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On solutions of the Gołąb-Schinzel functional equation

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Let X be a real topological linear space and let $\mathbb{R}$ denote the set of all reals. In this paper we are mainly concerned with solutions $f : X \to \mathbb{R}$ of the functional equation

$$f(x + f(x)^n y) = f(x)f(y), \tag{1}$$

where n is a given positive integer.

Equation (1) is a generalization of the well–known Gołąb–Schinzel functional equation

$$f(x + f(x)y) = f(x)f(y), \tag{2}$$

which has been considered and solved in several classes of functions. For details we refer e.g. to [2]–[6], [8]–[12], [17], [18], and [20].

Equation (1) is also a particular case of the functional equation

$$f(f(y)^k x + f(x)^n y) = tf(x)f(y) \tag{3}$$

studied by many authors in variuous case (see e.g. [5]–[7], [16], and [19]).

Finally, we must mention that equation (1) is tightly connected with some classes of subgroups of the Lie groups L_s^1, the one-dimensional affine group, and some other groups (see [8], cf. also [2], pp. 309–311, [3], [6], [16], and [20]).

We determine solutions of (1) having the Darboux property in the class of functions $f : X \to \mathbb{R}$. Such solutions $f : \mathbb{R} \to \mathbb{R}$ of equation (3) have already been studied in [19] for $k > 0$ and $t = 1$ and in [5] for $k = 0$. Our results (see Corollary 2 and Theorem 1) are interesting especially in view of the second part of Hilbert's fifth problem (cf. [1], p. 153).

We also prove that every linear functional $g : X \to \mathbb{R}$ having the Darboux property is continuous (see Corollary 1) and give an application of some of the results obtained to the question of finding subgroups.

Let us remind that a function $f : X \to \mathbb{R}$ has the Darboux property, whenever, for every non-empty connected set $D \subset X$, the set $f(D)$ is connected in $\mathbb{R}$.

We start with some facts concerning linear functionals, which are necessary in the proof of Theorem 1.

Proposition 1. *Let $g : X \to \mathbb{R}$ be a linear functional such that the set $g(D)$ is connected in $\mathbb{R}$ for every non-empty and connected (in X) set $D \subset B := g^{-1}((-1, +\infty))$. Then g is continuous.*

For the proof of Proposition 1 we need the following two lemmas.

Lemma 1. *Let $g : X \to \mathbb{R}$, $g \neq 0$ (i.e. $g(X) \neq \{0\}$), be a linear functional. Then $0 \in \mathrm{cl}(g^{-1}((-1, 0)))$.*

Proof. For the proof by contradiction suppose that this is not the case. Then the set $B_0 = X \setminus \mathrm{cl}(g^{-1}((-1, 0)))$ is open and $0 \in B_0$. Since $g \neq 0$, $\mathrm{int}(\ker g) = \emptyset$. Thus there is $z \in B_0$ with $g(z) \neq 0$. By the continuity of the function $R \ni a \to az \in X$ at 0, there exists a real $c > 0$ such that $bz \in B_0$ for every $b \in (-c, c)$. On the other hand, it is easily seen that there is $d \in (-c, c)$ satisfying the condition: $g(dz) = dg(z) \in (-1, 0)$. This yields a contradiction.

Lemma 2. *Suppose that $g : X \to \mathbb{R}$ is a linear functional such that the set $\ker g$ is not closed. Then the set $D = g^{-1}((-1, +\infty)) \setminus \ker g$ is connected.*

Proof. For the proof by contradiction suppose that D is not connected. Put $B_1 = g^{-1}((0, +\infty))$ and $B_2 = g^{-1}((-1, 0))$. B_1 and B_2 are connected, because they are convex. Thus $B_1 \cap \mathrm{cl}(B_2) = \emptyset$ (cf. e.g. [15]). This yields

$$\mathrm{cl}(B_2) \cap \mathrm{cl}(-B_2) = \mathrm{cl}(B_2) \cap (-\mathrm{cl}(B_2)) \subset X \setminus (B_1 \cup (-B_1)) = \ker g. \tag{4}$$

On the other hand, $x + \mathrm{cl}(B_2) = \mathrm{cl}(x + B_2)$ and $x + B_2 = B_2$ for every $x \in \ker g$. Hence $(\ker g) + \mathrm{cl}(B_2) = \mathrm{cl}(B_2)$. Since $\ker g = -\ker g$ and, by Lemma 1, $\ker g \subset (\ker g) + \mathrm{cl}(B_2)$, we obtain $\ker g \subset \mathrm{cl}(B_2) \cap \mathrm{cl}(-B_2)$, which, in view of (4), implies $\ker g = \mathrm{cl}(B_2) \cap \mathrm{cl}(-B_2)$. This contradicts the hypothesis on $\ker g$.

Now we are in a position to prove Proposition 1. So, suppose that g is not continuous. Then the set $\ker g$ is not closed. Thus, according to Lemma 2, the set $D = g^{-1}((-1, +\infty)) \setminus \ker g$ is connected. Since $g(D) = (-1, +\infty) \setminus \{0\}$, we get a contradiction. This completes the proof of Proposition 1.

In particular, from Proposition 1 we get the next

Corollary 1. *Every linear functional $g : X \to \mathbb{R}$ having the Darboux property is continuous.*

Remark 1. Corollary 1 is the more interesting as there are discontinuous additive functions $h : \mathbb{R} \to \mathbb{R}$ having the Darboux property (see [13], cf. also [14], pp. 286–291). Given a continuous linear functional $g : X \to \mathbb{R}$, $g \neq 0$, and a discontinuous additive function $h : \mathbb{R} \to \mathbb{R}$ having the Darboux property, we can obtain a discontinuous additive function $f : X \to \mathbb{R}$ having the Darboux property by putting $f(x) = h(g(x))$ for $x \in X$.

In the sequel we shall need some results from [8]. Let us recall them.

Lemma 4 (see [8], Corollary 1). *Let $f : X \to \mathbb{R}$, $f \neq 0$, be a function satisfying equation* (1)*. Put $A = f^{-1}(\{1\})$ and $W = f(X) \setminus \{0\}$. Then:*

(i) *A is an additive subgroup of X;*
(ii) *W is a multiplicative subgroup of $\mathbb{R}$;*
(iii) *$a^n A = A$ for every $a \in W$.*

Lemma 5 (see [8], Proposition 3). *Let $f : X \to \mathbb{R}$ be a function satisfying* (1)*. Let $A = f^{-1}(\{1\})$ and $W = f(X) \setminus \{0\}$. If there is $a_0 \in W$ such that $a_0^n \neq 1$ and $(a_0^n - 1)A \subset A$, then*

$$a^n \neq 1 \quad \text{for } a \in W \setminus \{1\} \tag{5}$$

and there exists $x_0 \in X \setminus \bigcup\{(a^n - 1)^{-1}A : a \in W \setminus \{1\}\}$ such that

$$f(x) = \begin{cases} a & \text{if } x \in (a^n - 1)x_0 + A \text{ and } a \in W; \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for } x \in X. \tag{6}$$

The next proposition will be very useful in the proof of Theorems 1 and 2. The proposition gives us, as well, some examples of solutions of equation (1).

Proposition 2. *A function $f : X \to \mathbb{R}$ satisfies equation* (1) *and $\operatorname{int}(f(X)) \neq \emptyset$ iff there exists a linear subspace $Y \subseteq X$, $Y \neq \{0\}$, and a linear functional $g : Y \to R$, $g \neq 0$, such that,*

1° *in the case when n is odd, either*

$$f(x) = \begin{cases} \sqrt[n]{(g(x) + 1)} & \text{for } x \in Y; \\ 0 & \text{for } x \in X \setminus Y, \end{cases} \tag{7}$$

or

$$f(x) = \begin{cases} \sqrt[n]{\sup(g(x) + 1, 0)} & \text{for } x \in Y; \\ 0 & \text{for } x \in X \setminus Y; \end{cases} \tag{8}$$

2° *in the case when n is even,* (8) *holds.*

Proof. Put $A = f^{-1}(\{1\})$ and $W = f(X) \setminus \{0\}$. According to Lemma 4(ii), we have either $W = (0, +\infty)$ or $W = \mathbb{R} \setminus \{0\}$. Thus, by

Lemma 4(i), (iii), A is a linear subspace of X. Hence $(a^n - 1)A \subset A$ for every $a \in W$. Consequently, in view of Lemma 5, conditions (5) and (6) are valid with some $x_0 \in X \setminus A$. Put $Y = \mathbb{R}x_0 + A$ and define a linear functional $g : Y \to \mathbb{R}$ by the formula:

$$g(ax_0 + y) = a \qquad \text{for } a \in \mathbb{R},\ y \in A.$$

It is easy to check that, according to (6),

$$g(x) = f(x)^n - 1 \qquad \text{for } x \in (W_n - 1)x_0 + A,$$

where $W_n = \{a^n : a \in W\}$. Further, in virtue of (5),

– in the case when n is odd, $W = \mathbb{R} \setminus \{0\}$ or $W = (0, +\infty)$;

– in the case when n is even $W = (0, +\infty)$.

Whence, by the definition of g and (6), conditions 1°, 2° hold.

The converse is easy to verify. This completes the proof.

Now we have all the tools to prove our main result.

Theorem 1. *A function $f : X \to \mathbb{R}$, $f \neq 0$, has the Darboux property and satisfies the functional equation* (1) *if and only if there exists a continuous linear functional $g : X \to \mathbb{R}$ such that,*

1° *in the case when n is odd,*

$$f(x) = \sqrt[n]{(g(x) + 1)} \qquad \text{for } x \in X \tag{9}$$

or

$$f(x) = \sqrt[n]{\sup(g(x) + 1, 0)} \qquad \text{for } x \in X; \tag{10}$$

2° *in the case when n is even, f is of form* (10).

Proof. The case $f = 1$ is trivial. So, assume that $f(X) \neq \{1\}$. Since X is connected, the set $f(X)$ is connected. Further, in virtue of Lemma 4 (ii), $1 \in f(X)$. Thus $\operatorname{int}(f(X)) \neq \emptyset$. Hence, according to Proposition 2, there exist a linear subspace Y of X and a linear functional $g : Y \to \mathbb{R}$ such that conditions 1°, 2° of Proposition 2 are valid.

Suppose that there is $x_0 \in X \setminus Y$. Then the set $\mathbb{R}x_0$ is connected and $f(\mathbb{R}x_0) = \{0, 1\}$. This is a contradiction. Consequently $Y = X$.

Notice that (7) or (8) implies $g(x) = f(x)^n - 1$ for $x \in g^{-1}((-1, +\infty))$. Thus the set $g(D)$ is connected in R for every non-empty connected set $D \subset g^{-1}((-1, +\infty))$. Hence, in view of Proposition 1, g is continuous. This completes the first part of the proof. The converse is easy to check.

Since the function $f = 0$ is continuous, from Theorem 1 we derive the following

Corollary 2. *Every function $f : X \to \mathbb{R}$ having the Darboux property and satisfying equation* (1) *is continuous.*

Remark 2. In the case $n = 0$ Corollary 2 is not valid. In fact, let $h : \mathbb{R} \to \mathbb{R}$ be a discontinuous additive function having the Darboux property and assume that there exists a continuous linear functional $g : X \to \mathbb{R}$, $g \neq 0$. Then the function $f : X \to \mathbb{R}$ given by the formula: $f(x) = e^{h(g(x))}$ for $x \in X$, is discontinuous, satisfies (1) with $n = 0$, and has the Darboux property.

Finally, we shall give *an example* for the application of Proposition 2 to the problem of finding subgroups of some groups.

In the set $P = (\mathbb{R} \setminus \{0\}) \times X$ we introduce a binary operation $\cdot : P \times P \to P$ as follows:

$$(a, x) \cdot (b, y) = (ab, y + b^n x) \qquad \text{for } (a, x), (b, y) \in P.$$

It is easy to verify that $(P, \cdot\,)$ is a group. In particular, in the case $X = \mathbb{R}$, $(P, \cdot\,)$ is isomorphic with a subgroup of the Lie group L^1_{n+1} (see [8], p.2). For further details concerning the group $(P, \cdot\,)$ we refer to [2] (pp. 310–311), [3], [5], [8], and [16].

We have the following description of a class of subgroups of the group $(P, \cdot\,)$:

Theorem 2. *Suppose that $f : X \to \mathbb{R}$ is a function with $f(X) \setminus \{0, 1\} \neq \emptyset$. Then the set $D = \{(f(x), x) : x \in X,\ f(x) \neq 0\}$ is a connected subgroup of the group $(P, \cdot\,)$ (P is endowed with the product topology) if and only if there exist a linear subspace Y of X, $Y \neq \{0\}$, and a linear functional $g : Y \to \mathbb{R}$ such that $g \neq 0$ and* (8) *holds, i.e.* $D = \left\{ \left(\sqrt[n]{g(x) + 1}, x \right) : x \in Y,\ g(x) > -1 \right\}$.

Proof. First, let us recall a result from [8]. Namely, we have the following

Lemma 6 (see [8], Theorem 1(ii))**.** *Let $f \neq 0$ be a function mapping X into $\mathbb{R}$. Then the set $D = \{(f(x), x) : x \in X,\ f(x) \neq 0\}$ is a subgroup of the group $(P, \cdot\,)$ iff f satisfies equation* (1)*.*

Assume that D is a connected subgroup of $(P, \cdot\,)$. Then, according to Lemma 6, f is a solution of equation (1). Further, notice that the function $p : P \to \mathbb{R}$, defined by: $p(a, x) = a$ for $(a, x) \in P$, is continuous. Thus the set $p(D) = f(X) \setminus \{0\}$ is connected. Hence $\operatorname{int}(f(X)) \neq \emptyset$, since $f(X) \setminus \{0, 1\} \neq \emptyset$ and, by Lemma 4(ii), $1 \in f(X)$. Consequently, in virtue of Proposition 2, there exist a linear subspace Y of X, $Y \neq \{0\}$, and a linear functional $g : Y \to \mathbb{R}$, $g \neq 0$, such that (7) or (8) holds. To complete the first part of the proof it suffices to notice that in the case when f is of form (7) the set $f(X) \setminus \{0\}$ is not connected.

For the converse, on account of Proposition 2 and Lemma 6, we must show that the set $D_0 = \left\{ \left(\sqrt[n]{g(x) + 1}, x \right) : x \in Y,\ g(x) > -1 \right\}$ is connected for every linear subspace $Y \neq \{0\}$ of X and every linear functional $g : Y \to \mathbb{R}$, $g \neq 0$.

Fix $x, y \in g^{-1}((-1, +\infty))$. Since the function

$$[0,1] \ni t \to (tg(y-x),\ t(y-x)) \in P$$

is continuous, the set $T = \{(g(x) + tg(y-x), x + t(y-x)) : t \in [0,1]\}$ is connected. Moreover $(g(x), x), (g(y), y) \in T \subset D_0$. So, we have proved that the set $\{(g(x), x) : x \in X,\ g(x) > -1\}$ is connected. Consequently D_0 is connected, because the function

$$(-1, +\infty) \times X \ni (a, x) \to \left(\sqrt[n]{a+1}, x\right) \in P$$

is continuous. This completes the proof.

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