Publ. Math. Debrecen
77/1-2 (2010), 115–123

Approximately convex functions on topological vector spaces

By JACEK TABOR (Kraków), JÓZEF TABOR (Rzeszów) and MAREK ŻOŁDAK (Rzeszów)

Abstract. Let X be a real topological vector space, let D be a subset of X and let $\alpha : X \to [0, \infty)$ be an even function locally bounded at zero.
A function $f : D \to \mathbb{R}$ is called (α, t)-*preconvex* (where $t \in (0, 1)$ is fixed), if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \alpha(x-y) \quad \text{for } x, y \in D \text{ such that } [x, y] \subset D.$$

We prove the Bernstein–Doetsch type theorem for (α, t)-preconvex functions.

1. Introduction

Let D be a convex subset of a real vector space X. A function $f : D \to \mathbb{R}$ is called *convex* if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \quad \text{for } x, y \in D, \ t \in [0, 1].$$

If the above inequality holds for all $x, y \in D$, $t = \frac{1}{2}$ then f is called *midconvex* (or *Jensen convex*) and if it holds for $x, y \in D$ and fixed $t \in (0, 1)$ then f is called t-*convex.*

The relation between convexity and midconvexity was established in the celebrated Bernstein–Doetsch Theorem [1]. An interesting version of Bernstein–Doestch Theorem for t-convex functions is presented in [8].

The notion of convex function was generalized by several authors. The main idea is based on modyfing the right hand side of defining inequality. The first

Mathematics Subject Classification: 26A51, 26B25.
Key words and phrases: midconvexity, t-convexity, approximate convexity, preconvexity, Bernstein–Doetsch theorem.

step in this direction was done by D. H. Hyers and S. M. Ulam [6]. They introduced the term "approximately convex function". Let $\delta > 0$. A function $f : D \to \mathbb{R}$ is called *δ-convex* if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \delta \quad \text{for } x, y \in D,\ t \in [0,1].$$

Further generalizations were made by S. Rolewicz who introduced in [10] the notions of paraconvex and strongly paraconvex function (for more information the reader is refered to [11], [12]). A different modification of approximate convexity can be found in the papers of A. Házy and Zs. Páles (see for example [4]).

Conditional version of approximate convexity (similar in the spirit to paraconvex functions of S. Rolewicz) has been studied by P. Cannarsa and C. Sinestrari [2]. In this case we resign from the convexity of the domain. Since our definition is inspired by that from [2] let us quote it.

Let S be a subset of $\mathbb{R}^n$. We say that a function $f : S \to \mathbb{R}$ is *semiconvex* if there exists a nondecreasing upper semicontinuous function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{\varrho \to 0^+} \omega(\varrho) = 0$ and

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + t(1-t)\|x-y\|\omega(\|x-y\|)$$

for any pair $x, y \in S$ such that the segment $[x,y] := \{sx + (1-s)y : s \in [0,1]\}$ is contained in S and for any $t \in [0,1]$.

It is worth mentioning that a semiconvex function defined on an open subset of $\mathbb{R}^n$ is locally Lipschitz [2, Theorem 2.1.7].

Our aim in this paper is to study approximately convex function on a Hausdorff real topological vector space X. From now on we assume that D is an open subset of X, $\alpha : X \to [0,\infty)$ is a even function locally bounded at zero and $t \in (0,1)$ is a fixed number.

Definition 1.1. We call a function $f : D \to \mathbb{R}$ *(α,t)-preconvex* if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \alpha(x-y) \quad \text{for } x, y \in D \text{ such } [x,y] \subset D.$$

In the case $t = \frac{1}{2}$ we say that f is *α-premidconvex.*

The above definition generalizes all the mentioned before versions of the notion of approximately convex function.

2. Bernstein–Doetsch theorem

We will generalize some results from [4] and [14]. Our general aim is to obtain an analogue of Bernstein–Doetsch theorem. To prove it we need some auxiliary results.

Lemma 2.1. *Let $N \in \mathbb{N}$, $a_k \in \mathbb{R}$ for $k = -N, \dots, N$, and let $b \in \mathbb{R}$ be such that*

$$a_k \le \frac{a_{k-1} + a_{k+1}}{2} + b \quad \text{for } k \in \{-N+1, \dots, N-1\}. \tag{1}$$

Then

$$-\left(\frac{a_{-N} - a_0}{N}\right) - (N+1)b \le a_1 - a_0 \le \frac{a_N - a_0}{N} + (N-1)b.$$

Proof. From (1) we directly obtain

$$\begin{aligned} a_1 - a_0 &\ge a_1 - a_0, \\ a_2 - a_1 &\ge a_1 - a_0 - 2b, \\ a_3 - a_2 &\ge a_2 - a_1 - 2b \ge a_1 - a_0 - 4b, \\ &\vdots \\ a_N - a_{N-1} &\ge \dots \ge a_1 - a_0 - 2(N-1)b. \end{aligned}$$

Summing the above inequalities up we obtain that

$$a_N - a_0 \ge N(a_1 - a_0) - N(N-1)b,$$

and consequently that

$$a_1 - a_0 \le \frac{a_N - a_0}{N} + (N-1)b.$$

We show the estimation from below. Making analogous reasoning as above for the sequence a_{-k} we get

$$a_{-1} - a_0 \le \frac{a_{-N} - a_0}{N} + (N-1)b.$$

From (1) we directly conclude that $(a_{-1} - a_0) \ge (a_0 - a_1) - 2b$, which by the above inequality gives

$$a_0 - a_1 - 2b \le \frac{a_{-N} - a_0}{N} + (N-1)b,$$

which makes the proof complete. □

As a direct corollary from Lemma 2.1 we obtain the following result.

Corollary 2.1. *Let $f : D \to \mathbb{R}$ be an α-premidconvex function, let $x \in D$, $b \in [0,\infty)$, $M \in \mathbb{R}$ and let U be a balanced neighbourhood of zero such that $x + U \subset D$ and*

$$\alpha(u) \le b \quad \text{for } u \in U,$$

$$f(x+u) \le M \quad \text{for } u \in U.$$

Then we have

$$-\frac{(M - f(x))}{N} - (N+1)b \le f(y) - f(x) \le \frac{M - f(x)}{N} + (N-1)b$$

for $y \in x + \frac{1}{N}U$, $N \ge 2$, $N \in \mathbb{N}$.

Proof. We apply Lemma 2.1 for the sequence $a_k = f(x + k \cdot (y - x))$. □

Remark 2.1. Corollary 2.1 means in particular that for α-premidconvex function local boundedness from above (at a point) implies local boundedness (at the same point).

Lemma 2.2. *Let $D \subset X$ be an open set, let $x \in D$, and let U be a balanced neighbourhood of zero such that $x + U + U \subset D$ and α is bounded on $U + U + U$. If $f : D \to \mathbb{R}$ is an α-premidconvex function locally bounded above at a point of $x + U$ then f is locally bounded above at x.*

Proof. Let $y \in x + U$, and let V be a balanced neighbourhood of zero such that $V \subset U$, $y + V \subset x + U$ and f is bounded above by M on $y + V$. We are going to show that f is bounded above on $x + \frac{1}{2}V$. Consider an arbitrary $h \in \frac{1}{2}V$. Let $z_0 := y + 2h$, $z_1 := 2x - y$. Then

$$z_0 \in y + V \subset x + U$$

and

$$z_1 = x - (y - x) \in x + U.$$

Moreover for arbitrary $t \in [0,1]$ we have

$$tz_0 + (1-t)z_1 = x + (2t-1)(y-x) + t(2h) \in x + U + V \subset x + U + U,$$

which means that $[z_0, z_1] \subset x + U + U \subset D$. By α-premidconvexity of f we obtain

$$\begin{aligned} f(x+h) = f\left(\frac{z_0 + z_1}{2}\right) &\le \frac{f(z_0) + f(z_1)}{2} + \alpha(z_0 - z_1) \\ &= \frac{f(y+2h) + f(2x-y)}{2} + \alpha(2(y-x) + 2h) \le \frac{M + f(2x-y)}{2} \\ &\quad + \alpha(2(y-x) + 2h). \end{aligned}$$

Furthermore $2(y-x)+2h \in 2U+V \subset U+U+U$.
It completes the proof. □

We will also need the following simple lemma.

Lemma 2.3. *Let $f : D \to \mathbb{R}$ be an (α, t)-preconvex function. Then f is α_t-premidconvex with*

$$\alpha_t(x) := \frac{1}{t(1-t)}\alpha(x/2) \quad \textit{for } x \in X.$$

Proof. We will use a similar method as in [3]. Let $x, y \in D$ such that $[x, y] \subset D$ be arbitrary. We begin with an obvious equality

$$\frac{x+y}{2} = t\left[t\frac{x+y}{2} + (1-t)x\right] + (1-t)\left[ty + (1-t)\frac{x+y}{2}\right].$$

From (α, t)-preconvexity of f we obtain

$$\begin{aligned} f\left(\frac{x+y}{2}\right) &\le tf\left(t\frac{x+y}{2} + (1-t)x\right) + (1-t)f\left(ty + (1-t)\frac{x+y}{2}\right) + \alpha\left(\frac{x-y}{2}\right) \\ &\le t\left[(tf\left(\frac{x+y}{2}\right) + (1-t)f(x) + \alpha\left(\frac{x-y}{2}\right)\right] \\ &\quad + (1-t)\left[tf(y) + (1-t)f\left(\frac{x+y}{2}\right) + \alpha\left(\frac{x-y}{2}\right)\right] + \alpha\left(\frac{x-y}{2}\right) \\ &= (2t^2 - 2t + 1)f\left(\frac{x+y}{2}\right) + t(1-t)(f(x) + f(y)) + 2\alpha\left(\frac{x-y}{2}\right). \end{aligned}$$

Whence we get $f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} + \frac{1}{t(1-t)}\alpha\left(\frac{x-y}{2}\right)$.
It is obvious that α_t is even and locally bounded above at zero. □

Now we are ready to prove the main result of this section.

Theorem 2.1. *Let D be an open connected subset of X. Let $f : D \to \mathbb{R}$ be an (α, t)-preconvex function locally bounded above at a point of D. Then f is locally bounded at every point.*

Proof. By Lemma 2.3 f is α_t-premidconvex. Let

$$B := \{x \in D : f \text{ is locally bounded above at } x\}.$$

Clearly B is open and nonempty. We are going to show that $B = D$. Since D is connected it is sufficient to prove that B is a closed subset of D. Consider an arbitrary $x \in (clB) \cap D$. Let U be a balanced neighbourhood of zero such that $x + U + U \subset D$ and that α_t is bounded on $U + U + U$. Since $x \in clB$ there exists a $y \in (x + U) \cap B$. By Lemma 2.2 we obtain that $x \in B$. We have proved that $B = D$. Corollary 2.1 completes the proof. □

To prove a full analogue of Bernstein–Doetsch Theorem we need to deal with the continuity of f. We will apply Corollary 2.1.

Theorem 2.2. *Let D be an open connected subset of X and let $f : D \to \mathbb{R}$ be an (α, t)-preconvex function locally bounded above at a point. We assume additionally that $\alpha(0) = 0$ and that α is continuous at zero. Then f is locally uniformly continuous.*

Proof. By Theorem 2.1 f is locally bounded at every point of D. Consider an arbitrary $x_0 \in D$. There exist an $M \in \mathbb{R}_+$ and a balanced neighbourhood U of zero such that $x_0 + U + U \subset D$ and

$$|f(x_0 + u)| < M \quad \text{for } u \in U.$$

We take an arbitrary $\delta > 0$, $\delta \le 2M$. We can find a balanced neighbourhood V of zero such that $V \subset U$ and

$$\alpha(v) \le \delta \quad \text{for } v \in V.$$

We choose an $N \in \mathbb{N}$ such that

$$N \in \left[\sqrt{\frac{2M}{\delta}}, 2\sqrt{\frac{2M}{\delta}}\right].$$

Let $x \in x_0 + U$ and $y \in x + \frac{1}{N}V$ be arbitrary. By Corollary 2.1 we have

$$|f(y) - f(x)| \le 2\left(\frac{2M}{N} + N\delta\right)$$
$$\le 2\left(\frac{2M}{\sqrt{\frac{2M}{\delta}}} + 2\sqrt{\frac{2M}{\delta}} \cdot \delta\right) = 6\sqrt{(2M)\delta} \le 6\sqrt{2M}\sqrt{\delta}. \qquad \square$$

3. Preconvexity

As it is well known midconvexity implies $\mathbb{Q}$-convexity (i.e. convexity with $t \in [0,1] \cap \mathbb{Q}$). Similar result can be obtained for generalized midconvexity [14, Theorem 2.2]. To prove analogue of such result in our settings we will need the function $d : \mathbb{R} \to \mathbb{R}$ defined as follows

$$d(x) := 2\,\mathrm{dist}(x, \mathbb{Z}) \quad \text{for } x \in \mathbb{R}.$$

We will also need the following

Lemma TT ([14, Corollary2.1])**.** *Let $\beta : [0,1] \to [0,\infty)$ be a nondecreasing function. We assume that $h : [0,1] \to \mathbb{R}$ satisfies the following conditions:*

$$h(0) = h(1) = 0,$$

$$h\left(\frac{x+y}{2}\right) \le \frac{h(x)+h(y)}{2} + \beta(|x-y|) \quad \text{for } x, y \in [0,1].$$

Then

$$h(r) \le \sum_{k=0}^{\infty} \frac{1}{2^k} \beta(d(2^k r)) \quad \text{for } r \in [0,1] \cap \mathbb{Q}.$$

Moreover, if h is upper bounded then the above inequality holds for all $r \in [0,1]$.

Proposition 3.1. *We assume that for each $x \in X$ the function $\mathbb{R}_+ \ni w \mapsto \alpha(wx)$ is nondecreasing. Let $f : D \to \mathbb{R}$ be a (α, t)-preconvex function. Then*

$$f(rx + (1-r)y) \le rf(x) + (1-r)f(y) + \frac{1}{t(1-t)} \sum_{k=0}^{\infty} \frac{1}{2^k} \alpha\left(d(2^k r)\frac{x-y}{2}\right) \tag{2}$$

for all $x, y \in D$ such that $[x, y] \subset D$ and all $r \in [0,1] \cap \mathbb{Q}$.

If additionally D is open and connected and f is locally bounded above at a point then (2) holds for all $x, y \in D$ such that $[x, y] \subset D$ and all $r \in [0,1]$.

Proof. From Lemma 2.3 we obtain that f is α_t-premidconvex with $\alpha_t(x) := \frac{1}{t(1-t)}\alpha(\frac{x}{2})$.

Fix arbitrarily $x, y \in D$ such that $[x, y] \subset D$. We define function $h : [0,1] \to \mathbb{R}$ by the formula $h(r) := f(rx+(1-r)y) - rf(x) - (1-r)f(y)$. Then $h(0) = h(1) = 0$ and we have for $r, s \in [0,1]$

$$h\left(\frac{r+s}{2}\right) - \frac{h(r)+h(s)}{2} \le \alpha_t((r-s)(x-y)).$$

We put

$$\beta(w) := \alpha_t(w(x-y)) = \frac{1}{t(1-t)}\alpha\left(w\frac{x-y}{2}\right) \quad \text{for } w \in [0,1].$$

Applying Lemma TT we get

$$h(r) \le \frac{1}{t(1-t)} \sum_{k=0}^{\infty} \frac{1}{2^k} \alpha\left(d(2^k r)\frac{x-y}{2}\right) \quad \text{for } r \in [0,1] \cap \mathbb{Q},$$

i.e.

$$f(rx+(1-r)y) \le rf(x)+(1-r)f(y)+\frac{1}{t(1-t)}\sum_{k=0}^{\infty}\frac{1}{2^k}\alpha\left(d(2^k r)\frac{x-y}{2}\right)$$

for $r \in [0,1] \cap \mathbb{Q}$.
Assume now that D is open and connected and that f is locally bounded above at a point. Then by Theorem 2.1 we obtain that f is locally bounded at each point. Consequently then h is locally bounded at each point. But h is defined on a compact set $[0,1]$. Therefore h is bounded. Lemma TT completes now the proof. □

S. Rolewicz proved in [10] that if a function $f : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, satisfies for certain $C > 0$, $r > 2$ the following inequality

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y)+C|x-y|^r \quad \text{for } x, y \in I,\ t \in [0,1],$$

then f is convex. This result was generalized in [14] for $\alpha(\cdot)$-midconvex function defined on an open convex subset of a normed space and locally bounded above at a point. We will prove an analogue of the result from [14] for (α, t)-preconvex functions.

Theorem 3.1. *We assume that*

$$\liminf_{n\to\infty} 4^n \alpha\left(\frac{1}{2^n}x\right) = 0 \quad \textit{for } x \in X. \tag{3}$$

Then every (α, t)-preconvex function is premidconvex.

Proof. Let $f : D \to \mathbb{R}$ be an (α, t)-preconvex function. Consider arbitrary $x, y \in D$ such that $[x, y] \subset D$. We will show that

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} + \frac{4^{n-1}}{t(1-t)}\alpha\left(\frac{x-y}{2^n}\right) \quad \text{for } n \in \mathbb{N}. \tag{4}$$

For $n = 1$ it follows directly from Lemma 2.3. Assume that (4) is valid for some $n \in \mathbb{N}$. Then applying Lemma 2.3 (with $t = \frac{1}{2}$) and (4) we obtain

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} + 4\frac{4^{n-1}}{t(1-t)}\alpha\left(\frac{x-y}{2^{n+1}}\right).$$

Hence (4) has been proved. Letting in (4) $n \to \infty$ and applying (3) we get

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2}.$$

□

References

[1] F. Bernstein and G. Doetsch, Zur theorie der convexen Funktionen, *Math. Ann.* **76** (1915), 514–526.

[2] P. Cannarsa and C. Sinestrari, Semiconvex Functions, Hamilton-Jacobi Equation and Optimal Control, Progress in Nonlinear Differential Equations and their Applications, *Birkhäuser*, *Boston*, 2004.

[3] Z. Daróczy and Zs. Páles, Convexity with given infinite weight sequences, *Stochastica* **XI-1** (1987), 5–12.

[4] A. Házy and Zs. Páles, On approximately t-convex functions, *Publ. Math. Debrecen* **66** (2005), 489–501.

[5] D. H. Hyers, G. Isac and Th. M. Rassias, Stability of Functional Equations in Several Variables, *Birkhäuser*, *Basel*, 1998.

[6] D. H. Hyers and S. M. Ulam, Approximately convex functions, *Proc. Amer. Math. Soc.* **3** (1952), 821–828.

[7] C. T. Ng and K. Nikodem, On approximately convex functions, *Proc. Amer. Math. Soc.* **118** (1993), 103–108.

[8] K. Nikodem and Zs. Páles, On t-convex functions, *Real Anal. Exchange* **29/1** (2003), 1–16.

[9] S. Rolewicz, Metric Linear Spaces, *PWN*, *Warszawa*, 1972.

[10] S. Rolewicz, On γ paraconvex multifunctions, *Math. Japonica* **24/3** (1979), 293–300.

[11] S. Rolewicz, On $\alpha(\cdot)$-paraconvex and strongly $\alpha(\cdot)$-paraconvex functions, *Control Cybernetics* **29** (2000), 367–377.

[12] S. Rolewicz, Paraconvex analysis, *Control Cybernetics* **34** (2005), 951–965.

[13] S. Rolewicz, An extension of Mazurs theorem on Gateaux differentiability to the class of strongly $\gamma()$-paraconvex functions, *Studia Math.* **172** (2006), 243–248.

[14] Jacek Tabor and Józef Tabor, Generalized approximate midconvexity, *Control and Cybernetics* **38**(3) (2009).

JACEK TABOR
INSTITUTE OF COMPUTER SCIENCE
JAGIELLONIAN UNIVERSITY
ŁOJASIEWICZA 6
30-348 KRAKÓW
POLAND

E-mail: tabor@ii.uj.edu.pl

JÓZEF TABOR
INSTITUTE OF MATHEMATICS
UNIVERSITY OF RZESZÓW
REJTANA 16A
35-959 RZESZÓW
POLAND

E-mail: tabor@univ.rzeszow.pl

MAREK ŻOŁDAK
INSTITUTE OF MATHEMATICS
UNIVERSITY OF RZESZÓW
REJTANA 16A
35-959 RZESZÓW
POLAND

E-mail: marek_z2@op.pl

(Received May 18, 2009)