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On absolutely conformal mappings

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Abstract. Let Ω be a domain in $\mathbb{R}^n$. It is proved that, if $u \in C^1(\Omega;\mathbb{R}^n)$ and there holds the formula $\|\nabla u(x)\|^n = n^{n/2}|\det\nabla u(x)|$ in Ω, then for $n \geq 3$ u is a restriction of a Möbius transformation, and for $n = 2$, u is an analytic function. This extends, partially, the well-known Liouville theorem ([6]), wich states that if $u \in ACL^n(\Omega;\mathbb{R}^n)$, $n \geq 3$, and the condition $\|\nabla u(x)\|^n = n^{n/2}\det\nabla u(x)$ is satisfied a.e. in Ω, then u is a restriction of a Möbius transformation.

1. Introduction

Let Ω be an open set of the Euclidean space $\mathbb{R}^n$. For $x = (x_1,\dots,x_n) \in \mathbb{R}^n$, we denote by $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$ the norm of x. Let $m = m_n$ denote the usual Lebesgue measure on $\mathbb{R}^n$. Sometimes we use notation $dx = dx_1 \dots dx_n$ and $|D|$ instead of dm and $m(D)$, where $x = (x_1,\dots,x_n) \in \mathbb{R}^n$ and D is a Lebesgue measurable set in $\mathbb{R}^n$, respectively. For a given domain $\Omega \subset \mathbb{R}^n$, we say that a continuous mapping $u : \Omega \to \mathbb{R}^n$ is quasiregular (abbreviated qr) if

(1) u is ACL^n, and

(2) there exists a real number K, $K \geq 1$, such that

$$|u'(x)|^n \leq KJ_u(x) \quad \text{a.e.} \quad \text{on} \quad \Omega, \tag{1.1}$$

where $|u'(x)| = \max_{|h|=1}|u'(x)h|$.

In this setting we shortly write that u is a K-qr mapping. For properties of qr-mappings see [1], [2], [3], [7] and [8]. If u is a K-qr and homeomophic mapping then it is called K-quasiconformal or shortly K-q.c.

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Let

$$\|u'(x)\| = \sqrt{\sum_{i,j=1}^{n} (\partial_j u_i(x))^2}$$

denote the Hilbert–Schmidt norm of $u'(x)$, where $\partial_j = \partial_{x_j}$ denotes $j-th$ partial derivative.

It is well known that if u is a K-qr mapping on Ω, then

$$\|u'(x)\|^n \le n^{n/2} K J_u(x) \quad \text{a.e.} \quad \text{on} \quad \Omega. \tag{1.2}$$

In this paper we will consider generalized 1-quasiregular mappings, i.e. continuous mappings u satisfying the conditions $u \in ACL^n$ and

$$\|u'(x)\|^n \le n^{n/2} |J_u(x)| \quad \text{a.e.} \quad \text{on} \quad \Omega. \tag{1.3}$$

2. The main result

Proposition 2.1 ([4, Section V.3]). *Let Ω be an open subset of $\mathbb{R}^n$ and let $u : \Omega \to \mathbb{R}^n$ be a ACL^n mapping satisfying the Lusin's condition (N) (The condition (N) means that a mapping maps sets of measure zero to sets of measure zero). Then the function $y \mapsto N(y, u)$ is measurable on $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} N(y, u)\, dy = \int_\Omega |J_u(x)|\, dx, \tag{2.1}$$

where $J_u(x)$ is the Jacobian of u at x and $N(y, u)$ denotes the cardinal number of the set $u^{-1}(y)$ if the last set is finite and it is $+\infty$ in the other case.

Corollary 2.2. *Under the condition of the previous proposition there holds the inequality*

$$\int_\Omega |J_u(x)|\, dx \ge |u(\Omega)|. \tag{2.2}$$

The equality holds in (2.2) if and only if u is univalent on Ω.

For 1-qr mapping we also say generalized conformal mapping. The generalized Liouville theorem ([6]) states: for $n \ge 3$ every 1-qr mapping on a domain $\Omega \subset \mathbb{R}^n$, is a restriction of a Möbius transformation or a constant.

We extend (partially) this theorem as follows:

Theorem 2.3. *Let Ω be a domain in $\mathbb{R}^n$ and let $u : \Omega \to \mathbb{R}^n$ be a C^1 mapping such that*

$$\|u'(x)\|^n = n^{n/2}|J_u(x)|, \ x \in \Omega. \text{ (We say that } u \text{ is absolutely conformal).} \quad (2.3)$$

Then, for $n = 2$, u is analytic or anti-analytic function. For $n \geq 3$, u is a restriction of a Möbius transformation or a constant.

Proof. We first consider the case $n = 2$. Let $\Omega_0 = \{z \in \Omega : J_u(z) = 0\}$, $\Omega_1 = \{z \in \Omega : J_u(z) > 0\}$, $\Omega_2 = \{z \in \Omega : J_u(z) < 0\}$. Let u^1 and u^2 be the restrictions of u on Ω_1 and Ω_2. Put $p = u_z$. Suppose that u is smooth and non constant function and the equality holds in (2.3). Then $u_z = u_{\overline{z}} = 0$ on Ω_0, u^1 is conformal on Ω_1, and u^2 is anti conformal on Ω_2 (and therefore $p = 0$ on Ω_2). Hence p is continuous on Ω and analytic on $\Omega_1 \cup \Omega_2$.

We will prove that p is analytic on Ω.

There are two cases:

(a) If $z_0 \in \Omega_0$ is an interior point of Ω_0, then p is analytic at z_0.

(b) If $z_0 \in \Omega_0$ is not an interior point of Ω_0 then $z_0 \in \partial\Omega_0 \setminus \partial\Omega = \partial(\Omega_1 \cup \Omega_2) \setminus \partial\Omega$. Hence then there exists a sequence $z_n \in \Omega_1 \cup \Omega_2$ such that $\lim_{n\to\infty} z_n = z_0$.

It follows that $p_{\bar{z}}(z_n) = 0$ and therefore $\lim_{n\to\infty} p_{\bar{z}}(z_n) = 0$ and hence, since p is continuous, we find $p_{\bar{z}}(z_0) = 0$. Hence p is analytic in Ω. If Ω_2 is not empty set, according to the uniqueness theorem, this gives that $u_z(z) \equiv 0$ on Ω and hence u is anti analytic on Ω. Since u is analytic on Ω_1, we first conclude that u is constant on Ω_1 and therefore that Ω_1 is empty set. In a similar way, if Ω_1 is not empty set, we conclude that u is analytic on Ω.

Hence u is analytic in Ω or it is anti-analytic in Ω. Note that the set $\Omega \setminus (\Omega_1 \cup \Omega_2)$ is discrete or it is equal to the set Ω.

We now consider the case $n > 2$.

Let $\Omega_0 = \{x \in \Omega : J_u(x) = 0\}$, $\Omega_1 = \{x \in \Omega : J_u(x) > 0\}$, $\Omega_2 = \{x \in \Omega : J_u(x) < 0\}$ and $\Omega^* = \Omega \setminus \Omega_0$. If u is a C^1 mapping, then the Hadamard inequality gives

$$|J_u| \leq \prod_{k=1}^{n} |\partial_k u| \quad (2.4)$$

and hence:

$$|J_u| \leq \left(\frac{\sum_{k=1}^{n} |\partial_k u|^2}{n}\right)^{\frac{n}{2}}, \text{ that is } n^{n/2}|J_u(x)| \leq \|u'(x)\|^n, \quad x \in \Omega. \quad (2.5)$$

Using the geometric interpretation of $J_u(x)$ one can show that the equality holds in (2.4) at a point $x \in \Omega^*$ if and only if the vectors $\partial_i u(x)$, $i = 1, 2, \dots, n$, are orthogonal. Observe that if the equality holds in (2.5), then the equality holds in (2.4). Hence, for $x \in \Omega^*$, the equality holds in (2.5) if and only if

$$\langle \partial_i u, \partial_j u \rangle = \begin{cases} 0 & \text{if } i \neq j, \\ s^2 & \text{if } i = j, \end{cases}$$

where $s = s(x)$ is a real function on Ω^*. Hence $\|u'(x)\|^n = n^{n/2}|J_u(x)|$ for $x \in \Omega^*$ and since $u \in C^1$, $\|u'(x)\|^n = n^{n/2}|J_u(x)|$ for $x \in \Omega$.

If u is not a constant function, then Ω^* is not empty set. Suppose, for example, that Ω_1 is not an empty set and let $\hat{\Omega}$ be a component of Ω_1.

Thus u is a generalized conformal mapping in $\hat{\Omega}$. Hence, by the generalized Liouville theorem i.e. Gehring-Reshetnyak's theorem see ([6]), every generalized conformal mapping in the space is a Möbius transformation. Hence u is the restriction to $\hat{\Omega}$ of a Möbius transformation A. Let $\omega \in \overline{\hat{\Omega}}$. Then there exists a sequence $z_n \in \hat{\Omega}$, $n \in \mathbb{N}$, such that z_n tends ω. Since u is a C^1 function, $J_u(\omega) = \lim J_u(z_n) = \lim J_A(z_n)$. It is clear that $J_A(\omega) = \lim J_A(z_n)$ and therefore $J_u(\omega) > 0$ $\omega \in \hat{\Omega}$. Hence $\hat{\Omega}$ is closed-open in Ω and therefore $\Omega = \hat{\Omega}$. □

Example 1. Let $n \geq 3$ and $x = (x_1, \ldots, x_n)$,

$$u(x) = \begin{cases} (x_1, \ldots, x_{n-1}, x_n) & \text{if } x_n \geq 0 \\ (x_1, \ldots, x_{n-1}, -x_n) & \text{if } x_n \leq 0. \end{cases}$$

Then there hold (2.3) almost everywhere but u is not a Möbius transformation. This means that the condition u is C^1 in Theorem 2.3 is important.

Corollary 2.4. *Let Ω be a domain in the Euclidean space $\mathbb{R}^n$, $n \geq 2$, and $u \in ACL^n(\Omega)$ satisfying the condition (N).*

Then

$$\int_\Omega \|u'(x)\|^n dx \geq n^{n/2}|u(\Omega)|\,. \tag{2.6}$$

If u is a C^1 mapping , then the equation in (2.6) holds if and only if u is an injective conformal mapping or a constant mapping.

Proof. Using (2.2) and (2.5) and Corollary 2.2 we obtain:

$$D_n(u) := \int_\Omega \left(\sum_{i=1}^n |\partial_i u|^2 \right)^{n/2} dm(x) \geq n^{n/2} \int_\Omega |J_u(x)| dm(x) \geq n^{n/2}\, |u(\Omega)|,$$

and consequently

$$D_n(u) \geq n^{n/2}\, |u(\Omega)|.$$

Suppose now that the equality holds in (2.6) and $u \in C^1$. Since u is C^1, u satisfies the condition (N) and the equality holds in (2.5) for every $x \in \Omega$ (that is u satisfies condition (2.3) and therefore u is absolutely conformal). Thus, by Theorem 2.3, it is an analytic (or anti-analytic) function in the plane or a Möbius transformation in the space. Thus if $n \geq 3$, then u is a Möbius transformation.

It remains to finish the proof in the case $n = 2$; that is to prove that if u is non-constant analytic (or anti-analytic) and

$$D_2(u) = 2|u(\Omega)|$$

then it is a univalent conformal (or anti-conformal) mapping.

Using the fact that u is an open mapping the assumption $u_0 = u(z_1) = u(z_2)$ for $z_1 \neq z_2$ has the consequence that there exist two disjoint open sets $U_1, U_2 \subset \Omega$ and a disk $D(u_0, r') \subset u(\Omega)$ such that $D(u_0, r') = u(U_1) = u(U_2)$. Hence

$$D_2(u) > \int_{\Omega \setminus U_1} \|u'(z)\|^2 \, dxdy \geq 2|(u(\Omega \setminus U_1))| = 2|\Omega'|$$

which is a contradiction. □

An open question. From the proofs of the main theorems of the paper a question emerges: Does there exist a number $K_o = K_o(n) > 1$ such that if $u \in C^1$ satisfies $|u'(x)|^n \leq K|J(x,u)|$ and $K < K_o$, then u is a K-quasiregular mapping?

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