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On weakly SS-permutable subgroups of a finite group

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Abstract. Suppose that G is a finite group and H is a subgroup of G. We say that H is SS-permutable in G if there is a supplement B of H to G such that H permutes with every Sylow subgroup of B; H is weakly SS-permutable in G if there exist a subnormal subgroup T of G and an SS-permutable subgroup H_{ss} of G contained in H such that $G = HT$ and $H \cap T \leq H_{ss}$. We investigate the influence of weakly SS-permutable subgroups on the structure of finite groups. Some recent results are generalized and unified.

1. Introduction

All groups considered in this paper are finite. G always denotes a finite group, $|G|$ the order of G, $\pi(G)$ the set of all primes dividing $|G|$, G_p a Sylow p-subgroup of G for some $p \in \pi(G)$. $M \cdot G$ means that M is a maximal subgroup of G.

Let $\mathcal{F}$ be a class of groups. We call $\mathcal{F}$ a formation provided that (i) if $G \in \mathcal{F}$ and $H \trianglelefteq G$, then $G/H \in \mathcal{F}$, and (ii) if G/M and G/N are in $\mathcal{F}$, then $G/(M \cap N)$ is in $\mathcal{F}$ for normal subgroups M, N of G. A formation $\mathcal{F}$ is said to be saturated if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$. In this paper, $\mathcal{U}$ will denote the class of all supersolvable groups. Clearly, $\mathcal{U}$ is a saturated formation (ref. [1, p. 713, Satz 8.6]).

A subgroup H of G is called *S-permutable* (or *π-quasinormal*) in G provided that H permutes with all Sylow subgroups of G, i.e., $HS = SH$ for any Sylow

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subgroup S of G [2]; H is said *c-normal* [3] in G if G has a normal subgroup T such that $G = HT$ and $H \cap T \leq H_G$, where H_G is the normal core of H in G. Recently, Skiba in [4] introduces the following concept, which covers both s-permutability and c-normality:

Definition 1.1. Let H be a subgroup of G. H is called weakly s-permutable in G if there is a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the maximal s-permutable subgroup of G contained in H, that is, the subgroup of H generated by all those subgroups of H which are s-permutable in G.

More recently, Li, etc. [5], introduced the concept of *SS-quasinormality* [5] which is a generalization of s-permutablity:

Definition 1.2. Let G be a finite group. A subgroup H of G is said to be an SS-quasinormal subgroup of G if there is a supplement B of H to G such that H permutes with every Sylow subgroup of B.

Remark. For convenience, it is suitable to call SS-quasinormal subgroups as SS-permutable subgroups.

In general, an SS-permutable subgroup need not be a subnormal subgroup. For instance, S_3 is an SS-permutable subgroup of the symmetric group S_4, but S_3 is not subnormal in S_4. Hence, we give a new concept which covers properly both SS-permutablity and Skiba's weakly s-permutability.

Definition 1.3. Let H be a subgroup of G. H is called weakly SS-permutable in G if there is a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{ss}$, where H_{ss} is an SS-permutable subgroup of G contained in H.

Remark. It is easy to see that weakly s-permutability (or SS-permutablity) implies weakly SS-permutability. The converse does not hold in general.

Example 1.4. 1. Let $G = A_5$, the alternative group of degree 5. Then A_4 is SS-permutable in G, certainly, weakly SS-permutable, but not weakly s-permutable in G.

2. Let $G = S_4$, the symmetric group of degree 4. Take $H = \langle (34) \rangle$. Then H is weakly SS-permutable in G, but not SS-permutable in G.

In the literature, authors usually put the assumptions on either the minimal subgroups (and cyclic subgroups of order 4 when $p = 2$) or the maximal subgroups of some kinds of subgroups of G when investigating the structure of G, such as in [5]–[12], ect. In the nice paper [4], Skiba provided a unified viewpoint for a series of similar problems.

For the sake of convenience of statement, we introduce the following notation.

Let P be a p-subgroup of G. We call P satisfies $(*)$ ($(*)'$, $(\Diamond_1)$, $(\Diamond_2)$ respectively) in G if

$(*)$: P has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ and with order $|H| = 2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) are weakly s-permutable in G.

$(*)'$: P has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ are weakly s-permutable in G. When P is a non-abelian 2-group and $|P : D| > 2$, in addition, suppose that H is weakly s-permutable in G if there exists $D_1 \trianglelefteq H \leq P$ with $2|D_1| = |D|$ and H/D_1 is cyclic of order 4.

$(\Diamond_1)$: P has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ are weakly SS-permutable in G. When $p = 2$ and $|P : D| > 2$, in addition, suppose that H is weakly SS-permutable in G if there exists $D_1 \trianglelefteq H \leq P$ with $2|D_1| = |D|$ and H/D_1 is cyclic of order 4.

$(\Diamond_2)$: P has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ are SS-quasinormal in G. When $p = 2$ and $|P : D| > 2$, in addition, suppose that H is SS-permutable in G if there exists $D_1 \trianglelefteq H \leq P$ with $2|D_1| = |D|$ and H/D_1 is cyclic of order 4.

Theorem 1.5 (4, Theorem 1.3). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups and G a group with E as a normal subgroup of G such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup P of $F^*(E)$ satisfies $(*)$ in G. Then $G \in \mathcal{F}$.*

Scrutinizing the proof of [4, Theorem 1.3], we can find that the following Theorem 1.6 holds:

Theorem 1.6 (4, Theorem 1.3). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with E as a normal subgroup of G such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup P of $F^*(E)$ satisfies $(*)'$ in G. Then $G \in \mathcal{F}$.*

In this paper, the main purpose is to generalize Theorem 1.6 as follows:

Theorem 1.7 (i.e. Theorem 3.5). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with E as a normal subgroup of G such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup of $F^*(E)$ satisfies $\Diamond_1$ in G. Then $G \in \mathcal{F}$.*

2. Preliminaries

Lemma 2.1 (5, Lemma 2.1 and Lemma 2.5)**.** *Let H be SS-permutable in a group G, $K \leq G$ and N a normal subgroup of G. We have:*

(1) *If $H \leq K$, then H is SS-permutable in K.*

(2) *HN/N is SS-permutable in G/N.*

(3) *If $N < K$, then K/N is SS-permutable in G/N if and only if K is SS-permutable in G.*

(4) *If K is quasinormal (or permutable) in G, then HK is SS-permutable in G.*

(5) *If a p-subgroup P of G is SS-permutable in G, where p is a prime, then P permutes with every Sylow q-subgroup of G with $q \neq p$.*

Lemma 2.2. *Let U be a weakly SS-permutable subgroup of G and N a normal subgroup of G. Then*

(1) *If $U \leq H \leq G$, then U is weakly SS-permutable in H.*

(2) *Suppose that U is a p-group for some prime p. If $N \leq U$, then U/N is weakly SS-permutable in G/N.*

(3) *Suppose that U is a p-group for some prime p and N is a p'-subgroup. Then UN/N is weakly SS-permutable in G/N.*

(4) *Suppose that U is a p-group for some prime p and U is not SS-permutable in G. Then G has a normal subgroup M such that $|G : M| = p$ and $G = MU$.*

(5) *If $U \leq O_p(G)$ for some prime p, then U is weakly s-permutable in G.*

Proof. By the hypotheses, there are a subnormal subgroup T of G and an SS-permutable subgroup U_{ss} of G contained in U such that $G = UT$ and $U \cap T \leq U_{ss}$.

(1) We can get that $H = U(H \cap T)$. Obviously, $H \cap T$ is subnormal in H and $U \cap (H \cap T) = U \cap T \leq U_{ss}$. By Lemma 2.1(1), U_{ss} is SS-permutable in H. Hence, U is weakly SS-permutable in H.

(2) We have that $G/N = (U/N)(TN/N)$. Obviously, TN/N is subnormal in G/N and $(U/N) \cap (TN/N) = (U \cap TN)/N = (U \cap T)N/N \leq (U_{ss}N)/N$. By Lemma 2.1(2), $(U_{ss}N)/N$ is SS-permutable in G/N. Hence, U/N is weakly SS-permutable in G/N.

(3) It is easy to see that $N \leq T$ and $G/N = (UN/N)(T/N)$. Since T/N is subnormal in G/N and $(UN/N) \cap T/N = (U \cap T)N/N \leq (U_{ss}N)/N$, $(U_{ss}N)/N$ is SS-permutable in G/N by Lemma 2.1(2). Hence, $(UN)/N$ is weakly SS-permutable in G/N.

(4) If $T = G$, then $U = U \cap T \leq U_{ss} \leq U$, therefore, $U = U_{ss}$ is SS-permutable in G, contrary to the hypotheses. Consequently, T is a proper subgroup of G. Hence, G has a proper normal subgroup K such that $T \leq K$. Since G/K is a p-group, G has a normal maximal subgroup M such that $|G : M| = p$ and $G = MU$.

(5) We can get that by Lemma 2.1(3). □

By Lemma 2.2(5) and [4, Lemma 2.11], we have that:

Lemma 2.3. *Let N be an elementary abelian normal subgroup of a group G. Assume that N has a subgroup D such that $1 < |D| < |N|$ and every subgroup H of N satisfying $|H| = |D|$ is weakly SS-permutable in G. Then some maximal subgroup of N is normal in G.*

Lemma 2.4 (13, A, 1.2)**.** *Let U, V and W be subgroups of a group G. Then the following statements are equivalent.*

(1) $U \cap VW = (U \cap V)(U \cap W)$.

(2) $UV \cap UW = U(V \cap W)$.

Applying Lemma 2.4, we have that:

Lemma 2.5. *Suppose that N is a normal subgroup of a group G and G_q is a Sylow q-subgroup of G and P is a p-subgroup of G, where $q \neq p$ for some primes p and q. If PG_q is a subgroup of G, then*

(1) $N \cap PG_q = (N \cap P)(N \cap G_q)$.

(2) $P \cap NG_q = P \cap N$.

Lemma 2.6 (1, VI, 4.10)**.** *Assume that A and B are two subgroups of a group G and $G \neq AB$. If $AB^g = B^g A$ holds for any $g \in G$, then either A or B is contained in a nontrivial normal subgroup of G.*

Lemma 2.7 (1, III, 5.2 and IV, 5.4)**.** *Suppose that p is a prime and G is a minimal non p-nilpotent, i.e., G is not a p-nilpotent group but whose proper subgroups are all p-nilpotent. Then*

(1) *G has a normal Sylow p-subgroup P for some prime p and $G = PQ$, where Q is a non-normal cyclic q-subgroup for some prime $q \neq p$.*

(2) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.*

(3) *The exponent of P is p or 4.*

Lemma 2.8 (14, X, 13)**.** *Let M be a subgroup of G.*

(1) *If M is normal in G, then $F^*(M) \leq F^*(G)$.*

(2) $F^*(G) \neq 1$ *if* $G \neq 1$*; in fact,* $F^*(G)/F(G) = \mathrm{Soc}(F(G)C_G(F(G))/F(G))$.

(3) $F^*(F^*(G)) = F^*(G) \geq F(G)$*; if* $F^*(G)$ *is solvable, then* $F^*(G) = F(G)$.

Lemma 2.9 (1, IV, Satz 4.7)**.** *If* P *is a Sylow* p*-subgroup of a group* G *and* $N \trianglelefteq G$ *such that* $P \cap N \leq \Phi(P)$*, then* N *is* p*-nilpotent.*

3. Main results

Theorem 3.1. *Let* p *be the smallest prime of* $\pi(G)$ *and* P *a Sylow* p*-subgroup of* G*. If all maximal subgroups of* P *are weakly* SS*-permutable in* G*, then* G *is* p*-nilpotent.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order, then we have:

(1) G has the unique minimal normal subgroup N such that G/N is p-nilpotent and $\Phi(G) = 1$.

Let N be a minimal normal subgroup of G. We consider the factor group G/N. Let M/N is a maximal subgroup of PN/N. It is easy to see that $M = P_1N$ for some maximal subgroup P_1 of P. It follows that $P \cap N = P_1 \cap N$ is a Sylow subgroup of N. By the hypotheses, there are a subnormal subgroup K_1 of G and an SS-permutable subgroup $(P_1)_{ss}$ of G contained in P_1 such that $G = P_1K_1$ and $P_1 \cap K_1 \leq (P_1)_{ss}$. Then $G/N = (P_1N/N)(K_1N/N) = (M/N)(K_1N/N)$. It is easy to see that K_1N/N is subnormal in G/N. Since $(|N : N \cap P_1|, |N : K_1 \cap N|) = 1$, $(P_1 \cap N)(K_1 \cap N) = N = N \cap G = N \cap (P_1K_1)$. By Lemma 2.4, $(P_1N) \cap (K_1N) = (P_1 \cap K_1)N$. Hence, $(P_1N)/N \cap (K_1N)/N = (P_1 \cap K_1)N/N \leq (P_1)_{ss}N/N$. It follows from Lemma 2.1(2) that $(P_1)_{ss}N/N \leq M/N$ is SS-permutable in G/N. Hence, M/N is weakly SS-permutable in G/N. Therefore, G/N satisfies the hypotheses of the theorem. The choice of G yields that G/N is p-nilpotent. The uniqueness of N and $\Phi(G) = 1$ are obvious.

(2) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then $N \leq O_{p'}(G)$ and $G/O_{p'}(G)$ is p-nilpotent by (1), G is p-nilpotent, a contradiction. Hence $O_{p'}(G) = 1$.

(3) $O_p(G) = 1$. Therefore, G is not solvable and N is a direct product of some isomorphic non-abelian simple groups.

If $O_p(G) \neq 1$, we have that $N \leq O_p(G)$ and $\Phi(O_p(G)) \leq \Phi(G) = 1$ by (1). Thus, G has a maximal subgroup M such that $G = MN$ and $M \cap N = 1$. Since $O_p(G) \cap M$ is normalized by N and M, hence, by G, the uniqueness of N yields $N = O_p(G)$. Clearly, $P = N(P \cap M)$. Since $P \cap M < P$, let P_1 be a maximal subgroup of P such that $P \cap M \leq P_1$. Then $P = NP_1$. By the hypotheses, there are a subnormal subgroup T of G and an SS-permutable subgroup $(P_1)_{ss}$ of G contained in P_1 such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{ss}$. Since $N \leq O^p(G) \leq T$ by (1), we have that $P_1 \cap N = (P_1)_{ss} \cap N$. For any Sylow q-subgroup G_q of G, there holds

$$\begin{aligned}[P_1 \cap N, G_q] &= [(P_1)_{ss} \cap N, G_q] = [(P_1)_{ss}G_q \cap N, G_q] \leq N \cap (P_1)_{ss}G_q \\ &= N \cap (P_1)_{ss} = N \cap P_1\end{aligned}$$

by Lemma 2.1(5) and Lemma 2.5(1), where $q \neq p$. Obviously, $P_1 \cap N$ is normalized by P. Therefore, $P_1 \cap N$ is normal in G. The minimality of N implies that $P_1 \cap N = 1$. Hence, N is of order p. Thus, G is p-nilpotent, a contradiction. Hence, $O_p(G) = 1$. Combining (2), we can see that G is not solvable and N is a direct product of some isomorphic non-abelian simple groups.

(4) The final contradiction.

If $N \cap P \leq \Phi(P)$, then N is p-nilpotent by Lemma 2.9, contrary to (3). Consequently, there is a maximal subgroup P_1 of P such that $P = (N \cap P)P_1$. Since P_1 is weakly SS-permutable in G, by the hypotheses, there are a subnormal subgroup T of G and an SS-permutable subgroup $(P_1)_{ss}$ of G contained in P_1 such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{ss}$.

For any Sylow q-subgroup N_q of N with $q \neq p$, we now claim that $((P_1)_{ss} \cap N)N_q = N_q((P_1)_{ss} \cap N)$. In fact, pick any Sylow q-subgroup G_q of G containing N_q. Then $(P_1)_{ss}G_q \cap NG_q = ((P_1)_{ss} \cap NG_q)G_q = ((P_1)_{ss} \cap N)G_q$ by Lemma 2.1(5) and Lemma 2.5(2). Hence,

$$((P_1)_{ss} \cap N)G_q \cap N = ((P_1)_{ss} \cap N)(G_q \cap N) = ((P_1)_{ss} \cap N)N_q$$

Therefore, $((P_1)_{ss} \cap N)N_q = N_q((P_1)_{ss} \cap N)$.

Applying Lemma 2.6, we know that N has a proper normal subgroup M such that either $(P_1)_{ss} \cap N \leq M$ or $N_q \leq M$. If $N_q \leq M$, this is contrary to [1, I, Satz 9.12(b)]. If $(P_1)_{ss} \cap N \leq M$, notice that $P_1 \cap N = (P_1)_{ss} \cap N \leq P_1 \cap M \leq P \cap M$, we have that

$$|N/M|_p = \frac{|N|_p}{|M|_p} = [P \cap N : P \cap M] \leq [P \cap N : P_1 \cap N] = [P : P_1] = p$$

$|N/M|_p = p$ with p minimum prime divisor implies that N/M is p-nilpotent.

By (3), N is a direct product of some isomorphic non-abelian simple groups, say, $N \cong N_1 \times \cdots \times N_k$. N/M is isomorphic to a direct product of some N_i. Hence N_1 is p-nilpotent. Contrary to that N_1 is a non-abelian simple group.

This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime of $\pi(G)$. If P satisfies $(\Diamond_1)$ in G, then G is p-nilpotent.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order, then:

(1) $O_{p'}(G) = 1$.

Denote $N = O_{p'}(G)$. If $N \neq 1$, then Sylow p-subgroup PN/N of G/N satisfies $(\Diamond_1)$ in G/N by Lemma 2.2(3). By the minimality of G, we have G/N is p-nilpotent. Then G is p-nilpotent, a contradiction. Hence, $N = O_{p'}(G) = 1$.

(2) $|D| > p$.

Suppose that $|D| = p$. Since G is not p-nilpotent, G has a minimal non p-nilpotent subgroup G_1. By Lemma 2.7(1), $G_1 = [P_1]Q$, where $P_1 \in \mathrm{Syl}_p(G_1)$ and $Q \in \mathrm{Syl}_q(G_1)$, $p \neq q$. Denote $\Phi = \Phi(P_1)$. Let X/Φ be a subgroup of P_1/Φ of order p, $x \in X \setminus \Phi$ and $L = \langle x \rangle$. Then L is of order p or 4 by Lemma 2.7(3). By the hypotheses, L is weakly SS-permutable in G, thus, in G_1 by Lemma 2.2(1). If L is not SS-permutable in G_1, then by Lemma 2.2(4), G_1 has a normal subgroup T such that $G_1 = LT$ and $|G_1 : T| = p$. Since G_1 is a minimal non p-nilpotent group, T is p-nilpotent. Then T_q char $T \trianglelefteq G_1$ and $T_q \trianglelefteq G_1$. Therefore, G_1 is p-nilpotent, a contradiction. Hence, L is SS-permutable in G_1. So $X/\Phi = L\Phi/\Phi$ is SS-permutable in G_1/Φ by Lemma 2.1(2). Now Lemma 2.3 and Lemma 2.7(2) imply that $|P_1/\Phi| = p$. It follows immediately that P_1 is cyclic. Thus, G_1 is p-nilpotent by [1, IV, Satz 2.8], contrary to the choice of G_1.

(3) $|P : D| > p$.

By Theorem 3.1.

(4) P satisfies $\Diamond_2$ in G.

Assume that $H \leq P$ such that $|H| = |D|$ and H is not SS-permutable in G. By Lemma 2.2(4), there is a normal subgroup M of G such that $|G : M| = p$. Since $|P : D| > p$ and Lemma 2.2(1), M satisfies the hypotheses of the theorem. The choice of G yields that M is p-nilpotent. It is easy to see that G is p-nilpotent, contrary to the choice of G.

(5) If $N \leq P$ and N is a minimal normal subgroup of G, then $|N| \leq |D|$.

Suppose that $|N| > |D|$. Since $N \leq O_p(G)$, N is elementary abelian. By Lemma 2.3, N has a maximal subgroup which is normal in G, contrary to the minimality of N.

(6) Suppose that $N \leq P$ and N is a minimal normal subgroup of G, then G/N is p-nilpotent.

If $|N| < |D|$, G/N satisfies the hypotheses of the theorem by Lemma 2.1(2). Thus, G/N is p-nilpotent by the minimal choice of G. So we may suppose that

$|N| = |D|$ by (5). We will show every cyclic subgroup of P/N of order p or order 4 (when P/N is a non-abelian 2-group) is SS-permutable in G/N. Let $K \leq P$ and $|K/N| = p$. By (2), N is non-cyclic, so are all subgroups containing N. Hence, there is a maximal subgroup $L \neq N$ of K such that $K = NL$. Of course, $|N| = |D| = |L|$. Since L is SS-permutable in G by the hypotheses, $K/N = LN/N$ is SS-permutable in G/N by Lemma 2.1(2). If $p = 2$ and P/N is non-abelian, take a cyclic subgroup X/N of P/N of order 4. Let K/N be maximal in X/N. Then K is maximal in X and $|K/N| = 2$. Since X is non-cyclic and X/N is cyclic, there is a maximal subgroup L of X such that N is not contained in L. Thus, $X = LN$ and $|L| = |K| = 2|D|$. Since $X/N = LN/N \cong L/(L \cap N)$ is cyclic of order 4, by the hypotheses, L is SS-permutable in G. By Lemma 2.1, $X/N = LN/N$ is SS-permutable in G/N. Hence, P/N satisfies $\lozenge_2$ in G/N. By the minimal choice of G, G/N is p-nilpotent.

(7) $O_p(G) = 1$.

If $O_p(G) \neq 1$. Take a minimal normal subgroup N of G contained in $O_p(G)$. By (6), G/N is p-nilpotent. It is easy to see that N is the unique minimal normal subgroup of G contained in $O_p(G)$. Furthermore, $O_p(G) \cap \Phi(G) = 1$. Hence, $O_p(G) = F(G)$ is an elementary abelian p-group. On the other hand, G has a maximal subgroup M such that $G = MN$ and $M \cap N = 1$. It is easy to deduce that $O_p(G) \cap M = 1$, $N = O_p(G)$ and $M \cong G/N$ is p-nilpotent and $N_G(M_{p'}) = M$. Then G can be written as $G = N(M \cap P)M_{p'}$, where $M_{p'}$ is the normal p-complement of M. Pick a maximal subgroup S of M_p. Then $SM_{p'} \leq M$ and $|M : SM_{p'}| = p$. Hence, $NSM_{p'} \leq G$ is a subgroup of G with index p. By the minimality of p, we know that $NSM_{p'} \trianglelefteq G$. Now by (3) and Lemma 2.1(1), we have that $NSM_{p'}$ is p-nilpotent. Therefore, G is p-nilpotent, a contradiction.

(8) The minimal normal subgroup L of G is not p-nilpotent.

If L is p-nilpotent, by the fact that $L_{p'}$ char $L \trianglelefteq G$, we have that $L_{p'} \leq O_{p'}(G) = 1$. Thus, L is a p-group. Then $L \leq O_p(G) = 1$ by Step (7), a contradiction.

(9) G is a non-abelian simple group.

Suppose that G is not a simple group. Take a minimal normal subgroup L of G. Then $L < G$. If $|L|_p > |D|$, then L is p-nilpotent by the minimal choice of G, contrary to (7). If $|L|_p \leq |D|$. Take $P_1 \geq L \cap P$ such that $|P_1| = p|D|$. Hence, P_1 is a Sylow p-subgroup of P_1L. Since every maximal subgroup of P_1 is of order $|D|$, every maximal subgroup of P_1 is SS-permutable in G by hypotheses, thus, in P_1L by Lemma 2.1(1). Now applying Theorem 3.1, we can get P_1L is p-nilpotent. Therefore, L is p-nilpotent, contrary to (8).

(10) The final contradiction.

Suppose that H is a subgroup of P with $|H| = |D|$ and Q is a Sylow q-subgroup of G with $q \neq p$. Then $HQ^g = Q^gH$ for any $g \in G$ by (4) and Lemma 2.1(5). Since G is simple by (9), $G = HQ$ from Lemma 2.6, the final contradiction. □

Corollary 3.3. *Suppose that G is a group. If every non cyclic Sylow subgroup of G satisfies $\Diamond_1$ in G, then G has the Sylow Tower of supersolvable type.*

Theorem 3.4. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with E as a normal subgroup of G such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup of E satisfies $\Diamond_1$ in G. Then $G \in \mathcal{F}$.*

Proof. Suppose that P is a Sylow p-subgroup of E, for any prime $p \in \pi(E)$. Since P satisfies $\Diamond_1$ in G by hypotheses, P satisfies $\Diamond_1$ in E by Lemma 2.2(1). Applying Corollary 3.3, we have that E has a Sylow tower of supersolvable type. Let q be the maximal prime divisor of $|E|$ and $Q \in \mathrm{Syl}_q(E)$. Then $Q \trianglelefteq G$. Since $(G/Q, E/Q)$ satisfies the hypotheses of the theorem, by induction, $G/Q \in \mathcal{F}$. For any subgroup H of Q with $|H| = |D|$, since $Q \leq O_q(G)$, H is weakly s-permutable in G by Lemma 2.2(5). Hence, Q satisfies $(*)'$ in G. Since $F^*(Q) = Q$ by Lemma 2.8, we get $G \in \mathcal{F}$ by applying Theorem 1.6. □

Theorem 3.5. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with E as a normal subgroup of G such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup of $F^*(E)$ satisfies $\Diamond_1$ in G. Then $G \in \mathcal{F}$.*

Proof. We distinguish two cases:

Case 1. $\mathcal{F} = \mathcal{U}$

Let G be a minimal counter-example.

(1) Every proper normal subgroup N (if it exists) of G containing $F^*(E)$ is supersolvable.

If N is a proper normal subgroup of G containing $F^*(E)$, we have that $N/N \cap E \cong NE/E$ is supersolvable. By Lemma 2.8(3), $F^*(E) = F^*(F^*(E)) \leq F^*(E \cap N) \leq F^*(E)$, so $F^*(E \cap N) = F^*(E)$. For any Sylow subgroup P of $F^*(E \cap N)$, P satisfies $\Diamond_1$ in G by hypotheses. Hence, P satisfies $\Diamond_1$ in N by Lemma 2.2(1). So $(N, N \cap E)$ satisfy the hypotheses of the theorem, the minimal choice of G implies that N is supersolvable

(2) $E = G$.

If $E < G$, then $E \in \mathcal{U}$ by (1). Hence $F^*(E) = F(E)$ by Lemma 2.8(3). It follows that every Sylow subgroup of $F^*(E)$ is normal in G. By Lemma 2.2(5), every Sylow subgroup of $F^*(E)$ satisfies $(*)'$ in G. Applying Theorem 1.6 for the special case $\mathcal{F} = \mathcal{U}$, $G \in \mathcal{U}$, a contradiction.

(3) $F^*(G) = F(G) < G$.

If $F^*(G) = G$, then $G \in \mathcal{F}$ by Theorem 3.4, contrary to the choice of G. So $F^*(G) < G$. By (1), $F^*(G) \in \mathcal{U}$ and $F^*(G) = F(G)$ by Lemma 2.8(3).

(4) The final contradiction.

Since $F^*(G) = F(G)$, each Sylow subgroup of $F^*(G)$ satisfies $(*)'$ in G by Lemma 2.2(5). Applying Theorem 1.4, $G \in \mathcal{U}$, a final contradiction.

Case 2. $\mathcal{F} \neq \mathcal{U}$.

By hypotheses, every non-cyclic Sylow subgroup of $F^*(E)$ satisfies $\Diamond_1$ in G, thus, in E by Lemma 2.2(1). Applying CASE 1, $E \in \mathcal{U}$. Then $F^*(E) = F(E)$ by Lemma 2.8(3). It follows that each Sylow subgroup of $F^*(E)$ is normal in G. By Lemma 2.2(5), each Sylow subgroup of $F^*(E)$ satisfies $(*)'$ in G. Applying Theorem 1.6, $G \in \mathcal{F}$. □

4. Some applications

From the definition of weakly SS-permutably subgroup, we can see that [4, Corollary 5.1–5.24] are corollaries of Theorem 3.5. Furthermore, we have

Corollary 4.1. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with E as a normal subgroup of G such that $G/E \in \mathcal{F}$. Suppose that all maximal subgroups of any Sylow subgroup of $F^*(E)$ are either SS-permutable or c-normal in G. Then $G \in \mathcal{F}$.*

Corollary 4.2. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with E as a normal subgroup of G such that $G/E \in \mathcal{F}$. Suppose that the cyclic subgroups of prime order or order 4 of $F^*(E)$ are either SS-permutable or c-normal in G. Then $G \in \mathcal{F}$.*

Corollary 4.3 (14,Theorem 3.3)**.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with E as a normal subgroup of G such that $G/E \in \mathcal{F}$. Suppose that all maximal subgroups of any Sylow subgroup of $F^*(E)$ are SS-permutable in G. Then $G \in \mathcal{F}$.*

Corollary 4.4 (14,Theorem 3.7)**.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with E as a normal subgroup of G such that $G/E \in \mathcal{F}$. Suppose that the cyclic subgroups of prime order or order 4 of $F^*(E)$ are SS-permutable in G. Then $G \in \mathcal{F}$.*

Theorem 3.2 is also interesting. Using routine way, we can generalize it as follows.

Corollary 4.5. *Let G be a group, H a normal subgroup of G such that G/H is p-nilpotent and P a Sylow p-subgroup of H, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If P satisfies $\lozenge_1$ in G, then G is p-nilpotent.*

Corollary 4.6. *Let G be a group, H a normal subgroup of G such that G/H is p-nilpotent and P a Sylow p-subgroup of H, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of P is either SS-permutable or c-normal in G, then G is p-nilpotent.*

Corollary 4.7. *Let G be a group, H a normal subgroup of G such that G/H is p-nilpotent and P a Sylow p-subgroup of H, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If the cyclic subgroups of prime order or order 4 of P are either SS-permutable or c-normal in G, then G is p-nilpotent.*

Corollary 4.8. *Let G be a group, H a normal subgroup of G such that G/H is p-nilpotent and P a Sylow p-subgroup of H, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of P is SS-permutable in G, then G is p-nilpotent.*

Corollary 4.9. *Let G be a group, H a normal subgroup of G such that G/H is p-nilpotent and P a Sylow p-subgroup of H, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If the cyclic subgroups of prime order or order 4 of P are either SS-permutable in G, then G is p-nilpotent.*

Corollary 4.10. *Let G be a group, H a normal subgroup of G such that G/H is p-nilpotent and P a Sylow p-subgroup of H, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If P satisfies $(*)'$ in G, then G is p-nilpotent.*

Corollary 4.11. *Let G be a group, H a normal subgroup of G such that G/H is p-nilpotent and P a Sylow p-subgroup of H, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If P satisfies $\lozenge_2$ in G, then G is p-nilpotent.*

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