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Title: Binary sequences generated by sequences $\{n\alpha\}$, $n = 1, 2, \ldots$

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Let α be an irrational number, I be a subinterval of the unit interval $(0, 1)$, and $\{x\}$ denote the fractional part of x. In this paper we shall study arithmetical properties of the set $A = \{n \in \mathbb{N}; \{n\alpha\} \in I\}$ and pseudorandom character of the sequence x_n, $n = 1, 2, \ldots$, where $x_n = 1$ when $\{n\alpha\} \in I$, and $x_n = -1$ otherwise. We prove, among others, that the gaps between successive elements of A are at most of three lengths, a, b and $a + b$ also in the case of an arbitrary interval $I \subset (0, 1)$, thereby extending the known Slater's results for intervals of the type $I = (0, t)$ with $t < 1/2$. Further we exactly describe the set of positive integers which are not equal to a difference of two arbitrary elements from A and we prove that A contains infinite double-arithmetic progressions. Then we find a new lower estimate of the Mauduit–Sárközy well distribution meaasure of x_n for an arbitrary interval I. We also prove that the sequence x_n is Sturmian for every interval I of length $\{\alpha\}$ or $1 - \{\alpha\}$ in the sense that the number of 1's in any pair of finite subsegments of the same length occurring in x_n can differ by at most one. We prove (Theorem **??**) that if $|I| \leq 1/2$ then any subsequence of x_n of the form x_{n+kK}, $k = 1, 2, \ldots$, splits into consecutive blocks of 1's and blocks of -1's whose lengths also differ by at most one. The proofs employ two geometric ideas: (i) a transposition of subintervals (cf. Lemma **??**) of I to construct arithmetic progressions of the set A, (ii) properties (cf. Lemma **??**) of line segments of the intersection of the graph of the sawtooth function $x + \{k\alpha\}$ with $I \times I$ to answer the question when two elements $\{n\alpha\}$ and $\{(n + k)\alpha\}$ simultaneously fall into I. This technique gives, for instance, a new proof of the mentioned Slater's three gap theorems.

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