotic expansion

$$\Psi(a, c; x) = \sum_{j=0}^{N-1} \frac{(-1)^j (a-c+1)_j (a)_j}{j!} x^{-a-j} + \rho_N(a, c; x) \tag{15}$$

([8, formula 6.13.1(1)]), where $(a)_j = \Gamma(a+j)/\Gamma(a)$ and $\rho_N(a,c;x)$ is the remainder term, implies that $\Psi(s_2, s_1+s_2; \pm 2\pi i k\tau)$ can be approximated by $(\pm 2\pi i k\tau)^{-s_2}$ (the term corresponding to $j = 0$ in (15)), hence $F_\pm(s_1, s_2; \tau)$ can be approximated by the Dirichlet series

$$\sum_{k=1}^{\infty} \frac{\sigma_{s_1+s_2-1}(k)}{(\pm 2\pi i k\tau)^{s_2}}.$$

Therefore $F_\pm(s_1, s_2; \tau)$ may be considered as a "generalized Dirichlet series". From this viewpoint, formula (13) can be regarded as a duality formula among (generalized) Dirichlet series.

In other words, since $n^{-s} = F(s, 1, 1; 1-n)$ (where the right-hand side is the usual notation of the Gauss hypergeometric function), we see that a Dirichlet series is a special case of infinite series of hypergeometric functions. Therefore "functional equations" in general setting are perhaps to be understood as duality relations among infinite series of (confluent or non-confluent) hypergeometric functions.

The special case $\omega_1 = 1$ and $\omega_2 \in \mathbb{R}$ with $\omega_2 > 0$ of formula (13) is essentially included in the previous paper [14] of the second-named author. More generally, let

$$\zeta_2(s_1, s_2; \alpha, \beta, \omega_2) = \sum_{m=0}^{\infty} (\alpha+m)^{-s_1} \sum_{n=1}^{\infty} e^{2\pi i n\beta} (\alpha + m + n\omega_2)^{-s_2}, \tag{16}$$

where $0 < \alpha \le 1$, $0 \le \beta \le 1$ and $\omega_2 > 0$. The function $F_\pm(s_1, s_2; \alpha, \beta, \tau)$ is defined by replacing $\sigma_{s_1+s_2-1}(k)$ on the right-hand side of (12) by

$$\sigma_{s_1+s_2-1}(k; \alpha, \beta) = \sum_{d|k} e^{2\pi i d\alpha} e^{2\pi i (k/d)\beta} d^{s_1+s_2-1}.$$

Then the formula

$$\begin{aligned} \zeta_2(s_1, s_2; \alpha, \beta, \omega_2) = &\frac{\Gamma(1-s_1)}{\Gamma(s_2)} \Gamma(s_1+s_2-1) \phi(s_1+s_2-1, \beta) \omega_2^{1-s_1-s_2} \\ &+ \Gamma(1-s_1) \omega_2^{1-s_1-s_2} \\ &\times \left\{ F_+(1-s_2, 1-s_1; \beta, \alpha, \omega_2) + F_-(1-s_2, 1-s_1; \beta, -\alpha, \omega_2) \right\} \end{aligned} \tag{17}$$

can be immediately deduced from Propositions 1 and 2 of [14]. Formula (17) itself was not stated in [14]; it was first explicitly stated in [15].

The proof of Theorem 2.1 is just a simple modification of the proof of (17), hence we omit it.

Next we show that, restricting (13) to certain hyperplanes, we obtain a functional equation of traditional type. A key to the proof of this theorem is the fact that $\zeta(-2k) = 0$ for $k \in \mathbb{N}$.

Theorem 2.2. *For $\omega_1, \omega_2 \in \mathbb{C}$ with $\Re\omega_1 > 0$, $\Re\omega_2 > 0$, the hyperplane*

$$\Omega_{2k+1} := \{(s_1, s_2) \in \mathbb{C}^2 \mid s_1 + s_2 = 2k+1\} \quad (k \in \mathbb{Z} \setminus \{0\}) \tag{18}$$

is not a singular locus of $\zeta_2(s_1, s_2; \omega_1, \omega_2)$. On this hyperplane the following functional equation holds:

$$\begin{aligned}\left(\frac{2\pi i}{\omega_1\omega_2}\right)^{\frac{1-s_1-s_2}{2}} &\Gamma(s_2)\{\zeta_2(s_1, s_2; \omega_1, \omega_2) - g_0(s_1, s_2; \omega_1, \omega_2)\} \\ = \left(\frac{2\pi i}{\omega_1\omega_2}\right)^{\frac{s_1+s_2-1}{2}} &\Gamma(1-s_1) \\ &\times \{\zeta_2(1-s_2, 1-s_1; \omega_1, \omega_2) - g_0(1-s_2, 1-s_1; \omega_1, \omega_2)\}\end{aligned} \tag{19}$$

for $(s_1, s_2) \in \Omega_{2k+1}$ $(k \in \mathbb{Z} \setminus \{0\})$.

Using the notation (3), we can rewrite (19) as

$$\xi(s_1, s_2; \omega_1, \omega_2) = \xi(1-s_2, 1-s_1; \omega_1, \omega_2) \tag{20}$$

for $(s_1, s_2) \in \Omega_{2k+1}$ $(k \in \mathbb{Z} \setminus \{0\})$. This is a reflection formula involving two variables, and an interesting feature is that the roles of s_1 and s_2 are interchanged on the other side of the formula. It is important that we can obtain this type of functional equation by considering the function (1) of two variables. In particular when $\omega_1 = \omega_2 = 1$, we see that (19) gives the functional equation for the ordinary double zeta-function of Euler type. Essentially this case has been obtained by the second-named author in [14, (4.2)].

Proof of Theorem 2.2. Denote by $g(s_1, s_2; \omega_1, \omega_2)$ the second term on the right-hand side of (13). As for the first assertion, our task is to check that Ω_{2k+1} is not a singular locus of both $g(s_1, s_2; \omega_1, \omega_2)$ and $g_0(s_1, s_2; \omega_1, \omega_2)$. From (12) and (15) we obtain that for any non-negative integer N,

$$\begin{aligned}F_\pm(s_1, s_2; \tau) = &\sum_{j=0}^{N-1} \frac{(-1)^j}{j!}(1-s_1)_j(s_2)_j(\pm 2\pi i\tau)^{-s_2-j}\zeta(1-s_1+j)\zeta(s_2+j) \\ &+ (\pm 2\pi i\tau)^{1-s_1-s_2}\sum_{k=1}^{\infty}\sigma_{1-s_1-s_2}(k)\rho_N(1-s_1, 2-s_1-s_2; \pm 2\pi ik\tau).\end{aligned} \tag{21}$$

The explicit form of $\rho_N(1-s_1, 2-s_1-s_2; \pm 2\pi ik\tau)$ is given as formula (3.3) of [14]. From that formula and (21) we can immediately see that Ω_{2k+1} is not a singular locus of $g(s_1,s_2;\omega_1,\omega_2)$. Concerning $g_0(s_1,s_2;\omega_1,\omega_2)$, we see that its possible singularities are determined by $s_1=l$ ($l\in\mathbb{Z};\ l\geq 1$), $s_1+s_2=l$ ($l\in\mathbb{Z};\ l\leq 1$) and $s_1+s_2=2$, caused by the gamma factors and $\zeta(s_1+s_2-1)$. Hence we have only to consider the case $s_1+s_2=-2k+1$ ($k\in\mathbb{N}$). In this case, we can see that $\zeta(s_1+s_2-1)=\zeta(-2k)=0$ which corresponds to a trivial zero of $\zeta(s)$ for each $k\in\mathbb{N}$. The zero cancels the singularity, and hence Ω_{2k+1} is not a singular locus of $g_0(s_1,s_2;\omega_1,\omega_2)$.

Next we will prove (19). By (13), we have

$$\begin{aligned}\frac{1}{\Gamma(1-s_1)\omega_1^{-1}\omega_2^{1-s_1-s_2}}&\{\zeta_2(s_1,s_2;\omega_1,\omega_2)-g_0(s_1,s_2;\omega_1,\omega_2)\}\\ &=F_+\left(1-s_2,1-s_1;\frac{\omega_2}{\omega_1}\right)+F_-\left(1-s_2,1-s_1;\frac{\omega_2}{\omega_1}\right).\end{aligned} \tag{22}$$

The functional equation

$$F_\pm(1-s_2,1-s_1;\tau)=(\pm 2\pi i\tau)^{s_1+s_2-1}F_\pm(s_1,s_2;\tau) \tag{23}$$

can be shown in much the same way as Proposition 2 of [14], based on the transformation formula for $\Psi(a,c;x)$. Now assume $(s_1,s_2)\in\Omega_{2k+1}$ for $k\in\mathbb{Z}\setminus\{0\}$, that is, $s_1+s_2=2k+1$. Then we have $(\pm1)^{s_1+s_2-1}=(\pm1)^{2k}=1$, hence $(\pm 2\pi i\tau)^{s_1+s_2-1}$ on the right-hand side of (23) is $(2\pi i\tau)^{s_1+s_2-1}$. Therefore the right-hand side of (22) is equal to

$$(2\pi i\omega_2/\omega_1)^{s_1+s_2-1}\{F_+(s_1,s_2;\omega_2/\omega_1)+F_-(s_1,s_2;\omega_2/\omega_1)\}. \tag{24}$$

On the other hand, replacing (s_1,s_2) in (22) by $(1-s_2,1-s_1)$, we find that the resulting right-hand side is equal to the quantity in the curly bracket appearing in (24). This implies (19), and completes the proof of Theorem 2.2. □

Remark 2.1. From the above observation, we have obtained that the singular loci of $g_0(s_1,s_2;\omega_1,\omega_2)$ are $s_1+s_2=2,1,0,-2,,-4,\ldots$, and $s_1=j$ ($j\in\mathbb{N}$). On the other hand, since ρ_N appearing in (21) is holomorphic in the convergent area (see [14, (3.3)]), we see that the singular loci of $g(s_1,s_2;\omega_1,\omega_2)$ are $s_1=0$ and $s_2=1$ which come from the first term corresponding to $j=0$ on the right-hand side of (21). However $s_1=j$ ($j\in\mathbb{N}_0$) cannot be a singular locus of $\zeta_2(s_1,s_2;\omega_1,\omega_2)$, because it intersects with the domain of absolute convergence of $\zeta_2(s_1,s_2;\omega_1,\omega_2)$. Therefore we obtain that the singular loci of $\zeta_2(s_1,s_2;\omega_1,\omega_2)$ are

$$s_1+s_2=2-2k\quad(k\in\mathbb{N}_0),\ \ s_1+s_2=1,\ \ s_2=1. \tag{25}$$

In particular when $(\omega_1,\omega_2)=(1,1)$, the result of (25) coincides with the known result given by Akiyama–Egami–Tanigawa [1] using the Euler–Maclaurin formula. We can also determine (25) by using the Mellin–Barnes formula (see the previous result [13] of the second-named author).

Assume $\Re\omega_1>0$, $\Re\omega_2>0$. We can see that

$$\zeta_2(s_1,s_2;\omega_1,\omega_2)=\omega_1^{-s_1-s_2}\zeta_2(s_1,s_2;1,\omega_2/\omega_1) \tag{26}$$

in the domain of absolute convergence (2). Conversely, for $\tau\in\mathbb{C}$ with $\Im\tau>0$ or $\tau\in(0,\infty)$, we can choose ω_1, $\omega_2\in\mathbb{C}$ satisfying that $\Re\omega_1>0$, $\Re\omega_2>0$ and $\tau=\omega_2/\omega_1$. In fact, if we write $\tau=re^{i\theta}$ with $r>0$ and $0\le\theta<\pi$, then we can choose $\omega_1=e^{-i\theta/2}$ and $\omega_2=re^{i\theta/2}$. Therefore, by (26), we see that $\zeta_2(s_1,s_2;1,\tau)$ can be also continued meromorphically to the whole space $\mathbb{C}^2$. We can rewrite (19) as follows.

Proposition 2.3. *For any $\tau\in\mathbb{C}$ with $\Im\tau>0$ or $\tau\in(0,\infty)$, the functional equation*

$$\left(\frac{2\pi i}{\tau}\right)^{\frac{1-s_1-s_2}{2}}\Gamma(s_2)\left\{\zeta_2(s_1,s_2;1,\tau)-g_0(s_1,s_2;1,\tau)\right\}=\left(\frac{2\pi i}{\tau}\right)^{\frac{s_1+s_2-1}{2}}\Gamma(1-s_1)$$
$$\times\left\{\zeta_2(1-s_2,1-s_1;1,\tau)-g_0(1-s_2,1-s_1;1,\tau)\right\} \tag{27}$$

holds for $(s_1,s_2)\in\Omega_{2k+1}$ $(k\in\mathbb{Z}\setminus\{0\})$.

Divide both sides of (19) (resp. (27)) by $\Gamma(1-s_1)\Gamma(s_2)$, and put $(s_1,s_2)=(-2p,-2q-1)$ and $(-2p-1,-2q)$ in (19) (resp. (27)) for $p,q\in\mathbb{N}_0$. Then, noting the fact $\zeta(-2p-2q-2)=0$, we have the following.

Corollary 2.4. *With the above notation, and for $p,q\in\mathbb{N}_0$,*

$$\zeta_2(-2p,-2q-1;\omega_1,\omega_2)=\frac{B_{2p+2q+2}}{4(p+q+1)}\omega_1^{2p+2q+1}, \tag{28}$$

$$\zeta_2(-2p-1,-2q;\omega_1,\omega_2)=\frac{B_{2p+2q+2}}{4(p+q+1)}\omega_1^{2p+2q+1}, \tag{29}$$

where $B_k=B_k(0)$ $(k\in\mathbb{N}_0)$ is the kth Bernoulli number. In particular,

$$\zeta_2(-2p,-2q-1;1,\tau)=\frac{B_{2p+2q+2}}{4(p+q+1)}, \tag{30}$$

$$\zeta_2(-2p-1,-2q;1,\tau)=\frac{B_{2p+2q+2}}{4(p+q+1)}. \tag{31}$$

Remark 2.2. We can regard (28)–(31) as generalizations of the known results

$$\zeta_2(-2p,-2q-1;1,1)=\zeta_2(-2p-1,-2q;1,1)=\frac{B_{2p+2q+2}}{4(p+q+1)}\quad (p,q\in\mathbb{N}_0)$$

proved by Akiyama–Egami–Tanigawa [1, Equation (8)] by a quite different method. It is to be noted that the left-hand sides of (30) and (31) are independent of τ.

3. Functional relations

Now we consider the series (5) and (6) for $\tau\in\mathbb{C}$ with $\Im\tau>0$. Write $\tau=\eta e^{i\theta_1}$ ($\eta>0$; $0<\theta_1<\pi$) and let $\rho=\sqrt{\eta}e^{i\theta_1/2}$. Note that $\Re\rho>0$, $\Re\rho^{-1}>0$, $\Re(i\rho^{-1})>0$, $\Re(-i\rho)>0$ and $\rho^2=\tau$. We have the following.

Lemma 3.1.

$$\begin{aligned}\widetilde{\zeta}_2(s_1,s_2;\tau)=\left(1+e^{-\pi i(s_1+s_2)}\right)\bigl\{&\rho^{-s_1-s_2}\zeta_2\left(s_1,s_2;\rho^{-1},\rho\right)\\&+(i\rho^{-1})^{s_1+s_2}\zeta_2\left(s_1,s_2;i\rho^{-1},-i\rho\right)+\zeta(s_1+s_2)\bigr\}\end{aligned}\tag{32}$$

holds in the region where all functions on the both sides are convergent. Therefore Theorem 2.1 and (32) *give the meromorphic continuation of* $\widetilde{\zeta}_2(s_1,s_2;\tau)$ *to* $\mathbb{C}^2$.

Proof. We separate the double sum on the right-hand side of (5) as

$$\sum_{m\ge1}\sum_{n\ge1}+\sum_{m\ge1}\sum_{n\le-1}+\sum_{m\le-1}\sum_{n\ge1}+\sum_{m\le-1}\sum_{n\le-1}+\sum_{\substack{m\ge1\\n=0}}+\sum_{\substack{m\le-1\\n=0}}.$$

Since $\rho^2=\tau$, we have $m+n\tau=\rho(m\rho^{-1}+n\rho)$ for $m,n\ge1$, and so on. By noting the slight modification of the definition of $\arg m$ for $m<0$ (see Section 1), we obtain the proof of this lemma by direct calculations. □

We can obtain the following theorem, and will give its proof in the next section.

Theorem 3.2. *For* $k\in\mathbb{N}$ *with* $k\ge2$ *and for* $\tau\in\mathbb{C}$ *with* $\Im\tau>0$,

$$\mathcal{S}_2(s,k;\tau)=\frac{\tau}{\pi i}\sum_{j=0}^{k}\frac{(2\pi i/\tau)^{k-j}}{(k-j)!}B_{k-j}\left(\frac12\right)\widetilde{\zeta}_2(s,j+1;\tau)\tag{33}$$

holds for $s\in\mathbb{C}$.

Now we aim to confirm that (33) in the case $(s,k,\tau)=(0,2p+1,i)$ $(p\in\mathbb{N})$ gives the known formulas (7), (8), and so on, as follows.

We recall Eisenstein series $G_{2k}(\tau)$ defined by

$$G_{2k}(\tau)=\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}}\frac{1}{(m+n\tau)^{2k}}\quad(k\in\mathbb{N};\ k\geq 2)\tag{34}$$

for $\tau\in\mathbb{C}$ with $\Im\tau>0$ (see Serre [18]). For example, when $\tau=i$, it is known that

$$G_4(i)=\frac{1}{15}\varpi^4,\quad G_8(i)=\frac{1}{525}\varpi^8,\quad G_{12}(i)=\frac{2}{53625}\varpi^{12},\dots\tag{35}$$

which were given by Hurwitz [10] (see also [11]), where ϖ is the lemniscate constant. For $k\in\mathbb{N}$ with $k\geq 2$, it follows from (5) that

$$\widetilde{\zeta}_2(0,2k;i)=\sum_{m\neq 0}\sum_{n\in\mathbb{Z}}\frac{1}{(m+ni)^{2k}}=G_{2k}(i)-2(-1)^k\zeta(2k).\tag{36}$$

Furthermore we can prove the following.

Proposition 3.3.

$$\widetilde{\zeta}_2(0,2;i)=-\pi+\frac{\pi^2}{3}.\tag{37}$$

In order to prove this proposition, we prepare the following lemma.

Lemma 3.4. *For $m\in\mathbb{Z}$, put*

$$b(m)=\sum_{n=1}^{\infty}\frac{m^2-n^2}{(m^2+n^2)^2}.$$

Then $b(m)=O\left(m^{-2}\right)$ as $|m|\to\infty$.

Proof. We see that

$$b(m)=\sum_{n=1}^{\infty}\frac{2m^2-(m^2+n^2)}{(m^2+n^2)^2}=2m^2\sum_{n=1}^{\infty}\frac{1}{(m^2+n^2)^2}-\sum_{n=1}^{\infty}\frac{1}{m^2+n^2}.\tag{38}$$

It is well-known (see, for example, [22, Chapter 7]) that

$$\sum_{n=1}^{\infty}\frac{1}{m^2+n^2}=\frac{\pi}{2m}\coth(\pi m)-\frac{1}{2m^2},\tag{39}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{(m^2+n^2)^2} = \frac{1}{m^4} \sum_{n=1}^{\infty} \frac{1}{((m^{-1}n)^2+1)^2}$$
$$= \frac{1}{m^4} \left\{ \frac{\pi^2 m^2}{4} \operatorname{cosech}^2(\pi m) + \frac{\pi m}{4} \coth(\pi m) - \frac{1}{2} \right\}, \tag{40}$$

where $\operatorname{cosech}(x) = 2/(e^x - e^{-x})$. Substituting (39) and (40) to (38), we have

$$b(m) = \frac{2}{m^2} \left\{ \frac{\pi^2 m^2}{4} \operatorname{cosech}^2(\pi m) + \frac{\pi m}{4} \coth(\pi m) - \frac{1}{2} \right\}$$
$$- \left\{ \frac{\pi}{2m} \coth(\pi m) - \frac{1}{2m^2} \right\} = \frac{\pi^2}{2} \operatorname{cosech}^2(\pi m) - \frac{1}{2m^2}.$$

Thus we have the assertion of this lemma. □

Proof of Proposition 3.3. For $m \in \mathbb{Z} \setminus \{0\}$, let

$$a(m) = \sum_{n \in \mathbb{Z}} \frac{1}{(m+ni)^2} = \sum_{n \in \mathbb{Z}} \left\{ \frac{m^2-n^2}{(m^2+n^2)^2} - i \frac{2mn}{(m^2+n^2)^2} \right\}.$$

Then the imaginary part is an odd function in n, so it vanishes. Hence, by Lemma 3.4, we have

$$a(m) = \sum_{n=1}^{\infty} \frac{2(m^2-n^2)}{(m^2+n^2)^2} + \frac{1}{m^2} = 2b(m) + \frac{1}{m^2} = O\left(m^{-2}\right) \quad (|m| \to \infty).$$

Therefore we see that

$$\widetilde{\zeta}_2(s, 2; i) = \sum_{m \neq 0} \frac{1}{m^s} a(m)$$

converges absolutely for $\Re s > -1$. In particular when $s = 0$,

$$\widetilde{\zeta}_2(0, 2; i) = \sum_{m \neq 0} \sum_{n \in \mathbb{Z}} \frac{1}{(m+ni)^2}$$
$$= \sum_{m \in \mathbb{Z}} \sum_{\substack{n \in \mathbb{N} \\ (m,n) \neq (0,0)}} \frac{1}{(m+ni)^2} + 2\zeta(2) = -\pi + \frac{\pi^2}{3},$$

where the last equality is well-known (see [18, Chapter 7]). This completes the proof. □

Combining (32) with $(s,\tau)=(0,i)$, (35) and (37), we obtain, for example, the explicit formulas (7), (8) and so on, which were given by the third-named author in [21]. In other words, the functional relation formula (32) includes the known value-relation formulas (7), (8) as special cases.

Substitute (13) into (32), we can show a "functional equation" for $\widetilde{\zeta}_2(s_1,s_2;\tau)$, whose right-hand side includes four $F_\pm$ terms. In particular, when $\tau=i$, since

$$F_\pm(1-s_2,1-s_1;-i)=F_\mp(1-s_2,1-s_1;i),$$

we obtain

$$\begin{aligned}\widetilde{\zeta}_2(s_1,s_2;i)&=\big(1+e^{-\pi i(s_1+s_2)}\big)\Big[\big(e^{(2-s_1-s_2)\pi i/4}+e^{(s_1+s_2-2)\pi i/4}\big)\\&\quad\times\left\{\frac{\Gamma(1-s_1)}{\Gamma(s_2)}\Gamma(s_1+s_2-1)\zeta(s_1+s_2-1)\right.\\&\quad\left.+\Gamma(1-s_1)\big(F_+(1-s_2,1-s_1;i)+F_-(1-s_2,1-s_1;i)\big)\right\}+\zeta(s_1+s_2)\Big].\end{aligned}\tag{41}$$

Furthermore, by combining (32), (33) and (41), we have the following functional equation between the hyperbolic-sine double series of Eisenstein type and infinite series of hypergeometric functions.

Theorem 3.5. *For $k\in\mathbb{N}$ with $k\geq 2$,*

$$\begin{aligned}\mathcal{S}_2(s,k;i)&=\frac{1}{\pi}\sum_{j=0}^{k}\frac{(2\pi)^{k-j}}{(k-j)!}B_{k-j}\left(\frac{1}{2}\right)\big(1+e^{-\pi i(s+j+1)}\big)\\&\quad\times\Big[\big(e^{(1-s-j)\pi i/4}+e^{-(1-s-j)\pi i/4}\big)\left\{\frac{\Gamma(1-s)}{j!}\Gamma(s+j)\zeta(s+j)\right.\\&\quad\left.+\Gamma(1-s)\big(F_+(-j,1-s;i)+F_-(-j,1-s;i)\big)\right\}+\zeta(s+j+1)\Big]\end{aligned}\tag{42}$$

holds for $s\in\mathbb{C}$ except for singularities of the functions on the both sides.

Remark 3.1. Gangl, Kaneko and Zagier [9] considered another kind of double Eisenstein series defined by

$$\begin{aligned}G_{r,s}(\tau)&=\sum_{\substack{a,b\in\mathbb{N}\\a>b}}\frac{1}{a^rb^s}+\sum_{\substack{a\in\mathbb{Z}\\m,b\in\mathbb{N}}}\frac{1}{(m\tau+a)^rb^s}\\&\quad+\sum_{\substack{m\in\mathbb{N}\\a,b\in\mathbb{Z}\\a>b}}\frac{1}{(m\tau+a)^r(m\tau+b)^s}+\sum_{\substack{m,n\in\mathbb{N}\\m>n\\a,b\in\mathbb{Z}}}\frac{1}{(m\tau+a)^r(n\tau+b)^s}\end{aligned}\tag{43}$$

for $r, s \in \mathbb{N}$ with $r \geq 3$ and $s \geq 2$, and $\tau \in \mathbb{C}$ with $\Im\tau > 0$. By virtue of $G_{r,s}(\tau)$, they discovered a correspondence between a certain subspace of the formal double zeta space and the space of modular forms, and further gave a Fourier expansion of $G_{r,s}(\tau)$.

It seems important to consider $G_{r,s}(\tau)$ with complex variables r, s and further to search for a certain functional equation and some relevant results corresponding to the facts stated in Sections 2 and 3. However we have no idea in this direction.

4. Proof of Theorem 3.2

We let

$$\mathfrak{G}(t) = \frac{1}{\sin(\pi t/\tau)} = \frac{2i}{e^{\pi it/\tau} - e^{-\pi it/\tau}} = \sum_{k=0}^{\infty} \mathcal{G}_k t^{k-1}. \tag{44}$$

We can see that $\tau\mathbb{Z}$ is the set of all poles of $\mathfrak{G}(t)$ and each pole is simple. Let C_N be the parallelogram the vertices of which consist $\pm(N+1/2) \pm (N+1/2)\tau$. Then for $k \in \mathbb{N}$ with $k \geq 2$,

$$\lim_{N\to\infty} \int_{C_N} t^{-k}\mathfrak{G}(t)dt = 0 \tag{45}$$

since $\mathfrak{G}(t)$ is bounded on $\bigcup_{N\in\mathbb{N}} C_N$ (see, for example, [19, Section 5]). Therefore

$$\begin{aligned} 0 &= \lim_{N\to\infty} \sum_{0<|n|\leq N} \operatorname*{Res}_{t=n\tau} t^{-k}\mathfrak{G}(t) + \operatorname*{Res}_{t=0} t^{-k}\mathfrak{G}(t) \\ &= \sum_{n\in\mathbb{Z}\setminus\{0\}} \operatorname*{Res}_{t=n\tau} t^{-k}\mathfrak{G}(t) + \operatorname*{Res}_{t=0} t^{-k}\mathfrak{G}(t) = \frac{\tau}{\pi} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^n}{(n\tau)^k} + \mathcal{G}_k \end{aligned}$$

due to the absolute convergence of the series.

By the definition (see, for example, [2])

$$\mathfrak{G}(t) = \frac{2ie^{\pi it/\tau}}{e^{2\pi it/\tau} - 1} = \frac{1}{\pi t/\tau} \frac{(2\pi it/\tau)e^{(2\pi it/\tau)\frac{1}{2}}}{e^{2\pi it/\tau} - 1} = \frac{\tau}{\pi t} \sum_{k=0}^{\infty} B_k\left(\frac{1}{2}\right) \frac{(2\pi it/\tau)^k}{k!}, \tag{46}$$

we have the following.

Lemma 4.1. *For $k \in \mathbb{N}$ with $k \geq 2$,*

$$\mathcal{G}_k = -\frac{\tau}{\pi} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^n}{(n\tau)^k} = \frac{\tau}{\pi} \frac{(2\pi i/\tau)^k}{k!} B_k\left(\frac{1}{2}\right). \tag{47}$$

Next assume $x \notin \tau\mathbb{Z}$, and let

$$\mathfrak{F}(t;x)=\frac{1}{e^{2\pi i(t-x)/\tau}-1}=\sum_{k=0}^{\infty}\mathcal{F}_k(x)t^k. \tag{48}$$

We can see that $x+\tau\mathbb{Z}$ is the set of all poles of $\mathfrak{F}(t;x)$ as a function in t and each pole is simple. Then we can similarly see that

$$\begin{aligned}0&=\lim_{N\to\infty}\sum_{-N\le n\le N}\operatorname*{Res}_{t=x+n\tau}t^{-k-1}\mathfrak{F}(t;x)+\operatorname*{Res}_{t=0}t^{-k-1}\mathfrak{F}(t;x)\\&=\frac{\tau}{2\pi i}\sum_{n\in\mathbb{Z}}\frac{1}{(x+n\tau)^{k+1}}+\mathcal{F}_k(x)\end{aligned}$$

for $k\in\mathbb{N}$.

Lemma 4.2. *For $k\in\mathbb{N}$,*

$$\begin{aligned}\mathcal{F}_0(x)&=\frac{1}{e^{-2\pi ix/\tau}-1},\\\mathcal{F}_k(x)&=-\frac{\tau}{2\pi i}\sum_{n\in\mathbb{Z}}\frac{1}{(x+n\tau)^{k+1}}.\end{aligned} \tag{49}$$

From the above consideration, we can give a proof of Theorem 3.2 as follows.

Proof of Theorem 3.2. Let $k\in\mathbb{N}$ with $k\ge 2$. Let

$$\mathfrak{H}(t;x)=\frac{1}{(e^{2\pi i(t-x)/\tau}-1)\sin(\pi t/\tau)}=\mathfrak{F}(t;x)\mathfrak{G}(t). \tag{50}$$

The set of poles of $\mathfrak{H}(t;x)$ are $(x+\tau\mathbb{Z})\cup\tau\mathbb{Z}$. Hence, similarly to the above, we have

$$\lim_{N\to\infty}\left(\frac{\tau}{2\pi i}\sum_{-N\le n\le N}\frac{(-1)^n}{(x+n\tau)^k\sin(\pi x/\tau)}+\frac{\tau}{\pi}\sum_{\substack{-N\le n\le N\\ n\ne 0}}\frac{(-1)^n}{(e^{-2\pi ix/\tau}-1)(n\tau)^k}\right)$$
$$+\operatorname*{Res}_{t=0}t^{-k}\mathfrak{H}(t;x)=0.$$

Therefore, by Lemma 4.1 we have

$$\begin{aligned}&\frac{\tau}{2\pi i}\sum_{n\in\mathbb{Z}}\frac{(-1)^n}{(x+n\tau)^k\sin(\pi x/\tau)}\\&\qquad-\frac{\tau}{\pi}\frac{(2\pi i/\tau)^k}{k!}B_k\left(\frac12\right)\frac{1}{e^{-2\pi ix/\tau}-1}+\operatorname*{Res}_{t=0}t^{-k}\mathfrak{H}(t;x)=0.\end{aligned} \tag{51}$$

By (44), (46) and Lemma 4.2, the last term is calculated as

$$\begin{aligned}\operatorname*{Res}_{t=0} t^{-k}\mathfrak{H}(t;x) &= \operatorname*{Res}_{t=0} t^{-k}\left(\sum_{j=0}^{\infty}\mathcal{F}_j(x)t^j\right)\left(\sum_{j=0}^{\infty}\mathcal{G}_j t^{j-1}\right) = \sum_{j=0}^{k}\mathcal{F}_j(x)\mathcal{G}_{k-j}\\ &= -\frac{\tau^2}{2\pi^2 i}\sum_{j=1}^{k}\frac{(2\pi i/\tau)^{k-j}}{(k-j)!}B_{k-j}\left(\frac{1}{2}\right)\sum_{n\in\mathbb{Z}}\frac{1}{(x+n\tau)^{j+1}}\\ &\quad+\frac{\tau}{\pi}\frac{(2\pi i/\tau)^k}{k!}B_k\left(\frac{1}{2}\right)\frac{1}{e^{-2\pi i x/\tau}-1}.\end{aligned}$$

Therefore we have

$$\begin{aligned}&\frac{\tau}{2\pi i}\sum_{n\in\mathbb{Z}}\frac{(-1)^n}{(x+n\tau)^k\sin(\pi x/\tau)}\\ &\qquad-\frac{\tau^2}{2\pi^2 i}\sum_{j=1}^{k}\frac{(2\pi i/\tau)^{k-j}}{(k-j)!}B_{k-j}\left(\frac{1}{2}\right)\sum_{n\in\mathbb{Z}}\frac{1}{(x+n\tau)^{j+1}}=0.\end{aligned} \tag{52}$$

Putting $x=m$ in (52), multiplying by m^{-s} the both sides, and summing them up in $m\in\mathbb{Z}\setminus\{0\}$ for sufficiently large $\Re s>0$, we obtain

$$\begin{aligned}&\frac{\tau}{2\pi i}\sum_{m\in\mathbb{Z}\setminus\{0\}}\sum_{n\in\mathbb{Z}}\frac{(-1)^n}{m^s(m+n\tau)^k\sin(m\pi/\tau)}\\ &\quad-\frac{\tau^2}{2\pi^2 i}\sum_{j=1}^{k}\frac{(2\pi i/\tau)^{k-j}}{(k-j)!}B_{k-j}\left(\frac{1}{2}\right)\sum_{m\in\mathbb{Z}\setminus\{0\}}\sum_{n\in\mathbb{Z}}\frac{1}{m^s(m+n\tau)^{j+1}}=0,\end{aligned}$$

where all the sums are absolutely convergent. Hence, by $i\sin x=\sinh(ix)$, we obtain the assertion of Theorem 3.2. □

Remark 4.1. Theorem 3.2 can also be proved by the same method as in [21].

Acknowledgements The authors would like to express their sincere gratitude to the referee for valuable comments.

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YASUSHI KOMORI
GRADUATE SCHOOL OF MATHEMATICS
NAGOYA UNIVERSITY
CHIKUSA-KU, NAGOYA 464-8602
JAPAN

E-mail: komori@math.nagoya-u.ac.jp

PRESENT ADDRESS:
DEPARTMENT OF MATHEMATICS
RIKKYO UNIVERSITY
NISHI-IKEBUKURO, TOSHIMA-KU
TOKYO 171-8501
JAPAN

E-mail: komori@rikkyo.ac.jp

KOHJI MATSUMOTO
GRADUATE SCHOOL OF MATHEMATICS
NAGOYA UNIVERSITY
CHIKUSA-KU, NAGOYA 464-8602
JAPAN

E-mail: kohjimat@math.nagoya-u.ac.jp

HIROFUMI TSUMURA
DEPARTMENT OF MATHEMATICS AND INFORMATION SCIENCES
TOKYO METROPOLITAN UNIVERSITY
1-1, MINAMI-OHSAWA, HACHIOJI
TOKYO 192-0397
JAPAN

E-mail: tsumura@tmu.ac.jp

(Received January 27, 2009; revised December 11, 2009)