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Simplices of maximum volume contained in the unit ball of a normed space

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Abstract. We prove that if Δ is a simplex of the maximum possible volume contained in the unit ball of an n-dimensional normed space, then $\sum_{i=0}^{n} w_i^{-1} = n$, where w_i is the width (in the sense of the norm) of Δ in the direction perpendicular to the i-th facet of Δ. Moreover, we prove that all the sides of any triangle of the maximum area contained in the unit disk of any 2-dimensional normed plane are of the lengths (in the sense of the norm) at least $\sqrt{2}$. This value cannot be increased as is shown by the example of the normed plane whose unit disk is the regular octagon. We also estimate the perimeter (in the sense of the norm) of this triangle.

By a strip G in an n-dimensional normed space we mean the set of points between two parallel hyperplanes. The distance of these hyperplanes measured by the norm (i.e. the minimum of $\|p_2 - p_1\|$, where p_1 and p_2 are points in these hyperplanes) is called the *width of G in the sense of the norm* or the *$\|\cdot\|$-width of G* and it is denoted by $\|\cdot\|$-width(G). By the *$\|\cdot\|$-width of a convex body C in a direction* we mean the $\|\cdot\|$-width of the narrowest strip containing C whose bounding hyperplanes are perpendicular (in the Euclidean sense) to this direction. For the Euclidean norm we obtain the classic notions of the widths width(G) of G and width(C) of C in a direction, respectively. By the *$\|\cdot\|$-length of a segment ab* we mean $\|b - a\|$ and by the *$\|\cdot\|$-perimeter of a triangle* we mean the sum of the $\|\cdot\|$-lengths of the sides of this triangle.

Recall that McKinney [5] estimated the volume of simplices of maximum

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volume in a centrally symmetric convex body, and Blaschke [3] in an arbitrary planar body.

We consider simplices of the maximum possible volume contained in the unit ball B of an n-dimensional normed space. We have in mind the usual Euclidean volume. But the properties presented in this note are also true for any kind of volume in normed space considered in [1], [2] and [8]. In Theorem 1 we prove that if Δ is a simplex of maximum possible volume contained in B, then $\sum_{i=0}^{n} w_i^{-1} = n$, where w_i means the $\|\cdot\|$-width of Δ in the direction perpendicular to the i-th facet of Δ. In Theorem 2 we prove that if Δ is a triangle of the maximum area contained in the unit disk B of a normed plane, then the sides of Δ are of $\|\cdot\|$-lengths at least $\sqrt{2}$. This value cannot be increased as is shown by the example of the normed plane whose unit disk is the regular octagon. In Proposition 1 we show that the $\|\cdot\|$-perimeter of any such a triangle is at least $\frac{9}{2}$. We conjecture that it is always at least $3\sqrt{3}$. Proposition 2 shows that the $\|\cdot\|$-perimeter of any triangle Δ contained in the unit disk of a normed plane is at most 6. Moreover, Proposition 2 explains when the $\|\cdot\|$-perimeter is exactly 6.

Lemma 1. *Let S be a simplex and let S' be the image of S under a homothety whose ratio is -1. Let $S^* = \operatorname{conv}(S \cup S')$ be the unit ball of a normed space whose norm is denoted by $\|\cdot\|^*$. Denote by w_i^* the $\|\cdot\|^*$-width of S in the direction perpendicular to the i-th facet of S. Then $\sum_{i=0}^{n}(w_i^*)^{-1} \geq n$ with equality if and only if the center of the homothety is in the simplex whose vertices are in the barycentres of the facets of S.*

Proof. Observe that the $\|\cdot\|$-width(G) of a strip G does not change under affine transformation of the unit ball and simultaneously of G. This is why it is sufficient to consider the case when S is a regular simplex.

First consider the special "central" case when the center of homothety is at the center of S. It divides every height of S, and also of S' in this special case, in the proportion $1:n$. Hence $(w_i^*)^{-1} = \frac{n}{n+1}$ for $i = 1, \ldots, n$. Since we consider $n+1$ widths, and since $(n+1)\cdot\frac{n}{n+1} = n$, we get $\sum_{i=0}^{n}(w_i^*)^{-1} = n$, which implies Lemma 1 in the considered case.

We obtain the general case from the above "central" case by some successive n translations of S' parallel to n edges of S having a common end-point at a vertex of S. Below we show that every of these translations does not change the sum $\sum_{i=0}^{n}(w_i^*)^{-1}$.

So consider any position of S' after k mentioned translations, where $0 \leq k \leq n-1$. Imagine that we again translate S'. Let this translation be parallel to an edge E of S. At every vertex of S' we provide the hyperplane parallel to the opposite facet of S'. The two of these hyperplanes which pass through the endpoints of E are not parallel to E, and all the remaining $n-1$ hyperplanes are parallel to E. Thus only for these two from amongst our pairs of hyperplanes the widths of S^* change during the considered process of translation. Observe that if one of the two corresponding numbers of the form $(w_i^*)^{-1}$ decreases, the second increases exactly by the same value, (hint: the non-oriented angles between E and the two hyperplanes are equal). Thus the sum $\sum_{i=0}^{n}(w_i^*)^{-1}$ does not decrease.

Moreover, it should be noticed that one of the two corresponding numbers of the form $(w_i^*)^{-1}$ decreases exactly by the same value, as the second increases, if and only if the center of the homothety (which transforms S into S') after every successive translation remains in the simplex whose vertices are in the centers of the facets of S. □

Remark 1. The assertion of Lemma 1 can also be derived analytically. Here is the outline. Apply barycentric coordinates. If $(a_0, \dots, a_n)$ is the center of the homothety, then $(w_i^*)^{-1} = \max\{a_i, 1-a_i\} \geq 1-a_i$ for $i = 0, \dots, n$. Hence $\sum_{i=0}^{n}(w_i^*)^{-1} \geq \sum_{i=0}^{n}(1-a_i) = n+1-\sum_{i=0}^{n} a_i = n+1-1 = n$. Clearly, the equality holds if and only if $\max\{a_i, 1-a_i\} = 1-a_i$ for $i = 0, \dots, n$, this is, if and only if $(a_0, \dots, a_n)$ is in the simplex whose vertices are in the barycentres of the facets of S.

Remark 2. Assume that in Lemma 1 the center a of the homothety belongs to S. Then $\sum_{i=0}^{n}(w_i^*)^{-1} \leq n+1$ with equality if and only if a is at a vertex of S. This follows from the fact that for every $i \in \{0, \dots, n\}$ we have $\max\{a_i, 1-a_i\} \leq 1$ with equality if and only if a_i is equal to 0 or 1.

From Lemma 1 we easily obtain the following Lemma 2.

Lemma 2. *Let S be a simplex contained in the unit ball B of an n-dimensional normed space. Denote by w_i the $\|\cdot\|$-width of S in the direction perpendicular to the i-th facet of S. Then $\sum_{i=0}^{n} w_i^{-1} \geq n$ with equality if and only if for every facet of S the parallel hyperplane through the opposite vertex of S supports B.*

Lemma 3. *Let S be a simplex of maximum volume contained in a convex body C of Euclidean space. Then S is inscribed in C and for every vertex of S the hyperplane parallel to the opposite facet of S supports C.*

Lemma 3 is very easy to show, see the proof of Proposition in [7]. From Lemma 2 and Lemma 3 we immediately obtain the following Theorem 1.

Theorem 1. *Let S be a simplex of maximum possible volume contained in the unit ball of an n-dimensional normed space. Denote by w_i the $\|\cdot\|$-width of S in the direction perpendicular to the i-th facet of S. Then $\sum_{i=0}^{n} w_i^{-1} = n$.*

Theorem 1 implies the following Proposition 1 and Remark 3.

Proposition 1. *The $\|\cdot\|$-perimeter of every triangle of the maximum area inscribed in the unit disk of a normed plane is at least $\frac{9}{2}$.*

Proof. Denote by Δ a triangle of maximum area contained in the unit disk. By x, y, z denote the $\|\cdot\|$-widths of Δ in the directions perpendicular to the sides of Δ. Since for every points a and b on the opposite lines bounding a strip the $\|\cdot\|$-length of the segment ab is at least the $\|\cdot\|$-width of this strip, we see that the $\|\cdot\|$-lengths of the three sides of Δ are at least x, y, z. Thus the $\|\cdot\|$-perimeter of Δ is at least $x+y+z$. Moreover, by the equality case of Theorem 1 for $n=2$ we have $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2$. These two facts and the fact that the arithmetic mean is at least the harmonic mean imply the inequality $\frac{x+y+z}{3} \geq \frac{3}{2}$. Consequently, the $\|\cdot\|$-perimeter of Δ is at least $\frac{9}{2}$. □

Remark 3. Similarly like in the above proof we show that the sum of the $\|\cdot\|$-lengths of any $n+1$ edges of every simplex S of the maximum volume inscribed in the unit ball of any normed n-space, where $n \geq 2$, is at least $(n+1)^2/n$. Consequently, we deduce that the the sum of the $\|\cdot\|$-lengths of all the $\frac{n(n+1)}{2}$ edges of S is at least $(n+1)^2/2$.

We conjecture that the $\|\cdot\|$-perimeter of any triangle of the maximum area inscribed in B is always at least $3\sqrt{3}$. Observe that this value is attained when B is an arbitrary ellipse and an arbitrary affine-regular $12k$-gon.

Proposition 2. *The $\|\cdot\|$-perimeter of any triangle Δ contained in the unit disk of a normed plane is at most 6. It equals 6 if and only if the unit disk is the set $D_\Delta = \mathrm{conv}(\Delta \cup \nabla)$, where ∇ is a homothetic image of Δ of ratio -1 and the homothety center in the triangle whose vertices are the midpoints of the sides of Δ.*

Proof. Since every side of any triangle contained in the unit disk is of the $\|\cdot\|$-length at most 2, the $\|\cdot\|$-perimeter is always at most 6.

Let the unit disk be D_Δ. Then the sides of Δ are of $\|\cdot\|$-length 2 and thus the $\|\cdot\|$-perimeter of Δ is 6.

Let Δ be a triangle inscribed in the unit disk B of a normed plane. The homothetic image ∇ of Δ with the center in the center of the unit disk and ratio -1 is also inscribed in B. If B is larger than D_Δ, then a segment through o and parallel to a side of Δ is longer (in the Euclidean sense) than the parallel side of Δ. Thus this side of Δ is of $\|\cdot\|$-length smaller than 2. Hence the $\|\cdot\|$-perimeter of Δ is smaller than 6. $\square$

Observe that Δ in Proposition 2 is a triangle of the maximum area in D_Δ. Moreover, the triangles Δ and ∇ are the unique triangles of the maximum area in the disk $D_\Delta$if and only if the disk is not a parallelogram.

For the proof of Theorem 2 and the comment after Theorem 2 we recall the following notion being a generalization of the notion of the distance of points in a finite-dimensional normed space. Let C be a convex body in Euclidean space. The *C-distance* $\mathrm{dist}_C(a,b)$ of points a and b is the ratio of the Euclidean distance $|ab|$ of a and b to the half of the Euclidean length of a longest parallel chord of C (see [6]). We also use the term the *C-length* of the segment ab. The above makes also sense if C is a strip and a and b are points such that the segment ab is not parallel to this strip.

Theorem 2. *Let Δ be a triangle of maximum area contained in the unit disk of a normed plane. The $\|\cdot\|$-lengths of the sides of Δ are at least $\sqrt{2}$. The estimate is sharp as it follows from the example of the normed plane whose unit disk is the regular octagon.*

Proof. Since the ratio of the areas of two convex bodies does not change under any affine transformation, it is sufficient to assume that $\Delta = abc$ is the regular triangle whose heights are equal 1 (see Figure 1). So the length of its sides is $\frac{2}{3}\sqrt{3}$.

Denote by $\nabla = a'b'c'$ the image of the triangle Δ under the homothety with ratio -1 and the center in the center o of the unit disk B of our normed plane. Here a' and a (respectively, b' and b, c' and c) are points symmetric with respect to o. Since Δ and ∇ are triangles of maximum area contained in the unit disk B, by Lemma 3 they are inscribed in B and the lines through their vertices parallel to the opposite sides of them support B. Between the three pairs of parallel lines we have three strips G_1, G_2 and G_3. We do not loose the generality assuming that $\mathrm{width}(G_1) \le \mathrm{width}(G_2) \le \mathrm{width}(G_3)$. Moreover, let for instance the lines bounding the strip G_1 be parallel to the side ab. Put $\lambda = \mathrm{width}(G_1)$. From the above inequalities and from

$$\mathrm{width}(G_1) + \mathrm{width}(G_2) + \mathrm{width}(G_3) = 4 \qquad (1)$$

(which follows from Theorem 1 for $n = 2$ and from the fact that Δ is a regular triangle of heights equal 1) we see that $\lambda \le \frac{4}{3}$. Clearly, $\lambda \ge 1$. Hence $\lambda \in [1, \frac{4}{3}]$.

The (Euclidean) height of the triangle abb' perpendicular to the segment ab is denoted by μ. Since the heights of Δ and ∇ are 1, we deduce that

$$\mu = 2 - \lambda. \tag{2}$$

We support B by lines ℓ and ℓ' parallel to the segment ab' and denote by $e \in \ell$ and $e' \in \ell'$ points from the line containing the segment ab. The above notation is chosen such that $b \in ae$. Since $\ell, \ell', a'b, ab'$ are parallel, since ℓ, ℓ' are symmetric with respect to o and since $a'b, b'a$ are symmetric with respect to o, we conclude that $|be| = |e'a|$. The strip between ℓ and ℓ' is denoted by H. Denote by $d \in \ell$ and $d' \in \ell'$ symmetric (with respect to o) points of support of B by these lines.

Since $\Delta = abc$ is a triangle of maximum area contained in B, since $\text{area}(abc) = \frac{1}{3}\sqrt{3}$ and since the triangle adb' is contained in B, we have

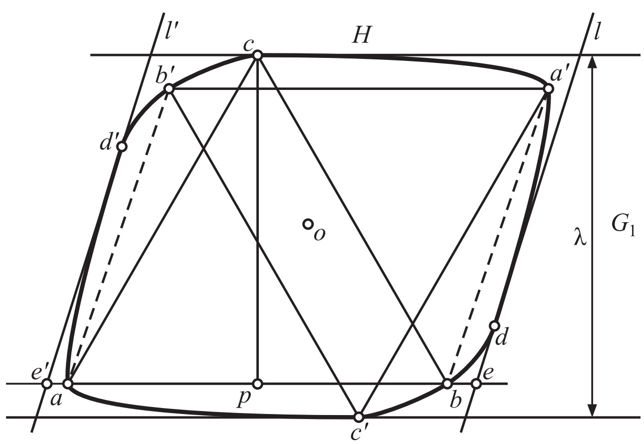

Figure 1

$$\text{area}(adb') \leq \frac{1}{3}\sqrt{3}. \tag{3}$$

Since the segments ab' and de are parallel, and since the segments ab and ae are parallel, by (2) we get

$$\text{area}(adb') = \text{area}(aeb') = \frac{1}{2}|ae|\mu = \frac{1}{2}|ae|(2 - \lambda). \tag{4}$$

From (3) and (4) we conclude that $\frac{1}{2}|ae|(2-\lambda) \leq \frac{1}{3}\sqrt{3}$. Hence $|ae| \leq \frac{2}{3}\sqrt{3} \cdot \frac{1}{2-\lambda}$. Moreover, we have $|ae| = |e'b|$ and $|ab| = \frac{2}{3}\sqrt{3}$. These facts and the fact that

e', a, b, e are collinear imply that $|be| = |e'a| \le \frac{2}{3}\sqrt{3} \cdot \frac{1}{2-\lambda} - \frac{2}{3}\sqrt{3} = \frac{2}{3}\sqrt{3} \cdot \frac{-1+\lambda}{2-\lambda}$. Thus $|e'e| = |e'a| + |ab| + |be| \le 2 \cdot \frac{2}{3}\sqrt{3} \cdot \frac{-1+\lambda}{2-\lambda} + \frac{2}{3}\sqrt{3} = \frac{2}{3}\sqrt{3} \cdot \frac{\lambda}{2-\lambda}$. This, $|ab| = \frac{2}{3}\sqrt{3}$ and $\lambda \ge 1$ imply that $\mathrm{dist}_H(a,b) \ge \left(\frac{2}{3}\sqrt{3}\right)/\left(\frac{1}{2} \cdot \frac{2}{3}\sqrt{3} \cdot \frac{\lambda}{2-\lambda}\right) = \frac{4}{\lambda} - 2$. Since $B \subset H$, we obtain $\mathrm{dist}_B(a,b) \ge \frac{4}{\lambda} - 2$, this is, $\|b-a\| \ge \frac{4}{\lambda} - 2$.

From (1) and from $\mathrm{width}(G_1) = \lambda$ we see that $\mathrm{width}(G_2) + \mathrm{width}(G_3) = 4 - \lambda$. This and $\mathrm{width}(G_2) \le \mathrm{width}(G_3)$ imply that $\mathrm{width}(G_2) \le \frac{1}{2} \cdot (4-\lambda) = 2 - \frac{1}{2}\lambda$. Consequently, the length of each segment parallel to the segment ab connecting the two lines bounding G_2 is at most $\frac{2}{3}\sqrt{3} \cdot (2 - \frac{1}{2}\lambda)$. From this and from $|ab| = \frac{2}{3}\sqrt{3}$ we obtain $\mathrm{dist}_{G_2}(a,b) \ge \left(\frac{2}{3}\sqrt{3}\right)/\left(\frac{1}{2} \cdot \frac{2}{3}\sqrt{3} \cdot (2 - \frac{1}{2}\lambda)\right) = \frac{4}{4-\lambda}$. Since $B \subset G_2$, we get $\mathrm{dist}_B(a,b) \ge \frac{4}{4-\lambda}$, this is, $\|b-a\| \ge \frac{4}{4-\lambda}$.

In the two preceding paragraphs we have established that $\|b-a\| \ge \frac{4}{\lambda} - 2$ and that $\|b-a\| \ge \frac{4}{4-\lambda}$. The only solution of the equation $\frac{4}{\lambda} - 2 = \frac{4}{4-\lambda}$ is $\lambda_0 = 4 - 2\sqrt{2}$ and the common value of $\frac{4}{\lambda_0} - 2$ and of $\frac{4}{4-\lambda_0}$ is $\sqrt{2}$. This and the facts that the function $\frac{4}{\lambda} - 2$ is decreasing and that the function $\frac{4}{4-\lambda}$ is increasing in the interval $[1, \frac{4}{3}]$ imply that for every $\lambda \in \left[1, \frac{4}{3}\right]$ we have $\max\left\{\frac{4}{\lambda} - 2, \frac{4}{4-\lambda}\right\} \ge \sqrt{2}$. Hence $\|b-a\| \ge \sqrt{2}$.

Denote by p the perpendicular projection of c onto the segment ab. The points a and p belong to a line parallel to the lines bounding G_1. From this, since the height of Δ is 1, and since $\mathrm{width}(G_1) = \lambda \le \frac{4}{3}$, we obtain that $\mathrm{dist}_{G_1}(a,c) = \mathrm{dist}_{G_1}(p,c) = \frac{1}{(1/2)\cdot\lambda} \ge \frac{3}{2}$. Consequently, from $B \subset G_1$ we conclude that $\|c-a\| = \mathrm{dist}_B(a,c) \ge \mathrm{dist}_{G_1}(a,c) \ge \frac{3}{2} \ge \sqrt{2}$. Analogously, $\|c-b\| \ge \frac{3}{2} \ge \sqrt{2}$.

Take the regular octagon $v_1v_2\ldots v_8$ in the part of B. The triangle $v_1v_3v_6$ is a triangle of maximum area inscribed in B. We have $\mathrm{dist}_B(v_1, v_3) = \sqrt{2}$, which shows that the estimate $\sqrt{2}$ in our Theorem cannot be increased. $\square$

Of course, instead of the regular octagon in the formulation of Theorem 2 we can take any affine-regular octagon. The question arises if there exist other centrally symmetric convex bodies with this property?

Recall some similar results for an arbitrary convex body C of Euclidean space in place of the unit disk in Theorem 2. If Δ is a triangle of maximum area contained in a planar convex body C, then all its edges are of C-length at least $\sqrt{5} - 1$ and this estimate cannot be improved for the regular pentagon as C (see [4]). If Δ is a simplex of maximum volume contained in a convex body C of n-space, where $n \ge 3$, then all its edges are of C-length at least 1 (see [7]). For $n = 3$ this bound cannot be improved. A problem is to improve the above bound 1 for $n \ge 3$ under the assumption that C is centrally symmetric.

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