Characterizations of Lie-skew multiplicative maps on operator algebras of indefinite inner product spaces

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Abstract. Let H and K be indefinite inner product spaces. In this paper, we show that a bijective map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ satisfies $\Phi(AB - BA^\dagger) = \Phi(A)\Phi(B) - \Phi(B)\Phi(A)^\dagger$ for every pair $A, B \in \mathcal{B}(H)$ if and only if there exist a unitary or conjugate unitary operator $U \in \mathcal{B}(H, K)$ such that $\Phi(A) = UAU^\dagger$ for all $A \in \mathcal{B}(H)$.

1. Introduction

Let $\mathcal{R}$ and $\mathcal{R}'$ be two rings. Recall that a bijective map $\Phi : \mathcal{R} \to \mathcal{R}'$ is called a Lie multiplicative isomorphism if $\Phi([A, B]) = [\Phi(A), \Phi(B)]$, where $[A, B] = AB - BA$. Of course, a Lie multiplicative isomorphism need not always be linear or additive. So, it is an interesting and challenging task to characterize all Lie multiplicative isomorphisms of an (operator) algebra without linearity or additivity assumption. Recently, BAI and DU showed in [1] that, if $\mathcal{R}, \mathcal{R}'$ are prime rings with $\mathcal{R}$ being unital and containing a nontrivial idempotent, and if $\Phi : \mathcal{R} \to \mathcal{R}'$ is a Lie multiplicative isomorphism, then $\Phi(T + S) = \Phi(T) + \Phi(S) + Z'_{T,S}$ for all $T, S \in \mathcal{R}$, where $Z'_{T,S}$ is an element in the center $\mathcal{Z}'$ of $\mathcal{R}'$ depending on T and S.

As a kind of new products in a $*$ ring, Lie-skew product $AB - BA^*$ was discussed in [3]. Lie skew product is found playing a more and more important

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role in some research topics, and it has attracted many author's attention recently. For example, it is related closely to the Jordan $*$-derivation (see [5]). This product was extensively studied because, by the fundamental theorem of ŠEMRL in [6], maps of the form $T \mapsto TA - AT^*$ naturally arise in the problem of representing quadratic functionals with sesquilinear functionals (see [6], [7]).

Because the indefinite inner product space may be useful both for the discussion of physical problems as well as for more genuine mathematical questions (see instructions in [2]), motivated by a work of MOLNÁR (see [4]), some preserver problems were studied and solved for operator algebras on indefinite inner product spaces. Motivated by results in [1] and work of MOLNÁR, In this paper, we consider lie skew multiplicative maps Φ (that is, $\Phi(AB - BA^\dagger) = \Phi(A)\Phi(B) - \Phi(B)\Phi(A)^\dagger$ for every pair A, B) on operator algebras of indefinite inner product spaces.

Let us recall some conceptions and fix some notations. Denote by $\mathbb{C}$ the complex field. An indefinite inner product space means a linear space H over $\mathbb{C}$ equipped with a non-degenerate bilinear Hermite functional $[\,\cdot\,,\cdot\,]$. Let $(H, [\,\cdot\,,\cdot\,])$ be an indefinite inner product space. If there exist a positive subspace H_+ and a negative subspace H_- which are orthogonal to each other such that

$$H = H_+ \oplus H_- \tag{1.1}$$

and $(H_+, [\,\cdot\,,\cdot\,])$ is a Hilbert space when $[\,\cdot\,,\cdot\,]$ is restricted to H_+; $(H_-, -[\,\cdot\,,\cdot\,])$ is a Hilbert space when $-[\,\cdot\,,\cdot\,]$ is restricted to H_-, then we say that H is a complete indefinite inner product space and the decomposition (1.1) is called a regular one of H (see [2] and [8]). In the sequel we always assume that *the indefinite inner product spaces over $\mathbb{C}$ are complete with dimension greater than 1.* If $H = H_+ \oplus H_-$ is a regular decomposition of an indefinite inner product space H, then for any $x, y \in H$, x and y can be uniquely represented as $x = x_+ + x_-$ and $y = y_+ + y_-$, where $x_\pm, y_\pm \in H_\pm$. Define an inner product on H by

$$\langle x, y \rangle = [x_+, y_+] - [x_-, y_-].$$

It is obvious that $(H, \langle\cdot,\cdot\rangle)$ is a Hilbert space. We call $\langle\cdot,\cdot\rangle$ the inner product induced by the regular decomposition $H = H_+ \oplus H_-$ (see [2]). A linear operator T from an indefinite inner product space H into an indefinite inner product space K is said to be bounded if T is bounded with respect to the inner products of H and K induced by some regular decompositions. The boundedness of T does not depend on the choice of the regular decompositions. We still denote $\mathcal{B}(H, K)$ ($\mathcal{B}(H)$ if $H = K$) the set of all bounded linear operators from H into K. For any

$T \in \mathcal{B}(H, K)$, the indefinite conjugate of T with respect to the indefinite inner products $[\cdot, \cdot]$ is an operator $T^{\dagger} \in \mathcal{B}(K, H)$ defined by the equation $[Tx, y] = [x, T^{\dagger}y]$ for all $x \in H$ and $y \in K$. On the other hand, assume that H_i $(i = 1, 2)$ are Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and $J_i \in \mathcal{B}(H_i)$ are self-adjoint invertible operators. Then, for each $i = 1, 2$, $(H_i, [\cdot, \cdot]_{J_i})$ is a complete indefinite inner product space, where $[\cdot, \cdot]_{J_i} = \langle J_i(\cdot), \cdot \rangle$, which is induced by J_i. It is clear that, with respect to $[\cdot, \cdot]_{J_i}$, the indefinite conjugate $T^{\dagger}$ of an operator $T \in \mathcal{B}(H_1, H_2)$ is of the form $T^{\dagger} = J_1^{-1}T^*J_2$, in which T^* stands for the usual conjugate of T related to the inner product $\langle \cdot, \cdot \rangle$. Sometimes we also call $T^{\dagger} = J_1^{-1}T^*J_2$ the (J_1, J_2)-conjugate of T. If $H_1 = H_2$ are the same Hilbert spaces and $J_1 = J_2 = J$, the (J_1, J_2)-conjugate of an operator T is often called the J-conjugate of T. Recall that $U \in \mathcal{B}(H, K)$ is called a unitary operator if $UU^{\dagger} = I_K$ and $U^{\dagger}U = I_H$, where $I_H \in \mathcal{B}(H)$ and $I_K \in \mathcal{B}(K)$ are identity operators.

We refer the reader to [2] and [8] for more details of indefinite inner product.

2. Results and proofs

In this section, we give a characterization of †-isomorphisms of indefinite inner product space only by Lie-skew product. The following is our main result.

Theorem 2.1. *Let H and K be complex complete indefinite inner product spaces and let $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a bijective map. Then Φ satisfies*

$$\Phi(AB - BA^{\dagger}) = \Phi(A)\Phi(B) - \Phi(B)\Phi(A)^{\dagger} \qquad (2.1)$$

for every pair $A, B \in \mathcal{B}(H)$ if and only if there exists a unitary or conjugate unitary operator $U \in \mathcal{B}(H, K)$ such that $\Phi(A) = UAU^{\dagger}$ for all $A \in \mathcal{B}(H)$.

The "if" part is obvious. We only check "only if" part. Now assume that Φ satisfies the assumptions of Theorem 2.1. We divide the proof into several lemmas.

Lemma 2.1. $\Phi(0) = 0$.

PROOF. Since Φ is surjective, we can find an $A \in \mathcal{B}(H)$ such that $\Phi(A) = 0$. Therefore $\Phi(0) = \Phi(A0 - 0A^{\dagger}) = \Phi(A)\Phi(0) - \Phi(0)\Phi(A)^{\dagger} = 0$. $\qquad \square$

Let $H = H_+ \oplus H_-$ be a regular decomposition of H. We may assume that both H_+ and H_- are nontrivial. We denote $P_1 \in \mathcal{B}(H)$ be the fixed non-trivial

self-adjoint idempotent operator which has the matrix form

$$P_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

according to the regular decomposition, where I is the identity operator on H_+ (In the case that H_+ or H_- is $\{0\}$, say $H_+ = 0$, one may take any nonzero projection P_1 in $\mathcal{B}(H_-)$ with inner product $\langle \cdot, \cdot \rangle = -[\cdot, \cdot]$, and the proof is almost the same). Let $P_2 = I - P_1$ and set $\mathcal{A}_{ij} = P_i \mathcal{B}(H) P_j$, $i, j = 1, 2$. Then we have $\mathcal{B}(H) = \mathcal{A}_{11} \dotplus \mathcal{A}_{12} \dotplus \mathcal{A}_{21} \dotplus \mathcal{A}_{22}$. In what follows, we write A_{ij}, B_{ij}, ... for the elements in $\mathcal{A}_{ij}$.

Lemma 2.2. *Let* $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathcal{B}(H)$. *The following statements are true.*

 (i) *Let* $1 \le j, k \le 2$ *be fixed. If* $T_{ij}S_{jk} = 0$ *holds for every* $T_{ij} \in \mathcal{A}_{ij}$ $(1 \le i \le 2)$, *then* $S_{jk} = 0$. *Dually, let* $1 \le k, i \le 2$ *be fixed. If* $S_{ki}T_{ij} = 0$ *for all* $T_{ij} \in \mathcal{A}_{ij}$ $(1 \le j \le 2)$, *then* $S_{ki} = 0$.

 (ii) *If* $T_{ij}S - ST_{ij}^{\dagger} \in \mathcal{A}_{ij}$ *for every* $T_{ij} \in \mathcal{A}_{ij}(1 \le i \ne j \le 2)$, *then* $S_{jj} = 0$; *if* $ST_{ij} - T_{ij}S^{\dagger} \in \mathcal{A}_{ij}$ *for every* $T_{ij} \in \mathcal{A}_{ij}(1 \le i \ne j \le 2)$, *then* $S_{ij} = 0$

(iii) *If* $T_{ii}S - ST_{ii}^{\dagger} \in \mathcal{A}_{ij}$ *for every* $T_{ii} \in \mathcal{A}_{ii}$ $(i = 1, 2)$, *then* $S_{ii} = 0$ *and* $S_{ji} = 0$ $(1 \le i \ne j \le 2)$.

PROOF. (i) It is an easy consequence of the fact that $\mathcal{B}(H)$ is prime in the sense that $S\mathcal{B}(H)T = 0$ implies either $S = 0$ or $T = 0$.

(ii) Since $T_{ij}S - ST_{ij}^{\dagger} \in \mathcal{A}_{ij}$, we have $P_j(T_{ij}S - ST_{ij}^{\dagger}) = -S_{jj}T_{ij}^{\dagger} = 0$. By (i), we see that $S_{jj} = 0$. Similarly, $ST_{ij} - T_{ij}S^{\dagger} \in \mathcal{A}_{ij}$ implies $(ST_{ij} - T_{ij}S^{\dagger})P_i = 0$. Hence $T_{ij}S_{ij}^{\dagger} = 0$, by (i) we get $S_{ij} = 0$.

(iii) Since $T_{ii}S - ST_{ii}^{\dagger} \in \mathcal{A}_{ij}$, we have $P_j(T_{ii}S - ST_{ii}^{\dagger}) = 0$. It follows $S_{ji}T_{ii}^{\dagger} = 0$ for all $T_{ii} \in \mathcal{A}_{ii}$, hence $S_{ji} = 0$ by (i). Using $T_{ii}S - ST_{ii}^{\dagger} \in \mathcal{A}_{ij}$ again, we see that $(T_{ii}S - ST_{ii}^{\dagger})P_i = 0$, that is $T_{ii}S_{ii} - S_{ii}T_{ii}^{\dagger} = 0$ since $S_{ji} = 0$. Taking $T_{ii} = \mathrm{i}P_i$, we get $S_{ii} = 0$. $\square$

The main technique we will use in Lemmas 2.3–2.9 is the following argument which will be termed as a "standard argument". Suppose $A, B, S \in \mathcal{B}(H)$ are such that $\Phi(S) = \Phi(A) + \Phi(B)$. Multiplying this equality by $\Phi(T)$ and $-\Phi(T)^{\dagger}$ $(T \in \mathcal{B}(H))$ from the left and from the right, respectively, we get $\Phi(T)\Phi(S) = \Phi(T)\Phi(A) + \Phi(T)\Phi(B)$ and $-\Phi(S)\Phi(T)^{\dagger} = -\Phi(A)\Phi(T)^{\dagger} - \Phi(B)\Phi(T)^{\dagger}$. Summing them, we get

$$\Phi(T)\Phi(S) - \Phi(S)\Phi(T)^{\dagger} = \Phi(T)\Phi(A) - \Phi(A)\Phi(T)^{\dagger} + \Phi(T)\Phi(B) - \Phi(B)\Phi(T)^{\dagger}.$$

It follows from equation (2.1) that

$$\Phi(TS - ST^\dagger) = \Phi(TA - AT^\dagger) + \Phi(TB - BT^\dagger).$$

Similarly, by multiplying $\Phi(S) = \Phi(A) + \Phi(B)$ and $-\Phi(S)^\dagger = -\Phi(A)^\dagger - \Phi(B)^\dagger$ by $\Phi(T)$ from the right and from the left respectively, we get

$$\Phi(ST - TS^\dagger) = \Phi(AT - TA^\dagger) + \Phi(BT - TB^\dagger).$$

Lemma 2.3. For every $A_{ii} \in \mathcal{A}_{ii}$ and $A_{ij} \in \mathcal{A}_{ij}$, $\Phi(A_{ii} + A_{ij}) = \Phi(A_{ii}) + \Phi(A_{ij})$ $(1 \le i \ne j \le 2)$.

PROOF. Let $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathcal{B}(H)$ be such that

$$\Phi(S) = \Phi(A_{ii}) + \Phi(A_{ij}). \tag{2.2}$$

For T_{jj}, applying standard argument to equation (2.2), we have

$$\Phi(T_{jj}S - ST_{jj}^\dagger) = \Phi(-A_{ij}T_{jj}^\dagger),$$

which implies that

$$T_{jj}S - ST_{jj}^\dagger = -A_{ij}T_{jj}^\dagger. \tag{2.3}$$

The lemma 2.2(iii) entails that $S_{jj} = 0$ and $S_{ji} = 0$. Multiplying P_i from the left in equation (2.3), we have $S_{ij} = A_{ij}$. For T_{ii}, applying standard argument to equation (2.2) again, we have

$$\Phi(ST_{ii} - T_{ii}S^\dagger) = \Phi(A_{ii}T_{ii} - T_{ii}A_{ii}^\dagger),$$

this implies that $S_{ii}T_{ii} - T_{ii}S_{ii}^\dagger = A_{ii}T_{ii} - T_{ii}A_{ii}^\dagger \Leftrightarrow (S_{ii} - A_{ii})T_{ii} = T_{ii}(S_{ii} - A_{ii})^\dagger$. So $(S_{ii} - A_{ii})^\dagger = S_{ii} - A_{ii}$ and there exists a real number $f_{P_i}(A)$ such that $S_{ii} = A_{ii} + f_{P_i}(A)P_i$. Thus for any A_{ii} and A_{ij}

$$\Phi(A_{ii} + A_{ij} + f_{P_i}(A)P_i) = \Phi(A_{ii}) + \Phi(A_{ij}). \tag{2.4}$$

Next we show that $f_{P_i}(A) = 0$. For T_{ij}, applying standard argument to equation (2.4), and from equation (2.4) there is a scalar α such that

$$\Phi(A_{ii}T_{ij} + f_{P_i}(A)T_{ij} - T_{ij}A_{ij}^\dagger) = \Phi(A_{ii}T_{ij}) + \Phi(-T_{ij}A_{ij}^\dagger)$$
$$= \Phi(\alpha P_i + A_{ii}T_{ij} - T_{ij}A_{ij}^\dagger),$$

and thus $f_{P_i}(A) = 0$. As desired. $\qquad\square$

Lemma 2.4. Φ *is additive on* $\mathcal{A}_{12}$.

PROOF. Note that

$$A_{12} + B_{12} - A_{12}^{\dagger} - B_{12}A_{12}^{\dagger} = (A_{12} + P_1)(P_2 + B_{12}) - (P_2 + B_{12})(A_{12} + P_1)^{\dagger}.$$

We have by Lemma 2.3

$$\begin{aligned}
\Phi(&A_{12} + B_{12} - A_{12}^{\dagger} - B_{12}A_{12}^{\dagger}) \\
&= \Phi[(A_{12} + P_1)(P_2 + B_{12}) - (P_2 + B_{12})(A_{12} + P_1)^{\dagger}] \\
&= \Phi(A_{12} + P_1)\Phi(P_2 + B_{12}) - \Phi(P_2 + B_{12})\Phi(A_{12} + P_1)^{\dagger} \\
&= (\Phi(A_{12}) + \Phi(P_1))(\Phi(P_2) + \Phi(B_{12})) - (\Phi(P_2) + \Phi(B_{12}))(\Phi(A_{12}) + \Phi(P_1))^{\dagger} \\
&= \Phi(A_{12})\Phi(P_2) - \Phi(P_2)\Phi(A_{12})^{\dagger} + \Phi(A_{12})\Phi(B_{12}) - \Phi(B_{12})\Phi(A_{12})^{\dagger} \\
&\quad + \Phi(P_1)\Phi(P_2) - \Phi(P_2)\Phi(P_1)^{\dagger} + \Phi(P_1)\Phi(B_{12}) - \Phi(B_{12})\Phi(P_1)^{\dagger} \\
&= \Phi(A_{12} - A_{12}^{\dagger}) + \Phi(B_{12} - B_{12}A_{12}^{\dagger}). \tag{2.5}
\end{aligned}$$

Applying standard argument with P_2 to equation (2.5), we see that

$$\begin{aligned}
\Phi(&A_{12} + B_{12} - A_{12}^{\dagger} - B_{12}^{\dagger}) \\
&= \Phi((A_{12} + B_{12} - A_{12}^{\dagger} - B_{12}A_{12}^{\dagger})P_2 - P_2(A_{12} + B_{12} - A_{12}^{\dagger} - B_{12}A_{12}^{\dagger})^{\dagger}) \\
&= \Phi((A_{12} - A_{12}^{\dagger})P_2 - P_2(A_{12} - A_{12}^{\dagger})^{\dagger}) + \Phi(B_{12}P_2 - P_2B_{12}^{\dagger}) \\
&= \Phi(A_{12} - A_{12}^{\dagger}) + \Phi(B_{12} - B_{12}^{\dagger}). \tag{2.6}
\end{aligned}$$

Let $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathcal{B}(H)$ such that

$$\Phi(S) = \Phi(A_{12}) + \Phi(B_{12}). \tag{2.7}$$

For P_1, applying standard argument to equation (2.7), we have

$$\begin{aligned}
\Phi(P_1S - SP_1) &= \Phi(P_1A_{12} - A_{12}P_1) + \Phi(P_1B_{12} - B_{12}P_1) \\
&= \Phi(A_{12}) + \Phi(B_{12}) = \Phi(S).
\end{aligned}$$

Thus $S_{11} = S_{21} = S_{22} = 0$. For P_2, by using standard argument in equation (2.7) and by equation (2.6), we have

$$\Phi(S_{12} - S_{12}^{\dagger}) = \Phi(S_{12}P_2 - P_2S_{12}^{\dagger})$$
$$= \Phi(A_{12}P_2 - P_2A_{12}^{\dagger}) + \Phi(B_{12}P_2 - P_2B_{12}^{\dagger}) = \Phi(A_{12} + B_{12} - A_{12}^{\dagger} - B_{12}^{\dagger}).$$

Thus $S_{12} - S_{12}^{\dagger} = A_{12} + B_{12} - A_{12}^{\dagger} - B_{12}^{\dagger}$, by multiplying P_1 from the left, we get $S_{12} = A_{12} + B_{12}$. $\square$

Similarly, we have:

Lemma 2.5. Φ is additive on $\mathcal{A}_{21}$.

Lemma 2.6. For every $A_{ii}, B_{ii} \in \mathcal{A}_{ii}$, $\Phi(A_{ii} + B_{ii}) = \Phi(A_{ii}) + \Phi(B_{ii})$, $i = 1, 2$.

PROOF. Without loss of generality, assume $i = 1$. Let $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathcal{B}(H)$ such that

$$\Phi(S) = \Phi(A_{11}) + \Phi(B_{11}). \tag{2.8}$$

For P_1, applying standard argument to equation (2.8), we have

$$\Phi(P_1 S - S P_1) = \Phi(P_1 A_{11} - A_{11} P_1) + \Phi(P_1 B_{11} - B_{11} P_1) = 0,$$

hence $S_{12} = S_{21} = 0$. For any $T_{22} \in \mathcal{A}_{22}$, applying standard argument to equation (2.8), we have $\Phi(T_{22} S_{22} - S_{22} T_{22}^{\dagger}) = 0$, thus $S_{22} = 0$ from Lemma 2.2 (iii). For any $T_{12} \in \mathcal{A}_{12}$, applying standard argument to equation (2.8) and by Lemma 2.4, we have

$$\Phi(S T_{12} - T_{12} S^{\dagger}) = \Phi(A_{11} T_{12} - T_{12} A_{11}^{\dagger}) + \Phi(B_{11} T_{12} - T_{12} B_{11}^{\dagger})$$
$$= \Phi(A_{11} T_{12}) + \Phi(B_{11} T_{12}) = \Phi(A_{11} T_{12} + B_{11} T_{12}).$$

Thus $S_{11} T_{12} = (A_{11} + B_{11}) T_{12}$ and from Lemma 2.2 (i), we get $S_{11} = A_{11} + B_{11}$. As desired. $\qquad\square$

Lemma 2.7. $\Phi(A_{11} + A_{22}) = \Phi(A_{11}) + \Phi(A_{22})$.

PROOF. Let $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathcal{B}(H)$ be such that

$$\Phi(S) = \Phi(A_{11}) + \Phi(A_{22}). \tag{2.9}$$

For T_{11}, by using standard argument in equation (2.9), we have

$$\Phi(T_{11} S - S T_{11}^{\dagger}) = \Phi(T_{11} A_{11} - A_{11} T_{11}^{\dagger}).$$

This implies that $T_{11} S - S T_{11}^{\dagger} = T_{11} A_{11} - A_{11} T_{11}^{\dagger}$. Multiplying P_2 from the left and the right in the above equality, we get that $S_{12} = S_{21} = 0$ by Lemma 2.2 (i). Moreover from Lemma 2.2 (iii) $T_{11}(S_{11} - A_{11}) = (S_{11} - A_{11}) T_{11}^{\dagger}$ implies that $S_{11} = A_{11}$. Similarly using standard argument with T_{22}, we get $S_{22} = A_{22}$. $\qquad\square$

Lemma 2.8. $\Phi(A_{12} + A_{21}) = \Phi(A_{12}) + \Phi(A_{21})$.

PROOF. Let $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathcal{B}(H)$ be such that

$$\Phi(S) = \Phi(A_{12}) + \Phi(A_{21}). \tag{2.10}$$

Just like preceding lemmas, we need only to check that $S_{11} = S_{22} = 0$, $S_{12} = A_{12}$ and $S_{21} = A_{21}$.

Using standard argument in equation (2.10), we have

$$\Phi(P_1 S - S P_1) = \Phi(P_1 A_{12} - A_{12} P_1) + \Phi(P_1 A_{21} - A_{21} P_1) = \Phi(A_{12}) + \Phi(-A_{21}).$$

For P_1, applying standard argument in the above equality again, we see

$$\Phi(P_1 S - 2 P_1 S P_1 + S P_1) = \Phi(P_1 (P_1 S - S P_1) - (P_1 S - S P_1) P_1)$$
$$= \Phi(P_1 A_{12} - A_{12} P_1) + \Phi(P_1(-A_{21}) - (-A_{21}) P_1) = \Phi(A_{12}) + \Phi(A_{21}) = \Phi(S),$$

hence $P_1 S - 2 P_1 S P_1 + S P_1 = S_{11} + S_{12} + S_{21} + S_{22}$ and $S_{11} = S_{22} = 0$. Note that $\Phi(S P_1 - P_1 S^\dagger) = \Phi(A_{12} P_1 - P_1 A_{12}^\dagger) + \Phi(A_{21} P_1 - P_1 A_{21}^\dagger) = \Phi(A_{21} - A_{21}^\dagger)$, thus $S_{21} - S_{21}^\dagger = A_{21} - A_{21}^\dagger$ and $S_{21} = A_{21}$. Using the same argument with P_2, we have $S_{12} = A_{12}$.   □

Lemma 2.9. $\Phi(A_{11} + A_{12} + A_{21} + A_{22}) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22})$.

PROOF. Let $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathcal{B}(H)$ be such that

$$\Phi(S) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22}). \tag{2.11}$$

For P_1, applying standard argument to equation (2.11), by Lemma 2.3, and we have $\Phi(S_{12} - S_{21}) = \Phi(P_1 S - S P_1) = \Phi(P_1 A_{12} - A_{12} P_1) + \Phi(P_1 A_{21} - A_{21} P_1) = \Phi(A_{12} - A_{21})$, thus $S_{12} - S_{21} = A_{12} - A_{21}$ and $S_{12} = A_{12}$, $S_{21} = A_{21}$. For any $T_{12} \in \mathcal{A}_{12}$, applying standard argument to equation (2.11) again, we have

$$\Phi(T_{12} S - S T_{12}^\dagger) = \Phi(-A_{12} T_{12}^\dagger + T_{12} A_{21}) + \Phi(T_{12} A_{22} - A_{22} T_{12}^\dagger). \tag{2.12}$$

For P_1, applying standard argument to equation (2.12), we get

$$\Phi(T_{12} S_{22} + S_{22} T_{12}^\dagger) = \Phi(P_1(T_{12} S - S T_{12}^\dagger) - (T_{12} S - S T_{12}^\dagger) P_1) = \Phi(T_{12} A_{22} + A_{22} T_{12}^\dagger),$$

thus $S_{22} = A_{22}$. For any $T_{21} \in \mathcal{A}_{12}$ and P_2, using similar computations we get $S_{11} = A_{11}$. As desired.   □

Now by Lemmas 2.4–2.7 and Lemma 2.9, we get the additivity of Φ. Next we show Φ is multiplicative.

Lemma 2.10. *For any non-trivial self adjoint idempotent P, $\Phi(P)$ is a non-trivial self adjoint idempotent.*

PROOF. First we show $\Phi(I) = I$ and $\Phi(iI) = \pm iI$. Suppose Φ satisfies equation (2.1), that is $\Phi(AB - BA^\dagger) = \Phi(A)\Phi(B) - \Phi(B)\Phi(A)^\dagger$, thus

$$AB = BA^\dagger \Leftrightarrow \Phi(A)\Phi(B) = \Phi(B)\Phi(A)^\dagger. \qquad (2.13)$$

Taking $A = I$ in equation (2.13), one get $\Phi(I)\Phi(B) = \Phi(B)\Phi(I)^\dagger$ for all $B \in \mathcal{B}(H)$ which implies that $\Phi(I) = aI$ for some $a \in \mathbb{R}$ since Φ is surjective. Taking $B = I$ in equation (2.13), we get $A = A^\dagger \Leftrightarrow \Phi(A) = \Phi(A)^\dagger$, that is, Φ preserves self adjoint operators in both directions. Now for any $a \in \mathbb{C}$ and self-adjoint operator $A \in \mathcal{B}(H)$, $\Phi(A)\Phi(aI) = \Phi(aI)\Phi(A)$ implies that there is $b \in \mathbb{C}$ such that $\Phi(aI) = bI$. From the equality

$$\Phi(iI) = \Phi\left((iI)\left(\frac{1}{2}I\right) - \left(\frac{1}{2}I\right)(iI)^\dagger\right) = \Phi(iI)\Phi\left(\frac{1}{2}I\right) - \Phi\left(\frac{1}{2}I\right)\Phi(iI)^\dagger,$$

we get

$$\Phi(iI)^\dagger = -\Phi(iI) \quad \text{and} \quad \Phi\left(\frac{1}{2}I\right) = \frac{1}{2}I \qquad (2.14)$$

Thus by the additivity of Φ, $\Phi(I) = I$ and $\Phi(aI) = aI$ for every integer number a. For $\Phi(iI)$, from equation (2.14) and

$$-2I = \Phi(-2I) = \Phi((iI)(iI) - (iI)(iI)^\dagger) = \Phi(iI)^2 - \Phi(iI)\Phi(iI)^\dagger = 2\Phi(iI)^2,$$

we have $\Phi(iI) = \pm iI$. Without loss of generality, we assume in the sequel that $\Phi(iI) = iI$.

For any $A \in \mathcal{B}(H)$, we have

$$\Phi(2iA) = \Phi(iIA - A(iI)^\dagger) = \Phi(iI)\Phi(A) - \Phi(A)\Phi(iI)^\dagger = 2i\Phi(A). \qquad (2.15)$$

This implies that

$$A^\dagger = -A \Leftrightarrow \Phi(A)^\dagger = -\Phi(A). \qquad (2.16)$$

Thus for $A = A^\dagger$, we get from equation (2.15)–(2.16) and

$$2i\Phi(2A^2) = \Phi(4iA^2) = \Phi(2iAA - A(2iA)^\dagger)$$
$$= \Phi(2iA)\Phi(A) - \Phi(A)\Phi(2iA)^\dagger = 4i\Phi(A)^2. \qquad (2.17)$$

Hence we get $\Phi(P)^2 = \Phi(P)$ for every self adjoint idempotent P in $\mathcal{B}(H)$. $\qquad \square$

Now we get when $A^\dagger = -A$, then by equation (2.16)

$$\Phi(2A) = \Phi(AI - IA^\dagger) = \Phi(A)\Phi(I) - \Phi(I)\Phi(A)^\dagger = 2\Phi(A); \qquad (2.18)$$

further for any $B \in \mathcal{B}(H)$,

$$\Phi(AB + BA) = \Phi(AB - BA^\dagger) = \Phi(A)\Phi(B) + \Phi(B)\Phi(A). \qquad (2.19)$$

Let $P, Q \in \mathcal{B}(H)$ be two idempotents, recall that P and Q are orthogonal if $PQ = QP = 0$; $P \le Q$ if $PQ = QP = P$.

Lemma 2.11. Φ *preserves the order, and the orthogonality of self-adjoint idempotents in both directions.*

PROOF. Let $P, Q \in \mathcal{B}(H)$ be two orthogonal self-adjoint idempotents. Then from equation (2.15) and equation (2.19) we obtain

$$0 = \Phi(iPQ - Q(iP)^\dagger) = \Phi(iP)\Phi(Q) + \Phi(Q)\Phi(iP) = i(\Phi(P)\Phi(Q) + \Phi(Q)\Phi(P))$$

which implies that $\Phi(P)\Phi(Q) + \Phi(Q)\Phi(P) = 0$. Multiplying this equality by $\Phi(Q)$ from the right and from the left respectively, we have $\Phi(Q)\Phi(P)\Phi(Q) = -\Phi(P)\Phi(Q)$ and $\Phi(Q)\Phi(P)\Phi(Q) = -\Phi(Q)\Phi(P)$. Hence

$$\Phi(P)\Phi(Q) = \Phi(Q)\Phi(P) = 0$$

and Φ preserves the orthogonality of self-adjoint idempotents. In the same way we see that Φ preserves orthogonality also in the other direction. We assert that Φ preserves the partial order relation $\le$ between self-adjoint idempotents. If $P, Q \in \mathcal{B}(H)$ are self-adjoint idempotents and $P \le Q$, then by equation (2.18-2.19) we obtain

$$2i\Phi(P) = \Phi(2iP) = \Phi(iPQ - Q(iP)^\dagger) = \Phi(iP)\Phi(Q) + \Phi(Q)\Phi(iP)$$
$$= i\Phi(P)\Phi(Q) + i\Phi(Q)\Phi(P).$$

Hence $2\Phi(P) = \Phi(P)\Phi(Q) + \Phi(Q)\Phi(P)$, multiplying this equality by $\Phi(Q)$ from both sides, we get that $\Phi(P)\Phi(Q) = \Phi(Q)\Phi(P) = \Phi(P)$. Thus Φ preserves the partial order. Similarly, we get Φ preserves the partial order in the other direction. As desired.   □

In the next Lemma, let $\Phi(P_i) = Q_i$, $i = 1, 2$. Set $\mathcal{B}_{ij} = Q_i \mathcal{B}(K) Q_j$, $i, j = 1, 2$. Then, $\mathcal{B}(K) = \mathcal{B}_{11} + \mathcal{B}_{12} + \mathcal{B}_{21} + \mathcal{B}_{22}$.

Lemma 2.12. $\Phi(\mathcal{A}_{ij}) = \mathcal{B}_{ij}$, $i, j = 1, 2$.

PROOF. If $i \neq j$, without loss of generality, assume $i = 1, j = 2$. Let $A_{12} \in \mathcal{A}_{12}$ and set $\Phi(A_{12}) = S$. Since $A_{12} = P_1 A_{12} - A_{12} P_1^{\dagger}$, we obtain $S = \Phi(A_{12}) = \Phi(P_1 A_{12} - A_{12} P_1^{\dagger}) = \Phi(P_1)\Phi(A_{12}) - \Phi(A_{12})\Phi(P_1)^{\dagger} = \Phi(P_1)S - S\Phi(P_1)$, thus $S_{11} + S_{12} + S_{21} + S_{22} = S_{12} - S_{21}$ and $S_{11} = S_{22} = S_{21} = 0$, $\Phi(A_{12}) \in \mathcal{B}_{12}$. Considering Φ^{-1}, we get $\Phi(\mathcal{A}_{12}) = \mathcal{B}_{12}$.

If $i = j$, without loss of generality, assume $i = j = 1$. Let $A_{11} \in \mathcal{A}_{11}$ and set $\Phi(A_{11}) = S$. Note that $0 = \Phi(P_2 A_{11} - A_{11} P_2^{\dagger}) = \Phi(P_2)\Phi(A_{11}) - \Phi(A_{11})\Phi(P_2)^{\dagger}$, hence $S_{21} - S_{12} = 0$ and $S_{12} = S_{21} = 0$. For any $B_{12} \in \mathcal{A}_{12}$, from $0 = \Phi(B_{12}A_{11} - A_{11}B_{12}^{\dagger}) = \Phi(B_{12})\Phi(A_{11}) - \Phi(A_{11})\Phi(B_{12})^{\dagger}$, we have $\Phi(B_{12})S_{22} = S_{22}\Phi(B_{12})^{\dagger}$ and $S_{22} = 0$. Thus $\Phi(A_{11}) \in \mathcal{B}_{11}$. Considering Φ^{-1}, we get $\Phi(A_{11}) = \mathcal{B}_{11}$. As desired. $\square$

Lemma 2.13. $\Phi(A_{ij}B_{jj}) = \Phi(A_{ij})\Phi(B_{jj})$ and $\Phi(A_{ii}B_{ij}) = \Phi(A_{ii})\Phi(B_{ij})$ $i \neq j \in \{1, 2\}$.

PROOF. Since $\Phi(A_{ij}B_{jj}) - \Phi(B_{jj}A_{ij}^{\dagger}) = \Phi(A_{ij}B_{jj} - B_{jj}A_{ij}^{\dagger}) = \Phi(A_{ij})\Phi(B_{jj}) - \Phi(B_{jj})\Phi(A_{ij})^{\dagger}$, we get $\Phi(A_{ij}B_{jj}) = \Phi(A_{ij})\Phi(B_{jj})$ from Lemma 2.12. Similarly we get $\Phi(A_{ii}B_{ij}) = \Phi(A_{ii})\Phi(B_{ij})$ $i \neq j \in \{1, 2\}$. $\square$

Lemma 2.14. $\Phi(A_{ij}B_{ji}) = \Phi(A_{ij})\Phi(B_{ji})$ $i \neq j \in \{1, 2\}$.

PROOF. From $\Phi(A_{ij}) - \Phi(A_{ij}^{\dagger}) = \Phi(A_{ij} - A_{ij}^{\dagger}) = \Phi(A_{ij}P_j - P_j A_{ij}^{\dagger}) = \Phi(A_{ij})\Phi(P_j) - \Phi(P_j)\Phi(A_{ij})^{\dagger} = \Phi(A_{ij}) - \Phi(A_{ij})^{\dagger}$, we get $\Phi(A_{ij}^{\dagger}) = \Phi(A_{ij})^{\dagger}$. For any B_{ji} there exists $B_{ij} \in \mathcal{A}_{ij}$ such that $B_{ij}^{\dagger} = B_{ji}$. Hence

$$-\Phi(A_{ij}B_{ji}) = \Phi(-A_{ij}B_{ij}^{\dagger}) = \Phi(B_{ij}A_{ij} - A_{ij}B_{ij}^{\dagger})$$
$$= \Phi(B_{ij})\Phi(A_{ij}) - \Phi(A_{ij})\Phi(B_{ij})^{\dagger} = -\Phi(A_{ij})\Phi(B_{ij})^{\dagger} = -\Phi(A_{ij})\Phi(B_{ji})$$

implies $\Phi(A_{ij}B_{ji}) = \Phi(A_{ij})\Phi(B_{ji})$. As desired. $\square$

Lemma 2.15. $\Phi(A_{ii}B_{ii}) = \Phi(A_{ii})\Phi(B_{ii})$ $i = 1, 2$.

PROOF. For any $T_{ij} \in \mathcal{A}_{ij}$, from Lemma 2.13, we get

$$\Phi(A_{ii}B_{ii})\Phi(T_{ij}) = \Phi(A_{ii}B_{ii}T_{ij}) = \Phi(A_{ii})\Phi(B_{ii}T_{ij}) = \Phi(A_{ii})\Phi(B_{ii})\Phi(T_{ij}).$$

It follows from Lemma 2.12 and Lemma 2.2 (i) that $\Phi(A_{ii}B_{ii}) = \Phi(A_{ii})\Phi(B_{ii})$. $\square$

Lemma 2.16. $\Phi(AB) = \Phi(A)\Phi(B)$ for all $A, B \in \mathcal{B}(H)$.

PROOF. Set $A = A_{11} + A_{12} + A_{21} + A_{22}$ and $B = B_{11} + B_{12} + B_{21} + B_{22}$. Then

$$
\begin{aligned}
\Phi(AB) &= \Phi((A_{11} + A_{12} + A_{21} + A_{22})(B_{11} + B_{12} + B_{21} + B_{22})) \\
&= \Phi(A_{11}B_{11} + A_{11}B_{12} + A_{12}B_{21} + A_{12}B_{22} + A_{21}B_{11} \\
&\quad + A_{21}B_{12} + A_{22}B_{21} + A_{22}B_{22}) \\
&= \Phi(A_{11})\Phi(B_{11}) + \Phi(A_{11})\Phi(B_{12}) + \Phi(A_{12})\Phi(B_{21}) + \Phi(A_{12})\Phi(B_{22}) \\
&\quad + \Phi(A_{21})\Phi(B_{11}) + \Phi(A_{21})\Phi(B_{12}) + \Phi(A_{22})\Phi(B_{21}) + \Phi(A_{22})\Phi(B_{22}) \\
&= \Phi(A)\Phi(B) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square
\end{aligned}
$$

.

PROOF OF THEOREM 2.1. From Lemmas 2.3-2.16, It remains to prove that $\Phi(A^\dagger) = \Phi(A)^\dagger$ for all $A \in \mathcal{B}(H)$ and that Φ is a linear or conjugate linear map.

For $B = I \in \mathcal{B}(H)$, from Lemmas 2.16–2.17 and the equality $\Phi(A)\Phi(B) - \Phi(B)\Phi(A^\dagger) = \Phi(AB) - \Phi(BA^\dagger) = \Phi(AB - BA^\dagger) = \Phi(A)\Phi(B) - \Phi(B)\Phi(A)^\dagger$, it is easy to check that $\Phi(A^\dagger) = \Phi(A)^\dagger$.

Next we show Φ is a linear or conjugate linear map. From Lemma 2.10, we get $\Phi(\mathbb{C}I) \subseteq \mathbb{C}I$, denote $\Phi(\lambda I) = f(\lambda)I$. It follows that f is a ring homomorphism of the real numbers and $f(r) = r$ for every rational number r. Since f sends squares to squares and it preserves order. It follows that for rational number r and real numbers x, y such that $-r < x - y < r$, we have $-r < f(x) - f(y) < r$. Hence, f is continuous and so must be identity map on the real number. Since $f(i) = i$ or $-i$, f must fix all complex number or send each complex number to its conjugate. So Φ is linear or conjugate linear.

Therefore Φ is a $\dagger$ isomorphism or conjugate $\dagger$ isomorphism. Thus there exists a linear or conjugate linear bounded invertible operator $U \in \mathcal{B}(H, K)$ such that $\Phi(A) = UAU^{-1}$ for all $A \in \mathcal{B}(H)$ or $\Phi(A) = UA^\dagger U^{-1}$ for all $A \in \mathcal{B}(H)$. Because Φ preserves the $\dagger$ operation and $\Phi(I) = I$, we see that $U^\dagger U = I_H$, $UU^\dagger = I_K$ and U is a unitary operator. Therefore, $\Phi(A) = UAU^{-1} = UAU^\dagger$ or $\Phi(A) = UA^\dagger U^{-1} = UA^\dagger U^\dagger$ for all $A \in \mathcal{B}(H)$. When $\Phi(A) = UA^\dagger U^\dagger$, it is easy to check that it dose not satisfy equation (2.1). Thus $\Phi(A) = UAU^\dagger$ for all $A \in \mathcal{B}(H)$. As desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

In particular, we have:

Corollary 2.2. *Let H and K be Hilbert spaces over the the complex field and let $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a bijective map. Then, Φ satisfies*

$$
\Phi(AB - BA^*) = \Phi(A)\Phi(B) - \Phi(B)\Phi(A)^*
$$

for every pair $A, B \in \mathcal{B}(H)$ if and only if there exists a unitary or a conjugate unitary operator $U \in \mathcal{B}(H, K)$ such that $\Phi(A) = UAU^*$ for all $A \in \mathcal{B}(H)$.

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