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Liouville numbers in the non-archimedean case

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Abstract. Basic results about real Liouville numbers are investigated in three non-archimedean settings, referred to as the non-archimedean case, comprising the field of p-adic numbers, the function field completed with respect to the degree valuation and the function field completed with respect to a prime-adic valuation. The result of Erdős that every real number is representable as a sum, and as a product of two real Liouville numbers is shown to hold in the non-archimedean case. The concept of Liouville continued fractions is also considered.

1. Introduction

Let $\mathbb{R}$ be the field of real numbers equipped with the usual absolute value $|\cdot|$. The classical *Liouville' s theorem*, see e.g. Theorem 1.2 of [1], states that if $\xi \in \mathbb{R}$ is an algebraic number of degree $n \geq 2$, then there exists a positive constant $c(\xi)$ depending only on ξ such that

$$\left|\xi - \frac{a}{b}\right| \geq \frac{c(\xi)}{b^n}$$

for all $a, b\,(> 0) \in \mathbb{Z}$. The existence of transcendental numbers is usually shown using Liouville's theorem. The transcendental numbers so constructed are referred to as real Liouville numbers. A number $\xi \in \mathbb{R}$ is called a *real Liouville number* if for any $w \in \mathbb{R}^+$, the set of positive real numbers, there exist $a, b\,(> 1) \in \mathbb{Z}$ such that

$$0 < \left|\xi - \frac{a}{b}\right| < \frac{1}{b^w};$$

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note that it suffices to take w in $\mathbb{N}$ if needed.

Real Liouville numbers possess a number of interesting properties, both algebraic and number theoretic. We are interested here in three of these properties, which will be referred to under the headings I, II and III.

I. It is well-known, Section 35 of [12], that real Liouville numbers are essentially closed under linear fractional transformation over $\mathbb{Z}$. Indeed, more is true. Let ξ be a real Liouville number. For each $w \in \mathbb{R}^+$, from the definition, there is a sequence of rational numbers (a_n/b_n), with $b_{n+1} > b_n > 1$, such that

$$0 < \left|\xi - \frac{a_n}{b_n}\right| < \frac{1}{b_n^w},$$

showing also that ξ is the limit of the sequence (a_n/b_n). In Chapitre II of [10], the concept of the class $H(\xi)$ of *real transcendental numbers associated with* ξ is defined as follows: a real number $\zeta \in H(\xi)$ if, for the given w and (a_n/b_n), there is a sequence of rational numbers (c_n/d_n) such that

$$0 < \left|\zeta - \frac{c_n}{d_n}\right| < \frac{1}{d_n^w} \quad \text{and} \quad d_n = b_n^{\sigma_n},$$

where (σ_n) is a sequence of real numbers bounded both above and below by fixed positive constants. Maillet, [10], proved that the set $H(\xi)$ enjoys the following properties.

- *Property* $H1$: $\frac{a}{b}\xi + \frac{c}{d} \in H(\xi)$ for all $a(\neq 0)$, $b(\neq 0)$, $c, d(\neq 0)$ in $\mathbb{Z}$.
- *Property* $H2$: The sum, difference, product and division of two numbers from $H(\xi)$ is either a number in $H(\xi)$ or a rational number.

II. In 1962, P. Erdős, [3], proved an amazing result that every nonzero real number can be represented as a sum and also a product of two real Liouville numbers.

III. It is well-known, [12], that each infinite simple continued fraction

$$[\bar{a}] := [a_0; a_1, a_2, a_3 \dots],$$

where $a_0 \in \mathbb{Z}, a_i \in \mathbb{N}$ $(i \geq 1)$, represents an irrational real number. Let h_n/k_n be its corresponding n^{th} convergents. We introduce here the notion of *real Liouville continued fraction.* We say that $[\bar{a}]$ is a real Liouville continued fraction if for each $w \in \mathbb{R}^+$ there is an $n \in \mathbb{N}$ such that

$$0 < \left|[\bar{a}] - \frac{h_n}{k_n}\right| < \frac{1}{k_n^w}.$$

We easily show in the following proposition that real Liouville numbers and real Liouville continued fractions are the same.

Proposition 1.1. *Each real Liouville continued fraction represents a real Liouville number and conversely.*

Proof. The first assertion is immediate from the definitions, while the converse follows from one of the approximation properties of simple continued fractions, see e.g. Satz 2.11, p. 39 of [12], which states that for an irrational $\xi \in \mathbb{R}$, if there is a rational number a/b with $b \geq 1$ such that

$$\left|\xi - \frac{a}{b}\right| < \frac{1}{2b^2},$$

then a/b equals one of the convergents of the simple continued fraction expansion of ξ. $\square$

There is also a concept of Liouville sequences introduced by J. Hančl in 2003, [4]. In brief, a sequence of positive numbers, (a_n), is a Liouville sequence if $\sum_{n\geq 1} 1/a_n c_n$ is a real Liouville number for every sequence of positive integers (c_n). We do not consider this notion here.

Our objectives here are first to derive basic properties of Liouville numbers, second to establish the result of Erdős and third to study Liouville continued fractions in the three settings of the field of p-adic numbers, the function field with degree valuation and the function field with τ-adic valuation, τ being a monic irreducible polynomial. Since the valuations in these settings are non-archimedean, we shall refer to them collectively as the *non-archimedean case*.

We start with the case of the function field with degree valuation which is closest to the case of real numbers.

2. The function field with degree valuation

Let k be a field, x an indeterminate, $k((1/x)) =: k_\infty$ the field of all formal Laurent series equipped with the degree valuation, $|\cdot|_\infty$, so normalized that $|x|_\infty = e^1$. *Liouvilles's theorem* for $k((1/x))$, see Theorem 1 of [8] or [14], states that if $\xi \in k((1/x))$ is an algebraic number, over $k(x)$, of degree $n \geq 2$, then there exists a positive constant $c(\xi)$ depending only on ξ such that

$$\left|\xi - \frac{a}{b}\right|_\infty \geq \frac{c(\xi)}{|b|_\infty^n}$$

for all $a, b\,(\neq 0) \in k[x]$. An element $\xi \in k((1/x))$ is called a k_∞*-Liouville number* if for each $w \in \mathbb{R}^+$, there exist $a, b \in k[x]\backslash\{0\}$ with $|b|_\infty > 1$ such that

$$0 < \left|\xi - \frac{a}{b}\right|_\infty < \frac{1}{|b|_\infty^w}.$$

Let ξ be a k_∞-Liouville number. For fixed $w \in \mathbb{R}^+$, from the definition, there are sequences (a_n) and (b_n) in $k[x]\backslash\{0\}$, with $|b_n|_\infty > 1$ such that

$$0 < \left|\xi - \frac{a_n}{b_n}\right|_\infty < \frac{1}{|b_n|_\infty^w}.$$

I. We define the class $H_\infty(\xi)$ of transcendental elements in $k((1/x))$ associated with ξ as follows: a transcendental element $\zeta \in H_\infty(\xi)$ if, for the given $w > 0$ and the sequence (a_n/b_n), there are sequences (c_n) and (d_n) in $k[x]\backslash\{0\}$, such that

$$0 < \left|\zeta - \frac{c_n}{d_n}\right|_\infty < \frac{1}{|d_n|_\infty^w} \quad \text{and} \quad |d_n|_\infty = |b_n|_\infty^{\sigma_n}\,, \tag{1}$$

where (σ_n) is a sequence of real numbers bounded both above and below by fixed positive constants. Clearly, each element in $H_\infty(\xi)$ is also a k_∞-Liouville number. Similar to the real case, we define:

- *Property* $H_\infty 1$: $\frac{a}{b}\xi + \frac{c}{d} \in H_\infty(\xi)$ for all $a(\neq 0)$, $b(\neq 0)$, $c, d(\neq 0)$ in $k[x]$.
- *Property* $H_\infty 2$: The sum, difference, product and division of two numbers from $H_\infty(\xi)$ is either a number in $H_\infty(\xi)$ or an element in $k(x)$.

Proposition 2.1. *Properties $H_\infty 1$ and $H_\infty 2$ hold for ξ being a k_∞-Liouville number.*

Proof. Fix $w > 0$. For $n \in \mathbb{N}$, $n \geq w$, since ξ is a k_∞-Liouville number, there exist $a_n, b_n \in k[x] \setminus \{0\}$ with $|b_n|_\infty > 1$ such that

$$0 < \left|\xi - \frac{a_n}{b_n}\right|_\infty < \frac{1}{|b_n|_\infty^n} \leq \frac{1}{|b_n|_\infty^w}.$$

Consider the element $\frac{a}{b}\xi + \frac{c}{d}$ with $a(\neq 0)$, $b(\neq 0)$, $c, d(\neq 0)$ in $k[x]$. Let

$$\frac{c_n}{d_n} = \frac{a}{b} \cdot \frac{a_n}{b_n} + \frac{c}{d} = \frac{a\,d\,a_n + b\,c\,b_n}{bd\,b_n},$$

where $d_n = b\,d\,b_n, |d_n|_\infty = |b_n|_\infty^{1+\epsilon_n}$. Clearly, (ϵ_n) is a bounded sequence of positive numbers. Thus,

$$0 < \left|\left(\frac{a}{b}\xi + \frac{c}{d}\right) - \frac{c_n}{d_n}\right|_\infty = \left|\frac{a}{b}\left(\xi - \frac{a_n}{b_n}\right)\right|_\infty < \left|\frac{a}{b\,b_n^n}\right|_\infty = \left|\frac{a}{b} \cdot \frac{1}{d_n^{n/(1+\epsilon_n)}}\right|_\infty < \frac{1}{|d_n|_\infty^w},$$

by choosing n large enough. Property $H_\infty 1$ follows with $\sigma_n = 1 + \epsilon_n$.

We remark that with the same proof just shown, we also have that if $\zeta \in H_\infty(\xi)$, then $\frac{a}{b}\zeta + \frac{c}{d} \in H_\infty(\xi)$ for all $a(\neq 0), b(\neq 0), c, d(\neq 0) \in k[x]$.

To establish Property $H_\infty 2$, let $\zeta_1, \zeta_2 \in H_\infty(\xi)$. Then for $n \in \mathbb{N}$, $n \geq w$, there exist $c_n(i), d_n(i) \in k[x] \setminus \{0\}$ $(i = 1, 2)$ with $|d_n(i)|_\infty > 1$ such that

$$0 < \left|\zeta_i - \frac{c_n(i)}{d_n(i)}\right|_\infty < \frac{1}{|d_n(i)|_\infty^n} \leq \frac{1}{|d_n(i)|_\infty^w}, \quad |d_n(i)|_\infty = |b_n|_\infty^{\sigma_n(i)} \quad (i = 1, 2).$$

For the proof of the sum and difference of two k_∞-Liouville numbers, it suffices by property $H_\infty 1$ to prove only for the case of the sum. Assuming $\zeta_1 + \zeta_2 \notin k(x)$, then

$$\begin{aligned} 0 &< \left|(\zeta_1+\zeta_2) - \left(\frac{c_n(1)d_n(2) + c_n(2)d_n(1)}{d_n(1)d_n(2)}\right)\right|_\infty = \left|\left(\zeta_1 - \frac{c_n(1)}{d_n(1)}\right) + \left(\zeta_2 - \frac{c_n(2)}{d_n(2)}\right)\right|_\infty \\ &< \max\left\{\frac{1}{|d_n(1)|_\infty^n}, \frac{1}{|d_n(2)|_\infty^n}\right\} = \frac{1}{|b_n|_\infty^{n \min\{\sigma_n(1), \sigma_n(2)\}}} \\ &= \frac{1}{|d_n(1)d_n(2)|_\infty^{n \min\{\sigma_n(1),\sigma_n(2)\}/(\sigma_n(1)+\sigma_n(2))}} < \frac{1}{|d_n(1)d_n(2)|_\infty^w}, \end{aligned}$$

when n is chosen large enough, i.e., $\zeta_1 + \zeta_2 \in H_\infty(\xi)$ with corresponding $\sigma_n = \sigma_n(1) + \sigma_n(2)$.

Next consider the product $\zeta_1 \zeta_2$ and assuming that it is not in $k(x)$. For n sufficiently large, using also $|\zeta_1|_\infty = |c_n(1)/d_n(1)|_\infty$, we have

$$\begin{aligned} 0 &< \left|\zeta_1 \zeta_2 - \frac{c_n(1)}{d_n(1)} \cdot \frac{c_n(2)}{d_n(2)}\right|_\infty = \left|\left(\zeta_1 - \frac{c_n(1)}{d_n(1)}\right)\zeta_2 + \left(\zeta_2 - \frac{c_n(2)}{d_n(2)}\right)\frac{c_n(1)}{d_n(1)}\right|_\infty \\ &< \max\left\{\left|\frac{\zeta_2}{d_n(1)^n}\right|_\infty, \left|\frac{\zeta_1}{d_n(2)^n}\right|_\infty\right\} \leq \frac{\max\{|\zeta_1|_\infty, |\zeta_2|_\infty\}}{|b_n|_\infty^{\min\{\sigma_n(1),\sigma_n(2)\}n}} \\ &= \frac{\max\{|\zeta_1|_\infty, |\zeta_2|_\infty\}}{\left|d_n(1)^{1/\sigma_n(1)} d_n(2)^{1/\sigma_n(2)}\right|_\infty^{\min\{\sigma_n(1),\sigma_n(2)\}n/2}} < \frac{1}{|d_n(1)d_n(2)|_\infty^w}, \end{aligned}$$

when n is chosen large enough, i.e., $\zeta_1 \zeta_2 \in H_\infty(\xi)$ with corresponding $\sigma_n = \sigma_n(1) + \sigma_n(2)$.

Finally for division, by the result about multiplication, it suffices to show that if $\zeta \in H_\infty(\xi)$, then $1/\zeta \in H_\infty(\xi)$. In addition, by the remark right after the proof of Property $H_\infty 1$ and multiplying with a suitable element in $k(x)$, it is sufficient to treat the case where $|\zeta|_\infty = 1$. Using the above notation, we have for $n \geq w$ sufficiently large $1 = |\zeta|_\infty = |c_n/d_n|_\infty$, and so $|c_n|_\infty = |d_n|_\infty = |b_n|_\infty^{\sigma_n}$. Consequently, for sufficiently large n,

$$\left|\frac{1}{\zeta} - \frac{d_n}{c_n}\right|_\infty < \frac{1}{\left|\zeta\, c_n\, d_n^{n-1}\right|_\infty} = \frac{1}{|\zeta\, c_n^n|_\infty} < \frac{1}{|c_n|_\infty^w}. \qquad \square$$

Immediate from Proposition 2.1 is the next corollary.

Corollary 2.2. *If ξ is a k_∞-Liouville number, then its linear fractional transformation $\frac{a\xi+b}{c\xi+d}$, where $a,b,c,d \in k[x]$ are such that $c\xi+d \neq 0$, is either a k_∞-Liouville number or belongs to the $k(x)$.*

II. Next, we prove Erdős' theorem.

Theorem 2.3. *Let $\xi \in k((1/x)) \setminus \{0\}$. Then there are k_∞-Liouville numbers α, β, μ, ν such that*

$$\xi = \alpha + \beta = \mu \cdot \nu.$$

Proof. The theorem is trivial for $\xi \in k(x)$. We assume then that $\xi \notin k(x)$. By Proposition 2.1, it suffices to consider those elements ξ for which $|\xi|_\infty < 1$. Write

$$\xi = \frac{a_{n_1}}{x^{n_1}} + \frac{a_{n_2}}{x^{n_2}} + \frac{a_{n_3}}{x^{n_3}} + \cdots \in k((1/x)),$$

where $n_i \in \mathbb{N}\,(i \geq 1)$, $n_1 < n_2 < n_3 < \dots$ and $a_{n_i} \neq 0\,(i \geq 1)$. Let

$$\alpha := \frac{a_{n_1}}{x^{n_1}} + \left(\frac{a_{n_{3!}}}{x^{n_{3!}}} + \cdots + \frac{a_{n_{4!-1}}}{x^{n_{4!-1}}}\right) + \left(\frac{a_{n_{5!}}}{x^{n_{5!}}} + \cdots + \frac{a_{n_{6!-1}}}{x^{n_{6!-1}}}\right) + \dots,$$

$$\beta := \left(\frac{a_{n_{2!}}}{x^{n_{2!}}} + \cdots + \frac{a_{n_{3!-1}}}{x^{n_{3!-1}}}\right) + \left(\frac{a_{n_{4!}}}{x^{n_{4!}}} + \cdots + \frac{a_{n_{5!-1}}}{x^{n_{5!-1}}}\right) + \dots .$$

Clearly, $\xi = \alpha + \beta$. To finish the first half, we need to show that α and β are k_∞-Liouville numbers. Let $r \in \mathbb{N}$ and

$$\alpha_r := \sum_{i=1}^{r} \left(\frac{a_{n_{(2i-1)!}}}{x^{n_{(2i-1)!}}} + \cdots + \frac{a_{n_{(2i)!-1}}}{x^{n_{(2i)!-1}}}\right) = \frac{A_r}{x^{n_{(2r)!-1}}} \qquad (A_r \in k[x])$$

$$\beta_r := \sum_{i=1}^{r} \left(\frac{a_{n_{(2i)!}}}{x^{n_{(2i)!}}} + \cdots + \frac{a_{n_{(2i+1)!-1}}}{x^{n_{(2i+1)!-1}}}\right) = \frac{B_r}{x^{n_{(2r+1)!-1}}} \qquad (B_r \in k[x])\,.$$

Thus,

$$|\alpha - \alpha_r|_\infty = \left|\alpha - \frac{A_r}{x^{n_{(2r)!-1}}}\right|_\infty = e^{-n_{(2r+1)!}} < e^{-rn_{(2r)!-1}},$$

$$|\beta - \beta_r|_\infty = \left|\beta - \frac{B_r}{x^{n_{(2r+1)!-1}}}\right|_\infty = e^{-n_{(2r+2)!}} < e^{-rn_{(2r+1)!-1}},$$

as desired.

To prove the other half, we assume, again without loss of generality, that ξ is of the form $\xi = 1 + \sum_{n\geq 1} c_n/x^n \neq 1$. By [5, Theorem 5.1], or [7, Theorem 4.4] for the case of finite base field, $\xi(\neq 1)$ has a unique product representation of the form

$$\xi = \prod_{n=1}^{\infty}\left(1 + \frac{b_n}{x^{e_n}}\right),$$

where $b_n \in k\backslash\{0\}$, $e_n \in \mathbb{N}$ and $e_{n+1} > e_n > n$. Let $m_0 = 1$ and $m_1 > 1$. For each $r \in \mathbb{N}$, choose m_{2r} in such a way that

$$e_{m_{2r}+1} > r((e_{m_0+1} + \cdots + e_{m_1}) + \cdots + (e_{m_{2r-2}+1} + \cdots + e_{m_{2r-1}})),$$

and choose m_{2r+1} in such a way that

$$e_{m_{2r+1}} > r((e_{m_1+1} + \cdots + e_{m_2}) + \cdots + (e_{m_{2r-1}+1} + \cdots + e_{m_{2r}})).$$

Let

$$\begin{aligned}\mu_r &:= \prod_{i=1}^{r}\left(1 + \frac{b_{m_{2i-2}+1}}{x^{e_{m_{2i-2}+1}}}\right)\cdots\left(1 + \frac{b_{m_{2i-1}}}{x^{e_{m_{2i-1}}}}\right) \\ &= C_r/x^{\sum_{i=1}^{r}(e_{m_{2i-2}+1}+\cdots+e_{m_{2i-1}})} \qquad (C_r \in k[x])\end{aligned}$$

and

$$\begin{aligned}\nu_r &:= \prod_{i=1}^{r}\left(1 + \frac{b_{m_{2i-1}+1}}{x^{e_{m_{2i-1}+1}}}\right)\cdots\left(1 + \frac{b_{m_{2i}}}{x^{e_{m_{2i}}}}\right) \\ &= D_r/x^{\sum_{i=1}^{r}(e_{m_{2i-1}+1}+\cdots+e_{m_{2i}})} \qquad (D_r \in k[x]).\end{aligned}$$

Clearly, $\mu_r \to \mu \in k((1/x))$ $(r \to \infty)$, with $|\mu|_\infty = 1$, and

$$|\mu_r|_\infty^{-r} = q^{-r\sum_{i=1}^{r}(e_{m_{2i-2}+1}+\cdots+e_{m_{2i-1}})}.$$

Thus, for each $r \in \mathbb{N}$,

$$\begin{aligned}&\left|\mu - C_r/x^{\sum_{i=1}^{r}(e_{m_{2i-2}+1}+\cdots+e_{m_{2i-1}})}\right|_\infty = |\mu - \mu_r|_\infty = |\mu_r|_\infty\left|\frac{\mu}{\mu_r} - 1\right|_\infty \\ &\quad = \left|\prod_{i=r+1}^{\infty}\left(1 + \frac{b_{m_{2i-2}+1}}{x^{e_{m_{2i-2}+1}}}\right)\cdots\left(1 + \frac{b_{m_{2i-1}}}{x^{e_{m_{2i-1}}}}\right) - 1\right|_\infty = e^{-e_{m_{2r}+1}} \\ &\quad < \exp\left(-r\sum_{i=1}^{r}(e_{m_{2i-2}+1} + \cdots + e_{m_{2i-1}})\right)\end{aligned}$$

showing that μ is a k_∞-Liouville number. Similarly, if $\nu = \lim_{r\to\infty} \nu_r$, then

$$\left|\nu - D_r/x^{\sum_{i=1}^{r}(e_{m_{2i-1}+1}+\cdots+e_{m_{2i}})}\right|_\infty = |\nu - \nu_r|_\infty = e^{-e_{m_{2r+1}+1}},$$

i.e., ν is also a k_∞-Liouville number. Since $\xi = \mu \cdot \nu$, this completes the proof. □

III. In $k((1/x))$, there is a continued fraction expansion ([2] or [13]), whose construction and basic properties are similar to the one in the real case, which we now review. Since each element $\xi \in k((1/x)) \setminus \{0\}$ has a unique representation of the form

$$\xi = c_m x^m + c_{m-1}x^{m-1} + c_{m-2}x^{m-2} + \dots \quad (m \in \mathbb{Z}),$$

with coefficients $c_m (\neq 0), c_{m-1}, c_{m-2}, \dots \in k$. Define

$$\xi = [\xi] + (\xi),$$

where

$$[\xi] := c_m x^m + c_{m-1}x^{m-1} + \dots + c_1 x + c_0, \quad \text{and}$$
$$(\xi) := c_{-1}x^{-1} + c_{-2}x^{-2} + \dots,$$

with the customary convention that empty sums are interpreted as 0. Clearly, $[\xi]$ and (ξ) are uniquely determined. Let $\beta_0 = [\xi] \in k[x]$, so that $|\beta_0|_\infty = |\xi|_\infty \geq 1$, provided $[\xi] \neq 0$. If $(\xi) = 0$, then the process stops. If $(\xi) \neq 0$, then write

$$\xi = \beta_0 + \frac{1}{\xi_1},$$

where $\xi_1^{-1} = (\xi)$ with $|\xi_1|_\infty > 1$. Next write $\xi_1 = [\xi_1] + (\xi_1)$ and let $\beta_1 = [\xi_1] \in k[x] \setminus k$, so that $|\beta_1|_\infty = |\xi_1|_\infty > 1$. If $(\xi_1) = 0$, then the process stops. If $(\xi_1) \neq 0$, then write

$$\xi_1 = \beta_1 + \frac{1}{\xi_2},$$

where $\xi_2^{-1} = (\xi_1)$ with $|\xi_2|_\infty > 1$ and let $\beta_2 = [\xi_2] \in k[x] \setminus k$, so that $|\beta_2|_\infty = |\xi_2|_\infty > 1$. Again, if $(\xi_2) = 0$, then the process stops; otherwise, continue in the same manner. By so doing, we obtain the unique representation

$$\xi = [\beta_0; \beta_1, \dots, \beta_{n-1}, \xi_n] := \beta_0 + \cfrac{1}{\beta_1 + \cfrac{1}{\ddots \cfrac{1}{\beta_{n-1} + \frac{1}{\xi_n}}}},$$

where $\beta_i \in k[x] \setminus k$ $(i \geq 1)$, $\xi_n \in k((1/x))$, $|\xi_n|_\infty > 1$ if the process does not stop before, and ξ_n is referred to as the n^{th} *complete quotient*. The sequence (β_n) is uniquely determined, called the sequence of *partial quotients* of ξ.

The two sequences of partial numerators, (C_n), and partial denominators, (D_n), are defined by

$$C_{-1} = 1, \quad C_0 = \beta_0, \;\; C_{n+1} = \beta_{n+1}C_n + C_{n-1} \quad (n \geq 0)$$
$$D_{-1} = 0, \quad D_0 = 1, \quad D_{n+1} = \beta_{n+1}D_n + D_{n-1} \quad (n \geq 0).$$

The following basic properties are easily verified by induction.

Lemma 2.4. *For $\xi \in k((1/x)) \setminus \{0\}$, with the above notation we have, for $n \geq 0$*

$$\frac{\xi_{n+1}C_n + C_{n-1}}{\xi_{n+1}D_n + D_{n-1}} = [\beta_0; \beta_1, \beta_2, \ldots, \beta_n, \xi_{n+1}] \tag{2}$$

$$|D_n|_\infty = |\beta_1\beta_2 \ldots \beta_n|_\infty, \tag{3}$$

$$\left|\xi - \frac{C_n}{D_n}\right|_\infty = \frac{1}{|D_n D_{n+1}|_\infty} = \frac{1}{|D_n|_\infty^2 |\beta_{n+1}|_\infty}, \tag{4}$$

$$C_n D_{n-1} - D_n C_{n-1} = (-1)^{n-1}. \tag{5}$$

From (2), we have

$$\frac{C_n}{D_n} = \frac{\beta_n C_{n-1} + C_{n-2}}{\beta_n D_{n-1} + D_{n-2}} = [\beta_0; \beta_1, \beta_2, \ldots, \beta_n] \quad (n \geq 0),$$

and so C_n/D_n is called the n^{th} *convergent*. If $(\xi_n) = 0$ for some n, then $\xi = [\beta_0; \beta_1, \beta_2, \ldots, \beta_{n-1}]$, i.e., the expansion terminates. Otherwise, $(\xi_n) \neq 0$ for all n implying that the expansion is infinite. Equation (3) shows then that

$$|D_n^2\, \beta_{n+1}|_\infty = |\beta_1 \ldots \beta_n|_\infty^2 |\beta_{n+1}|_\infty \geq e^{2n+1}.$$

and so

$$\left|\xi - \frac{C_n}{D_n}\right|_\infty \leq \frac{1}{e^{2n+1}} \to 0 \quad (n \to \infty),$$

which enables us to write $\xi = [\beta_0; \beta_1, \beta_2, \beta_3, \ldots]$; the right hand side is referred to as the k_∞-*continued fraction* of ξ.

An infinite k_∞-continued fraction $[\bar{\beta}] = [\beta_0; \beta_1, \beta_2, \beta_3, \ldots] \in k((1/x))$ is called a k_∞-*Liouville continued fraction* if for $w \in \mathbb{R}^+$, there is an $n \in \mathbb{N}$ such that

$$0 < \left|[\bar{\beta}] - \frac{C_n}{D_n}\right|_\infty < \frac{1}{|D_n|_\infty^w}.$$

In $k((1/x))$, there is an analogue of the approximation property mentioned in the proof of Proposition 1.1.

Lemma 2.5. *Let $C, D (\neq 0)$ be relatively prime polynomials in $k[x]$. Then*

$$\left|\xi - \frac{C}{D}\right|_\infty < \frac{1}{|D|_\infty^2}$$

if and only if the rational function C/D is a convergent of the k_∞-continued fraction of ξ.

Proof. The ‘ if ’ part follows immediately from (4). To establish the converse, take $n \in \mathbb{N}$ so that $\deg D_{n-1} \leq \deg D < \deg D_n$, where D_n is the n^{th} partial denominator of the k_∞-continued fraction of ξ. From Lemma 2.4 (vi), i.e., $C_n D_{n-1} - D_n C_{n-1} = \pm 1$, there are $a, b \in k[x]$, both nonzero provided C/D is not equal to either C_{n-1}/D_{n-1} or C_n/D_n, satisfying the system of equations

$$D = aD_{n-1} + bD_n$$

$$C = aC_{n-1} + bC_n,$$

and so $D\xi - C = a(D_{n-1}\xi - C_{n-1}) + b(D_n\xi - C_n)$. From the first equation of the system, observe that $\deg aD_{n-1} = \deg bD_n$, i.e., $\deg a - \deg D_n = \deg b - \deg D_{n-1} > \deg b - \deg D_{n+1}$, i.,e., $|a/D_n|_\infty > |b/D_{n+1}|_\infty$. Equation (4) implies then that

$$|D\xi - C|_\infty = \left|\frac{a}{D_n}\right|_\infty > \left|\frac{b}{D_{n-1}}\right|_\infty \geq \frac{1}{|D_{n-1}|_\infty} \geq \frac{1}{|D|_\infty}$$

which is a contradiction. □

We next show that the two concepts of k_∞-Liouville numbers and k_∞-Liouville continued fractions coincide.

Proposition 2.6. *Let* $\xi = [\beta_0; \beta_1, \beta_2, \ldots] \in k((1/x))$ *with convergent* C_n/D_n. *The following assertions are equivalent:*

(i) ξ *is a* k_∞*-Liouville number.*

(ii) ξ *is a* k_∞*-Liouville continued fraction.*

(iii) *To each* $w \in \mathbb{N}$*, there is an* $\nu \in \mathbb{N}$ *such that* $|\beta_{\nu+1}|_\infty > |D_\nu|_\infty^w$.

Proof. That (i) $\Leftrightarrow$ (ii) is immediate from their corresponding definitions and Lemma 2.5.

To prove (ii) $\Rightarrow$ (iii), let $w \in \mathbb{N}$. Since ξ is a k_∞-Liouville continued fraction, for $m \geq w + 2$, there is a convergent C_m/D_m such that

$$0 < \frac{1}{|D_m|_\infty^2 |\beta_{m+1}|_\infty} = \left|\xi - \frac{C_m}{D_m}\right|_\infty < \frac{1}{|D_m|_\infty^m},$$

and so

$$|D_m|_\infty^w \leq |D_m|_\infty^{m-2} < |\beta_{m+1}|_\infty.$$

To prove (iii) $\Rightarrow$ (ii), let $w \in \mathbb{R}^+$. For $n \geq w$, since there is $\nu \in \mathbb{N}$ such that $|\beta_{\nu+1}|_\infty > |D_\nu|_\infty^n$, we have

$$0 < \left|\xi - \frac{C_\nu}{D_\nu}\right|_\infty = \frac{1}{|D_\nu|_\infty^2 |\beta_{\nu+1}|_\infty} < \frac{1}{|D_\nu|_\infty^{n+2}} < \frac{1}{|D_\nu|_\infty^w}. \qquad \square$$

3. The field of p-adic numbers

Let $\mathbb{Q}_p$ be the field of p-adic numbers equipped with the p-adic valuation, $|\cdot|_p$ so normalized that $|p|_p = 1/p$. The *p-adic Liouville's theorem*, see e.g. p. 46 of [9], states that if $\xi \in \mathbb{Q}_p$ is an algebraic number of degree n, then there exists a positive constant $c(\xi)$ depending only on ξ such that

$$|b\xi - a|_p \geq \frac{c(\xi)}{|a,b|^n},$$

for all $a, b\,(\neq 0) \in \mathbb{Z}$, where $|a,b| := \max\{|a|,|b|\}$. A number $\xi \in \mathbb{Q}_p$ is called a *p-adic Liouville number* if for any $w \in \mathbb{R}^+$, there exist $a, b \in \mathbb{Z}\backslash\{0\}$ with $|a,b| > 1$ such that

$$0 < |b\xi - a|_p < \frac{1}{|a,b|^w}.$$

Let ξ be a p-adic Liouville number. For each fixed $w \in \mathbb{R}^+$, from the definition, there are sequences (a_n) and (b_n) in $\mathbb{Z}\backslash\{0\}$, with $b_n \in \mathbb{N}$, $|a_{n+1}, b_{n+1}| > |a_n, b_n| > 1$ such that

$$0 < |b_n\xi - a_n|_p < \frac{1}{|a_n,b_n|^w}.$$

I. We define the class $H_p(\xi)$ of *p-adic transcendental numbers associated with* ξ as follows: a p-adic transcendental number $\zeta \in H_p(\xi)$ if, for the given $w > 0$ and (a_n/b_n), there are sequences (c_n) and (d_n) in $\mathbb{Z}\backslash\{0\}$ such that

$$0 < |d_n\zeta - c_n|_p < \frac{1}{|c_n,d_n|^w} \quad \text{and} \quad |c_n,d_n| = |a_n,b_n|^{\sigma_n},$$

where (σ_n) is a sequence of real numbers bounded both above and below by fixed positive constants. Clearly, each element in $H_p(\xi)$ is also a p-adic Liouville number.

In addition, define

- *Property H_p1*: $\frac{a}{b}\xi + \frac{c}{d} \in H_p(\xi)$ for all $a(\neq 0)$, $b(\neq 0)$, $c, d(\neq 0)$ in $\mathbb{Z}$.
- *Property H_p2*: The sum, difference, product and division of two numbers from $H_p(\xi)$ is either a number in $H_p(\xi)$ or an element in $\mathbb{Q}$.

Proposition 3.1. *Properties H_p1 and H_p2 hold for ξ being a p-adic Liouville number.*

Proof. Fix $w > 0$. For $n \in \mathbb{N}$, $n \geq w$, since ξ is a p-adic Liouville number, there exist $a_n, b_n \in \mathbb{Z}\backslash\{0\}$ with $|a_n, b_n| > 1$ such that

$$0 < |b_n\xi - a_n|_p < \frac{1}{|a_n,b_n|^n} \leq \frac{1}{|a_n,b_n|^w}.$$

Consider the element $\frac{a}{b}\xi + \frac{c}{d}$ with $a(\neq 0)$, $b(\neq 0)$, $c, d(\neq 0)$ in $\mathbb{Z}$. Let

$$\frac{c_n}{d_n} := \frac{a}{b}\,\frac{a_n}{b_n} + \frac{c}{d} = \frac{ada_n + bcb_n}{bdb_n},$$

where $c_n = ada_n + bcb_n$ and $d_n = bdb_n$. Let (ϵ_n) be a sequence of positive numbers such that

$$|c_n| \leq \max\left(|ada_n|, |bcb_n|\right) \leq |a_n, b_n|^{1+\epsilon_n}$$

and

$$|d_n| = |bdb_n| \leq |a_n, b_n|^{1+\epsilon_n}.$$

Clearly, (ϵ_n) is a bounded sequence of positive numbers. Thus

$$\begin{aligned} 0 < \left| d_n\left(\frac{a}{b}\xi + \frac{c}{d}\right) - c_n \right|_p &= |ad|_p |b_n\xi - a_n|_p < \frac{|ad|_p}{|a_n, b_n|^n} \\ &\leq \frac{|ad|_p}{|c_n, d_n|^{n/(1+\epsilon_n)}} \leq \frac{1}{|c_n, d_n|^w}, \end{aligned}$$

by choosing n large enough. Property H_p1 follows with $\sigma_n = 1+\epsilon'_n$ where $\epsilon'_n \leq \epsilon_n$.

We remark that with the same proof just shown, we also have that if $\zeta \in H_p(\xi)$, then $\frac{a}{b}\zeta + \frac{c}{d} \in H_p(\xi)$ for all $a(\neq 0), b(\neq 0), c, d(\neq 0) \in \mathbb{Z}$.

To establish Property H_p2, let $\zeta_1, \zeta_2 \in H_p(\xi)$. Then for $n \in \mathbb{N}, n \geq w$, there exist $c_n(i), d_n(i) \in \mathbb{Z} \setminus \{0\}$ $(i = 1, 2)$ with $|d_n(i)| > 1$ such that

$$0 < |d_n(i)\zeta_i - c_n(i)|_p < \frac{1}{|c_n(i), d_n(i)|^n} \leq \frac{1}{|c_n(i), d_n(i)|^w},$$

$$|c_n(i), d_n(i)| = |a_n, b_n|^{\sigma_n(i)} \quad (i = 1, 2). \tag{6}$$

For the proof of the sum and difference of two p-adic Liouville numbers, it suffices by property H_p1 to prove only for the case of the sum. Assuming $\zeta_1 + \zeta_2 \notin \mathbb{Q}$, then

$$\begin{aligned} 0 &< |d_n(1)d_n(2)\{\zeta_1 + \zeta_2\} - \{c_n(1)d_n(2) + c_n(2)d_n(1)\}|_p \\ &= |d_n(1)d_n(2)|_p \left| \left(\zeta_1 - \frac{c_n(1)}{d_n(1)}\right) + \left(\zeta_2 - \frac{c_n(2)}{d_n(2)}\right) \right|_p \\ &< \max\left\{ \frac{1}{|c_n(1), d_n(1)|^n}, \frac{1}{|c_n(2), d_n(2)|^n} \right\} = \frac{1}{|a_n, b_n|^{n\min\{\sigma_n(1), \sigma_n(2)\}}} \\ &= \frac{1}{\{|c_n(1), d_n(1)| \cdot |c_n(2), d_n(2)|\}^{n\min\{\sigma_n(1), \sigma_n(2)\}/(\sigma_n(1)+\sigma_n(2))}} \end{aligned}$$

$$\leq \frac{1}{|c_n(1)c_n(2), c_n(1)d_n(2), d_n(1)c_n(2), d_n(1)d_n(2)|^w}$$
$$\leq \frac{1}{|d_n(1)d_n(2), \{c_n(1)d_n(2) + c_n(2)d_n(1)\}|^w}$$

when n is chosen large enough, i.e., $\zeta_1 + \zeta_2 \in H_p(\xi)$ with corresponding $\sigma_n = \sigma'_n(1) + \sigma'_n(2)$ where $\sigma'_n(1) + \sigma'_n(2) \leq \sigma_n(1) + \sigma_n(2)$.

Next consider the product $\zeta_1 \zeta_2$ and assuming that $\zeta_1\zeta_2 \notin k(x)$. For n sufficiently large, since both $|d_n(1)\zeta_1|_p$ and $|c_n(1)|_p$ are $> 1/|c_n(i), d_n(i)|^n$, the strong triangle inequality and (3) yield $|\zeta_1|_p = |c_n(1)/d_n(1)|_p$. Thus,

$$\begin{aligned}
0 &< |d_n(1)d_n(2)\zeta_1 \zeta_2 - c_n(1)c_n(2)|_p \\
&= |\{d_n(1)\zeta_1 - c_n(1)\}\, d_n(2)\zeta_2 + \{d_n(2)\zeta_2 - c_n(2)\}\, c_n(1)|_p \\
&< \max\left\{\frac{|\zeta_2|_p}{|c_n(1), d_n(1)|^n}, \frac{1}{|c_n(2), d_n(2)|^n}\right\} \leq \frac{\max\{1, |\zeta_2|_p\}}{|a_n, b_n|^{n \min\{\sigma_n(1), \sigma_n(2)\}}} \\
&< \frac{1}{|c_n(1)c_n(2), d_n(1)d_n(2)|^w},
\end{aligned}$$

when n is chosen large enough, i.e., $\zeta_1\zeta_2 \in H_p(\xi)$ with corresponding $\sigma_n = \sigma'_n(1) + \sigma'_n(2)$ where $\sigma'_n(1) + \sigma'_n(2) \leq \sigma_n(1) + \sigma_n(2)$.

Finally for division, by the result about multiplication, it suffices to show that if $\zeta \in H_p(\xi)$, then $1/\zeta \in H_p(\xi)$. In addition, by the remark right after the proof of Property H_p1 and multiplying with a suitable element in $k(x)$, it is sufficient to treat the case where $|\zeta|_p = 1$. Using the above notation, we have for $n \geq w$ sufficiently large $1 = |\zeta|_p = |c_n/d_n|_p$, and so $|c_n|_p = |d_n|_p = |b_n|^{\sigma_n}$. Consequently, for sufficiently large n,

$$\left|c_n \cdot \frac{1}{\zeta} - d_n\right|_p = |c_n - d_n\zeta|_p < \frac{1}{|c_n, d_n|^n} < \frac{1}{|c_n, d_n|^w},$$

i.e., $1/\zeta \in H_p(\xi)$ with the same σ_n. □

Corollary 3.2. *If ξ is a p-adic Liouville number, then its linear fractional transformation $\frac{a\xi+b}{c\xi+d}$, where $a, b, c, d \in \mathbb{Z}$ are such that $c\xi + d \neq 0$, is either a p-adic Liouville number or a rational number.*

II. Next, we prove the p-adic analogue of Erdős' theorem.

Theorem 3.3. *Let $\xi \in \mathbb{Q}_p \setminus \{0\}$. Then there are p-adic Liouville numbers α, β, μ, ν such that*

$$\xi = \alpha + \beta = \mu \cdot \nu.$$

Proof. The part corresponding to the sum of two p-adic Liouville numbers was proved by Menken in [11]. To prove the other half, we assume, again without loss of generality, that $\xi(\neq 0)$ is of the form $\xi = 1 + \sum_{n\geq 1} c_n p^n \neq 1$, $c_n \in \{0, 1, \dots, p-1\}$. By [5, Theorem 1.4], ξ has a unique product representation of the form

$$\xi = \prod_{n=1}^{\infty}(1 + b_n p^{e_n}),$$

where $1 \leq b_n \leq p-1$, $e_n \in \mathbb{N}$ and $e_{n+1} > e_n$. Let $m_0 = 1$ and $m_1 > 1$. For each $r \in \mathbb{N}$, choose m_{2r} such that

$$e_{m_{2r}+1} \geq r \sum_{i=1}^{r} \left((e_{m_{2i-2}+1} + 1) + \cdots + (e_{m_{2i-1}} + 1)\right),$$

and choose m_{2r+1} such that

$$e_{m_{2r+1}} \geq r \sum_{i=1}^{r} \left((e_{m_{2i-1}+1} + 1) + \cdots + (e_{m_{2i}} + 1)\right).$$

Let

$$\mu_r := \prod_{i=1}^{r}(1 + b_{m_{2i-2}+1} p^{e_{m_{2i-2}+1}}) \dots (1 + b_{m_{2i-1}} p^{e_{m_{2i-1}}})$$

and

$$\nu_r := \prod_{i=1}^{r}(1 + b_{m_{2i-1}+1} p^{e_{m_{2i-1}+1}}) \dots (1 + b_{m_{2i}} p^{e_{m_{2i}}}).$$

Clearly $\mu_r \to \mu \in \mathbb{Q}_p$ $(r \to \infty)$, with $|\mu|_p = 1$. We have

$$|\mu_r| < p^{\sum_{i=1}^{r}\left((e_{m_{2i-2}+1}+1)+\cdots+(e_{m_{2i-1}}+1)\right)}.$$

Thus, for each $r \in \mathbb{N}$,

$$\begin{aligned}
|\mu - \mu_r|_p &= |\mu_r|_p \left| \frac{\mu}{\mu_r} - 1 \right|_p \\
&= \left| \prod_{i=r+1}^{\infty} (1 + b_{m_{2i-2}+1} p^{e_{m_{2i-2}+1}}) \dots (1 + b_{m_{2i-1}} p^{e_{m_{2i-1}}}) - 1 \right|_p \\
&= |p^{e_{m_{2r}+1}}|_p = p^{-e_{m_{2r}+1}} \leq p^{-r\sum_{i=1}^{r}((e_{m_{2i-2}+1}+1)+\cdots+(e_{m_{2i-1}}+1))} < |\mu_r|^{-r}
\end{aligned}$$

showing that μ is a p-adic Liouville number. Similarly, if $\nu = \lim_{r\to\infty} \nu_r$, then

$$|\nu - \nu_r|_p = e^{-e_{m_{2r+1}+1}} < |\nu_r|^{-r},$$

i.e., ν is also a p-adic Liouville number. Since $\xi = \mu \cdot \nu$, this completes the proof. $\square$

III. The so-called Ruban continued fraction expansion in $\mathbb{Q}_p$, see e.g. [6], is now considered. Without loss of generality, it suffices to consider numbers in $p\mathbb{Z}_p$. Let $\xi \in p\mathbb{Z}_p \backslash \{0\}$. Since $\xi \neq 0$, we have $|\xi^{-1}|_p > 1$ and we uniquely write

$$\xi^{-1} = b_{-n}p^{-n} + b_{-n+1}p^{-n+1} + \cdots + b_0 + b_1 p + b_2 p^2 + \dots$$

where $n \in \mathbb{N}$ and $b_i \in \{0, 1, \dots, p-1\}$ for all $i \geq -n$ with $b_{-n} \neq 0$. Define

$$\left[\xi^{-1}\right] := \sum_{i=-n}^{0} b_i p^i, \quad \left(\xi^{-1}\right) := \sum_{i=1}^{\infty} b_i p^i,$$

Uniquely, we have $\xi^{-1} = [\xi^{-1}] + (\xi^{-1})$. Next write

$$\xi^{-1} = a_1 + \xi_2,$$

where $a_1 = [\xi^{-1}]$, $\xi_2 = (\xi^{-1})$. If $\xi_2 = 0$, the algorithm stops. If $\xi_2 \neq 0$, since $|\xi_2^{-1}|_p > 1$, repeating the step just described, we can uniquely write

$$\xi_2^{-1} = a_2 + \xi_3,$$

where $a_2 = [\xi_2^{-1}]$, $\xi_3 = (\xi_2^{-1})$. Again, if $\xi_3 = 0$, the algorithm stops, otherwise it proceeds in the same manner with ξ_3 replacing ξ_2 and so on. Since the a_i's $(i \geq 0)$ obtained are unique, each $\xi \in p\mathbb{Z}_p \backslash \{0\}$ has a unique Ruban continued fraction expansion of the form

$$\xi = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \; \frac{1}{a_k + \xi_{k+1}}}}},$$

where the a_i's are of the form $\sum_{i=-n}^{0} b_i p^i \in \mathbb{Z}[1/p]$ $(n \in \mathbb{N})$, $b_i \in \{0, 1, \dots, p-1\}$ for all i, with $b_{-n} \neq 0$. Define the two sequences of partial numerators and partial numerators by

$$C_{-1} = 1, \quad C_0 = 0, \quad C_{k+1} = a_{k+1}C_k + C_{k-1} \in \mathbb{Z}[1/p] \quad (k = 0, 1, 2, \dots)$$
$$D_{-1} = 0, \quad D_0 = 1, \quad D_{k+1} = a_{k+1}D_k + D_{k-1} \in \mathbb{Z}[1/p] \quad (k = 0, 1, 2, \dots),$$

and the so k^{th} convergent to ξ is

$$\frac{C_k}{D_k} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \; \frac{1}{a_{k-1} + \frac{1}{a_k}}}}}.$$

If $(\xi_k) = 0$ for some k, then $\xi = [0; a_1, a_2, \ldots, a_{k-1}]$, i.e., the Ruban continued fraction of ξ terminates. Otherwise, $(\xi_k) \neq 0$ for all k and its Ruban continued fraction is infinite and in this case, and we write

$$\xi = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}} := [0; a_1, a_2, \ldots], \tag{7}$$

where the right hand side is referred to as the Ruban continued fraction of ξ. Since the partial quotients, a_i, partial numerators C_k, and partial denominators D_k are all rational, not necessarily integral, it is convenient to make use of its *associated Ruban continued fraction*

$$\xi = \cfrac{\gamma_1}{\beta_1 + \cfrac{\gamma_2}{\beta_2 + \cfrac{\gamma_3}{\beta_3 + \ldots}}} := [0; \gamma_1/\beta_1, \gamma_2/\beta_2, \gamma_3/\beta_3, \ldots], \tag{8}$$

where

$$\gamma_1 = |a_1|_p, \quad \gamma_n = |a_n a_{n-1}|_p \quad (n \geq 2), \quad \beta_n = a_n |a_n|_p \quad (n \geq 1).$$

Clearly, the partial numerators γ_n and the partial denominators β_n of the associated Ruban continued fraction are positive integers. We similarly define the k^{th} convergent of (8) as

$$\mathcal{C}_{-1} = 1, \quad \mathcal{C}_0 = 0, \quad \mathcal{C}_{k+1} = \beta_{k+1}\mathcal{C}_k + \gamma_{k+1}\mathcal{C}_{k-1} \in \mathbb{Z} \qquad (k = 0, 1, 2, \ldots)$$

$$\mathcal{D}_{-1} = 0, \quad \mathcal{D}_0 = 1, \quad \mathcal{D}_{k+1} = \beta_{k+1}\mathcal{D}_k + \gamma_{k+1}\mathcal{D}_{k-1} \in \mathbb{Z} \quad (k = 0, 1, 2, \ldots).$$

The Ruban continued fraction (7) and its associated Ruban continued fraction (8) are equivalent in the sense that $C_k/D_k = \mathcal{C}_k/\mathcal{D}_k$ for every $k \geq 0$, which can easily proved by induction. The following lemma summarizes basic properties of the Ruban continued fraction of ξ and its associated Ruban continued fraction.

Lemma 3.4. *For $n \geq 1$, we have*

$$|C_n|_p = |a_2 \cdots a_n|_p < |D_n|_p = |a_1 \cdots a_n|_p \tag{9}$$

$$\left|\xi - \frac{\mathcal{C}_n}{\mathcal{D}_n}\right|_p = \left|\xi - \frac{C_n}{D_n}\right|_p = \frac{1}{|D_n D_{n+1}|_p} = \frac{1}{|a_{n+1} D_n^2|_p} < \frac{1}{|D_n^2|_p}, \tag{10}$$

$$|\mathcal{C}_n, \mathcal{D}_n| := \max(|\mathcal{C}_n|, |\mathcal{D}_n|) < 2^n \left|a_1^2 a_2^2 \cdots a_{n-1}^2 a_n\right|_p, \tag{11}$$

$$|D_n|_p < |D_{n+1}|_p, \ |\mathcal{D}_n|_p = 1. \tag{12}$$

$$D_n C_{n-1} - C_n D_{n-1} = (-1)^{n-1}. \tag{13}$$

An infinite Ruban continued fraction $[\bar{\beta}] = [\beta_0; \beta_1, \beta_2, \beta_3, \ldots]$ is a *p-adic Liouville continued fraction* if for any $w \in \mathbb{R}^+$, there is an $n \in \mathbb{N}$ such that

$$0 < \left|D_n \left[\bar{\beta}\right] - C_n\right|_p < \frac{1}{\left|\mathcal{C}_n, \mathcal{D}_n\right|^w}.$$

In $\mathbb{Q}_p$, we do not have an analogue of the approximation property mentioned in Proposition 1.1. This is due to the fact that we only have a weaker analogoue of Lemma 2.5.

Lemma 3.5. *Let $\xi \in p\mathbb{Z}_p$ and let $C, D \in \mathbb{Z}[1/p]$ with $|C|_p, |D|_p \geq 1$.*

(i) *If C/D is a convergent of the continued fraction of ξ, then*

$$\left|\xi - \frac{C}{D}\right|_p < \frac{1}{|D|_p^2}.$$

(ii) *If $|C_{n-1}D - CD_{n-1}|_p \geq 1$ and $n \in \mathbb{N}$ is so chosen that $|D_{n-1}|_p \leq |D|_p < |D_n|_p$, where D_n is the n^{th} partial denominator of the continued fraction of ξ, then either C/D is the n^{th} convergent of the continued fraction of ξ, or*

$$\left|\xi - \frac{C_{n-1}}{D_{n-1}}\right|_p \leq \left|\xi - \frac{C}{D}\right|_p.$$

Proof. The first part follows immediately from (10). To establish the second part, note that

$$\frac{C}{D} \neq \frac{C_{n-1}}{D_{n-1}}, \quad \frac{C}{D} \neq \frac{C_n}{D_n}. \tag{14}$$

Using (13), we deduce that there are $a, b \in \mathbb{Z}[1/p] \setminus \{0\}$ satisfying the system of equations

$$\begin{aligned} D &= aD_{n-1} + bD_n \\ C &= aC_{n-1} + bC_n, \end{aligned} \tag{15}$$

namely, $a = (-1)^n(C_nD - CD_n) \neq 0$ and $b = (-1)^{n-1}(C_{n-1}D - CD_{n-1}) \neq 0$, and so

$$D\xi - C = a(D_{n-1}\xi - C_{n-1}) + b(D_n\xi - C_n). \tag{16}$$

From (15), using the strong triangle inequality and $|D_{n-1}|_p \leq |D|_p < |D_n|_p$, we get $|aD_{n-1}|_p = |bD_n|_p$ and so

$$|D|_p \leq |aD_{n-1}|_p = |bD_n|_p. \tag{17}$$

From (16), using (17), $|\xi - C_n/D_n|_p < |\xi - C_{n-1}/D_{n-1}|_p$ and (10), we have

$$\begin{aligned}\left|\xi - \frac{C}{D}\right|_p &= \left|\frac{aD_{n-1}}{D}\left(\xi - \frac{C_{n-1}}{D_{n-1}}\right)\right|_p = \left|\frac{a}{DD_n}\right| \\ &= \left|\frac{b}{DD_{n-1}}\right|_p \geq \frac{1}{|D_nD_{n-1}|_p} = \left|\xi - \frac{C_{n-1}}{D_{n-1}}\right|_p . \qquad \square\end{aligned}$$

Regarding connections between p-adic Liouville numbers and p-adic Liouville continued fractions, we have:

Proposition 3.6. *Let $\xi = [\beta_0; \beta_1, \beta_2, \ldots] \in p\mathbb{Z}_p$. Consider the following assertions.*

(i) *ξ is a p-adic Liouville number.*

(ii) *ξ is a p-adic Liouville continued fraction.*

(iii) *To each $w \in \mathbb{N}$, there is an $\nu \in \mathbb{N}$ such that $|a_{\nu+1}|_p > |D_\nu|_p^w$.*

Then we have (ii) $\Rightarrow$ (i) *and* (ii) $\Longleftrightarrow$ (iii).

Proof. That (ii) $\Rightarrow$ (i) is immediate from the definitions.

To prove (ii) $\Rightarrow$ (iii), let $w \in \mathbb{N}$. Since ξ is a p-adic Liouville continued fraction, for $m \geq w + 2$, there is a convergent C_m/D_m such that

$$0 < \frac{1}{|a_{m+1}D_m|_p} = |D_m\xi - C_m|_p < \frac{1}{|\mathcal{C}_m, \mathcal{D}_m|^m} < \frac{(2|a_m|_p)^{1/2}}{|D_m|_p^m},$$

and so for m large enough

$$|C_m, D_m|_p^w = |D_m|_p^w \leq \frac{|D_m|_p^{m-1}}{(2|a_m|_p)^{1/2}} < |a_{m+1}|_p.$$

To prove (iii) $\Rightarrow$ (ii), let $w \in \mathbb{R}^+$. For $n \geq w$, since there is $\nu \in \mathbb{N}$ such that $|a_{\nu+1}|_p > |D_\nu|_p^n$, we have for n sufficiently large, using also (11),

$$\begin{aligned}0 < |D_\nu\xi - C_\nu|_p &= \frac{1}{|a_{\nu+1}D_\nu|_p} < \frac{1}{|D_\nu|_p^{n+1}} = \frac{1}{|a_1 \cdots a_n|_p^{n+1}} \\ &< \frac{1}{2^{nw}|a_1^2 \cdots a_{n-1}^2 a_n|_p^w} < \frac{1}{|\mathcal{C}_n, \mathcal{D}_n|^w}. \qquad \square\end{aligned}$$

4. The function field with τ-adic valuation

Let k be a field, x an indeterminate, τ a monic irreducible element of $k[x]$ and let $k((\tau)) =: k_\tau$ be the field of all formal Laurent series in τ equipped with the τ-adic valuation, $|\cdot|_\tau$, so normalized that $|\tau|_\tau = e^{-\deg \tau}$. The *Liouville's theorem* for k_τ, see e.g. [14], states that if $\xi \in k((\tau))$ is an algebraic number, over $k(x)$, of degree $n \geq 2$, then there exists a positive constant $c(\xi)$ depending only on ξ such that

$$|b\xi - a|_\tau \geq \frac{c(\xi)}{|a,b|_\infty^n}$$

for all $a, b\,(\neq 0) \in k[x]$, where $|a,b|_\infty := \max\{|a|_\infty, |b|_\infty\}$. An element $\xi \in k((\tau))$ is called a k_τ-*Liouville number* if for any $w \in \mathbb{R}^+$, there exist $a, b \in k[x]\backslash\{0\}$ with $|a,b|_\infty > 1$ such that

$$0 < |b\xi - a|_\tau < \frac{1}{|a,b|_\infty^w}.$$

Let ξ be a k_τ-Liouville number. For fixed $w \in \mathbb{R}^+$, from the definition, there are sequences (a_n) and (b_n) in $k[x]\backslash\{0\}$, with $|a_n, b_n|_\infty > 1$ such that

$$0 < |b_n\xi - a_n|_\tau < \frac{1}{|a_n, b_n|_\infty^w}.$$

I. We define the class $H_\tau(\xi)$ of transcendental elements associated with ξ as follows: a k_τ-transcendental element $\zeta \in H_\tau(\xi)$ if, for the given $w > 0$ and the corresponding sequence (a_n/b_n), there are sequences (c_n) and (d_n) in $k[x]\backslash\{0\}$ such that

$$0 < |d_n\zeta - c_n|_\tau < \frac{1}{|c_n, d_n|_\infty^w} \quad \text{and} \quad |c_n, d_n|_\infty = |a_n, b_n|_\infty^{\sigma_n},$$

where (σ_n) is a sequence of real numbers bounded both above and below by fixed positive constants. Clearly, each element in any of the class $H_\tau(\xi)$ is also a k_τ-Liouville number. Moreover, we define:

- *Property* $H_\tau 1$: $\frac{a}{b}\xi + \frac{c}{d} \in H_\tau(\xi)$ for all $a(\neq 0), b(\neq 0), c, d(\neq 0)$ in $k[x]$.
- *Property* $H_\tau 2$: The sum, difference, product and division of two numbers from $H_\tau(\xi)$ is either a number in $H_\tau(\xi)$ or an element in $k(x)$.

Slightly modified proofs of Theorem 3.1 and Corollary 3.2 yield:

Proposition 4.1. *Properties $H_\tau 1$ and $H_\tau 2$ hold for ξ being a k_τ-Liouville number.*

Corollary 4.2. *If ξ is a k_τ-Liouville number, then its linear fractional transformation $\frac{a\xi+b}{c\xi+d}$, where a, b, c, d belong to $k[x]$ are such that $c\xi + d \neq 0$, is either a k_τ-Liouville number or belong to the $k(x)$.*

II. We now prove the result of Erdős.

Theorem 4.3. *To each number ξ in $k((\tau)) \setminus \{0\}$, there are k_τ-Liouville numbers α, β, μ, ν such that*

$$\xi = \alpha + \beta = \mu \cdot \nu.$$

Proof. The theorem is trivial for $\xi \in k(x)$. Thus we assume that $\xi \notin k(x)$. By Proposition 2.1, we also assume that $|\xi|_\tau < 1$. Write

$$\xi = a_{n_1}\tau^{n_1} + a_{n_2}\tau^{n_2} + a_{n_3}\tau^{n_3} + \dots \quad \in k((\tau)),$$

where $n_i \in \mathbb{N}$, $n_1 < n_2 < n_3 < \dots$ and $a_{n_i} \neq 0$ for all $i \geq 1$. Let

$$\alpha := a_{n_1}\tau^{n_1} + (a_{n_{3!}}\tau^{n_{3!}} + \dots + a_{n_{4!-1}}\tau^{n_{4!-1}}) + (a_{n_{5!}}\tau^{n_{5!}} + \dots + a_{n_{6!-1}}\tau^{n_{6!-1}}) + \dots,$$
$$\beta := (a_{n_{2!}}\tau^{n_{2!}} + \dots + a_{n_{3!-1}}\tau^{n_{3!-1}}) + (a_{n_{4!}}\tau^{n_{4!}} + \dots + a_{n_{5!-1}}\tau^{n_{5!-1}}) + \dots.$$

Then $\xi = \alpha + \beta$. To finish the first half, we need to show that α and β are k_τ-Liouville numbers. Let $r \in \mathbb{N}$ and

$$\alpha_r := \sum_{i=1}^{r} \left(a_{n_{(2i-1)!}}\, \tau^{n_{(2i-1)!}} + \dots + a_{n_{(2i)!-1}}\, \tau^{n_{(2i)!-1}}\right),$$
$$\beta_r := \sum_{i=1}^{r} \left(a_{n_{(2i)!}}\, \tau^{n_{(2i)!}} + \dots + a_{n_{(2i+1)!-1}}\, \tau^{n_{(2i+1)!-1}}\right).$$

Thus,
$$|\alpha_r|_\infty^{-r} = e^{-rn_{(2r)!-1}\deg\tau}, \quad |\beta_r|_\infty^{-r} = e^{-rn_{(2r+1)!-1}\deg\tau},$$
yielding
$$|\alpha - \alpha_r|_\tau = e^{-n_{(2r+1)!}\deg\tau} < e^{-rn_{(2r)!-1}\deg\tau} = |\alpha_r|_\infty^{-r},$$
and
$$|\beta - \beta_r|_\tau = e^{-n_{(2r+2)!}\deg\tau} < e^{-rn_{(2r+1)!-1}\deg\tau} = |\beta_r|_\infty^{-r},$$
as desired.

To prove the other half, we assume, again without loss of generality, that ξ is of the form $\xi = 1 + \sum_{n\geq 1} c_n\tau^n \neq 1$, $\deg c_n < \deg\tau$. By [5, Theorem 5.1] or

[7, Theorem 4.4] for the case of finite base field, ξ has a unique product representation of the form

$$\xi = \prod_{n=1}^{\infty} (1 + b_n \tau^{e_n}),$$

where $b_n \in k[x]\backslash\{0\}, \deg b_n < \deg \tau$, $e_n \in \mathbb{N}$ and $e_{n+1} > e_n > n$. Let $m_0 = 1$ and $m_1 > 1$. For each $r \in \mathbb{N}$, choose m_{2r} such that

$$e_{m_{2r}+1} > r\big((e_{m_0+1} + \cdots + e_{m_1}) + \cdots + (e_{m_{2r-2}+1} + \cdots + e_{m_{2r-1}})\big),$$

and choose m_{2r+1} such that

$$e_{m_{2r+1}} > r\big((e_{m_1+1} + \cdots + e_{m_2}) + \cdots + (e_{m_{2r-1}+1} + \cdots + e_{m_{2r}})\big).$$

Let

$$\mu_r := \prod_{i=1}^{r} (1 + b_{m_{2i-2}+1} \tau^{e_{m_{2i-2}+1}}) \dots (1 + b_{m_{2i-1}} \tau^{e_{m_{2i-1}}})$$

and

$$\nu_r := \prod_{i=1}^{r} (1 + b_{m_{2i-1}+1} \tau^{e_{m_{2i-1}+1}}) \dots (1 + b_{m_{2i}} \tau^{e_{m_{2i}}}).$$

Clearly $\mu_r \to \mu \in k((\tau))$ $(r \to \infty)$, with $|\mu|_\tau = 1$. We have

$$|\mu_r|_\infty^{-r} = q^{-r \sum_{i=1}^{r} (e_{m_{2i-2}+1} + \cdots + e_{m_{2i-1}})}.$$

Thus, for each $r \in \mathbb{N}$,

$$\begin{aligned}
|\mu - \mu_r|_\tau &= |\mu_r|_\tau \left| \frac{\mu}{\mu_r} - 1 \right|_\tau \\
&= \left| \prod_{i=r+1}^{\infty} (1 + b_{m_{2i-2}+1} \tau^{e_{m_{2i-2}+1}}) \dots (1 + b_{m_{2i-1}} \tau^{e_{m_{2i-1}}}) - 1 \right|_\tau \\
&= |\tau^{e_{m_{2r}+1}}|_\tau = e^{-e_{m_{2r}+1}} < e^{-r \sum_{i=1}^{r} (e_{m_{2i-2}+1} + \cdots + e_{m_{2i-1}})} = |\mu_r|_\infty^{-r}
\end{aligned}$$

showing that μ is a k_τ-Liouville element. Similarly, if $\nu = \lim_{r \to \infty} \nu_r$, then

$$|\nu - \nu_r|_\tau = e^{-e_{m_{2r+1}+1}} < |\nu_r|_\infty^{-r},$$

i.e., ν is also a k_τ-Liouville element. Since $\xi = \mu \cdot \nu$, this completes the proof. $\square$

III. There is a continued fraction algorithm in $k((\tau))$ very much like the Ruban continued fraction expansion in $\mathbb{Q}_p$ which is now briefly described. Without loss of generality, it suffices to consider numbers $\xi \in k((\tau)) \setminus \{0\}$ for which $|\xi|_\tau < 1$, i.e.,

$$\xi = \frac{c_{-m}}{\tau^m} + \frac{c_{-m+1}}{\tau^{m-1}} + \frac{c_{-m+2}}{\tau^{m-2}} + \cdots + c_0 + c_1\tau + c_2\tau^2 + \cdots \in k((\tau)) \quad (m \in \mathbb{N}),$$

with coefficients $c_{-m}(\neq 0), c_{-m+1}, c_{-m+2}, \cdots \in k[x]$, $\deg c_i < \deg \tau$. Define

$$\xi = [\xi] + (\xi),$$

where

$$[\xi] := \frac{c_{-m}}{\tau^m} + \frac{c_{-m+1}}{\tau^{m-1}} + \frac{c_{-m+2}}{\tau^{m-2}} + \cdots + c_0, \quad \text{and} \quad (\xi) := c_1\tau + c_2\tau^2 + \ldots.$$

Let $\beta_0 = [\xi] \in k[1/\tau]$, so that $|\beta_0|_\tau = |\xi|_\tau \geq 1$, provided $[\xi] \neq 0$. If $(\xi) = 0$, then the process stops. If $(\xi) \neq 0$, then write

$$\xi = \beta_0 + \frac{1}{\xi_1},$$

where $\xi_1^{-1} = (\xi)$ with $|\xi_1|_\tau > 1$. Next write $\xi_1 = [\xi_1] + (\xi_1)$ and let $\beta_1 = [\xi_1] \in k[1/\tau] \setminus k$, so that $|\beta_1|_\tau = |\xi_1|_\tau > 1$. If $(\xi_1) = 0$, then the process stops. If $(\xi_1) \neq 0$, continue in the same manner. We thus obtain the unique representation

$$\xi = [\beta_0; \beta_1, \ldots, \beta_{n-1}, \xi_n] := \beta_0 + \cfrac{1}{\beta_1 + \cfrac{1}{\ddots \cfrac{1}{\beta_{n-1} + \frac{1}{\xi_n}}}},$$

where $\beta_i \in k[1/\tau] \setminus k \ (i \geq 1), \xi_n \in k((\tau))$, $|\xi_n|_\tau > 1$ if the process does not stop before, and ξ_n is referred to as the n^{th} complete quotient. The sequence (β_n) is uniquely determined, called the sequence of partial quotients of ξ. The two sequences of partial numerators, (C_n), and partial denominators, (D_n), are defined by

$$C_{-1} = 1, \quad C_0 = \beta_0, \quad C_{n+1} = \beta_{n+1}C_n + C_{n-1} \in k[1/\tau] \quad (n \geq 0)$$
$$D_{-1} = 0, \quad D_0 = 1, \quad D_{n+1} = \beta_{n+1}D_n + D_{n-1} \in k[1/\tau] \quad (n \geq 0).$$

As in the p-adic case, we have have

$$\frac{C_n}{D_n} = \frac{\beta_n C_{n-1} + C_{n-2}}{\beta_n D_{n-1} + D_{n-2}} = [\beta_0; \beta_1, \beta_2, \ldots, \beta_n] \quad (n \geq 0),$$

and so C_n/D_n is called the n^{th} convergent. If $(\xi_n) = 0$ for some n, then $\xi = [\beta_0; \beta_1, \beta_2, \dots, \beta_{n-1}]$, i.e., the expansion terminates. Otherwise, $(\xi_n) \neq 0$ for all n implying that the expansion is infinite, i.e.,

$$\xi = [\beta_0; \beta_1, \beta_2, \beta_3, \dots]; \tag{18}$$

the right hand side is called the k_τ-continued fraction of ξ. As in the case of p-adic numbers, here the partial quotients, partial numerators and partial denominators are genarally rational functions, not necessary polynomials in $k[x]$. We resort to its equivalent continued fractions in order to define k_τ-Liouville continued fractons. To this end, without loss of generality, it suffices to consider only those ξ with $|\xi|_\tau < 1$. Since $(\xi_n) \neq 0$ for all n, we have

$$\beta_{n+1} = [\xi_{n+1}] = \frac{c_{-\delta_{n+1}}}{\tau^{\delta_{n+1}}} + \frac{c_{-\delta_{n+1}+1}}{\tau^{\delta_{n+1}-1}} + \frac{c_{-\delta_{n+1}+2}}{\tau^{\delta_{n+1}-2}} + \dots + c_0, c_{-\delta_{n+1}} \neq 0, \ (n \geq 0).$$

Its (equivalent) *associated k_τ-continued fraction* is

$$\xi = \cfrac{\gamma_1}{\alpha_1 + \cfrac{\gamma_2}{\alpha_2 + \cfrac{\gamma_3}{\alpha_3 + \dots}}} := [0; \gamma_1/\alpha_1, \gamma_2/\alpha_2, \gamma_3/\alpha_3, \dots], \tag{19}$$

where

$$\gamma_1 = \tau^{\delta_1}, \quad \gamma_n = \tau^{\delta_n \delta_{n-1}} \ (n \geq 2), \quad \alpha_n = \beta_n \tau^{\delta_n} \ (n \geq 1).$$

Clearly, the partial numerators γ_n and the partial denominators α_n of the associated k_τ-continued fraction (19) are in $k[\tau] = k[x]$. Define the k^{th} convergent of (19) by

$$\mathcal{C}_{-1} = 1, \quad \mathcal{C}_0 = 0, \quad \mathcal{C}_{n+1} = \beta_{n+1}\mathcal{C}_n + \gamma_{n+1}\mathcal{C}_{n-1} \in k[x] \qquad (n = 0, 1, 2, \dots)$$

$$\mathcal{D}_{-1} = 0, \quad \mathcal{D}_0 = 1, \quad \mathcal{D}_{n+1} = \beta_{n+1}\mathcal{D}_n + \gamma_{n+1}\mathcal{D}_{n-1} \in k[x] \quad (n = 0, 1, 2, \dots).$$

The k_τ-continued fraction (18) and its associated k_τ-continued fraction (19) are equivalent in the sense that $C_n/D_n = \mathcal{C}_n/\mathcal{D}_n$ for every $n \geq 0$. An infinite k_τ-continued fraction $[\bar{\beta}] = [\beta_0; \beta_1, \beta_2, \beta_3, \dots]$ is called a *k_τ-Liouville continued fraction* if for $w \in \mathbb{R}^+$, there is an $n \in \mathbb{N}$ such that

$$0 < \left| D_n \left[\bar{\beta}\right] - C_n \right|_\tau < \frac{1}{|\mathcal{C}_n, \mathcal{D}_n|_\infty^w}.$$

In $k((\tau))$, an analogue of the approximation property mentioned in Proposition 1.1 also holds.

Lemma 4.4. *Let $\xi \in k((\tau))$ and $C, D(\neq 0) \in k[1/\tau]$. Then*

$$\left|\xi - \frac{C}{D}\right|_\tau < \frac{1}{|D|_\tau^2}$$

if and only if C/D is a convergent of the k_τ-continued fraction of ξ.

Proof. Both the proof of Lemma 2.5 and that of Lemma 3.5 work with slight modifications. □

Similar proofs as in Propositions 2.6 and 3.6 yield:

Proposition 4.5. *Let $\xi = [\beta_0; \beta_1, \beta_2, \dots] \in k((\tau))$. The following assertions are equivalent:*

(i) *ξ is a k_τ-Liouville number.*

(ii) *ξ is a k_τ-Liouville continued fraction.*

(iii) *To each $w \in \mathbb{N}$, there is an $\nu \in \mathbb{N}$ such that $|\beta_{\nu+1}|_\tau > |D_\nu|_\tau^w$.*

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