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Binary sequences generated by sequences $\{n\alpha\}$, $n = 1, 2, \dots$

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Abstract. Let α be an irrational number, I be a subinterval of the unit interval $(0, 1)$, and $\{x\}$ denote the fractional part of x. In this paper we shall study arithmetical properties of the set $A = \{n \in \mathbb{N}; \{n\alpha\} \in I\}$ and pseudorandom character of the sequence x_n, $n = 1, 2, \dots$, where $x_n = 1$ when $\{n\alpha\} \in I$, and $x_n = -1$ otherwise. We prove, among others, that the gaps between successive elements of A are at most of three lengths, a, b and $a + b$ also in the case of an arbitrary interval $I \subset (0, 1)$, thereby extending the known Slater's results for intervals of the type $I = (0, t)$ with $t < 1/2$. Further we exactly describe the set of positive integers which are not equal to a difference of two arbitrary elements from A and we prove that A contains infinite double-arithmetic progressions. Then we find a new lower estimate of the Mauduit–Sárközy well distribution meaasure of x_n for an arbitrary interval I. We also prove that the sequence x_n is Sturmian for every interval I of length $\{\alpha\}$ or $1 - \{\alpha\}$ in the sense that the number of 1's in any pair of finite subsegments of the same length occurring in x_n can differ by at most one. We prove (Theorem 26) that if $|I| \leq 1/2$ then any subsequence of x_n of the form x_{n+kK}, $k = 1, 2, \dots$, splits into consecutive blocks of 1's and blocks of -1's whose lengths also differ by at most one. The proofs employ two geometric ideas: (i) a transposition of subintervals (cf. Lemma 1) of I to construct arithmetic progressions of the set A, (ii) properties (cf. Lemma 4) of line segments of the intersection of the graph of the sawtooth function $x + \{k\alpha\}$ with $I \times I$ to answer the question when two elements $\{n\alpha\}$ and $\{(n + k)\alpha\}$ simultaneously fall into I. This technique gives, for instance, a new proof of the mentioned Slater's three gap theorems.

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1. Introduction

Throughout the paper we shall suppose that α is an irrational number, and that $\{n\alpha\}$ denotes the fractional part of $n\alpha$, $n = 1, 2, \dots$. Given a fixed interval $I \subset [0, 1]$, define the set

$$A = A(\alpha, I) = \{n \in \mathbb{N} : \{n\alpha\} \in I\}, \tag{1}$$

and the function

$$\chi_I(y) = \begin{cases} 1, & \text{if } y \in I, \text{ and} \\ -1, & \text{if } y \notin I. \end{cases} \tag{2}$$

Ch. Mauduit and A. Sárközy [4] started to investigate some distribution properties of binary $\{-1, 1\}$-sequences generated by formula

$$x_n = \chi_I(\{n\alpha\}), \quad n = 1, 2, \dots, \tag{3}$$

in the case when $I = (0, 1/2)$. Independently of this, a related class of sequences is a subject of an extensive interest for a longer time. Namely, if $I = (0, \{\alpha\})$, α irrational, then $A = A(\alpha, I)$ and $x_n = \chi_I(\{n\alpha\})$ yield the so called Sturmian set and Sturmian sequence, respectively[1]. The aim of this paper is to study

(I) the pseudorandomness of the sequence $x_n = \chi_I(\{n\alpha\})$, and

(II) arithmetical properties of the set $A = A(\alpha, I)$

not only for intervals I of length $|I| = 1/2$ but for intervals $I \subset (0, 1)$ of arbitrary length. Our approach is based on two geometrical ideas: The first one uses a transposition of subintervals (cf. Lemma 1) of I to construct arithmetic progressions in the set (1). The second one (cf. Lemma 4) employs properties of line segments of the intersection of the graph of the sawtooth function $x + \{k\alpha\}$ with $I \times I$ to answer the question when two elements $\{n\alpha\}$ and $\{(n + k)\alpha\}$ simultaneously fall into I.

In what follows we assume that the interval $I \subset (0, 1)$ is open, since the inclusion of the endpoins of I into consideration causes only finitely many changes in A. We start with two groups of notes related to our aims:

(I) In [4] Mauduit and Sárközy introduced three measures for measuring the pseudorandomness of a binary $\{-1, 1\}$-sequences. One of them is the so called well distribution measure

$$W_M = W_M(x_n) = \max_{n,K,D} \left| \sum_{k=1}^{D} x_{n+kK} \right|$$

[1]Sturmian sequences are usually defined as $\{0, 1\}$-sequences rather than $\{-1, 1\}$-sequences, but the mutual transformations of these forms are straightforward.

where the maximum is taken over all n, K, D such that $n, K, D \in \mathbb{N}$ and $1 \le n + K \le n + DK \le M$.

If α is an irrational number, $I = [0, 1/2)$ and $x_n = \chi_{[0,1/2)}(\{n\alpha\})$, $n = 1, \dots, M$, then they proved [6, Theorem 1,2]:

(a) $W_M(x_n) \ge \sqrt{M/2}$,

(b) $W_M(x_n) \le 6\big(\frac{K}{\log(K+1)}\big)^{1/2}(M \log M)^{1/2} + 1$, provided α is an irrational number such that the partial quotients of its continued fraction expansion are bounded by K.

Mauduit and Sárközy also proved estimations of this measure for other sequences, e.g. for Champernowne, Rudin–Schapiro and Thue–Morce ones, see [5] for more details.

In Section 3 we prove two lower bounds of W_M for sequences $x_n = \chi_I(\{n\alpha\})$ with an arbitrary interval $I \subset [0, 1)$. More precisely, let $\lfloor\cdot\rfloor$ be the greatest integer function, $D_{\lfloor\sqrt{M}\rfloor}$ denote the discrepancy of the sequence $\{1.\alpha\}, \{2.\alpha\}, \dots, \{\lfloor\sqrt{M}\rfloor.\alpha\}$, and $\alpha = [a_0; a_1, a_2, \dots]$ be the continued fraction expansion of α. Then in Theorem 6 we prove that $W_M \ge \sqrt{M}\sqrt{|I|/2}\sqrt{a_{n+1}}$ for infinitely many M, where M depends on the denominator q_n of the nth convergent of α. On the other hand, in Theorem 7 we prove that $W_M \ge |I|/(2D_{\lfloor\sqrt{M}\rfloor})$ for sufficiently large M. In Theorem 8 we give an upper bound of W_M for irrational algebraic numbers α.

(II) Our motivation to study of the set $A = A(\alpha, I) = \{a_1 < a_2 \dots\}$ has three historical sources:

(i) Three gaps problem saying that the differences $a_{i+1} - a_i$ attain at most three distinct values of the form $a, b, a+b$, where a is the first positive integer for which $\{a\alpha\} \in (0, |I|)$ and b is first one such that $\{b\alpha\} \in (1 - |I|, 1))$. This problem was introduced and proved by N. B. Slater [11], see also comments by K. Florek [1] and Slater [12]. [2]

(ii) There are many equivalent definitions of Sturmian sequences, and these sequences are subject of extensive investigation from various points of view (e.g. as symbolic dynamic systems) [8]. A Sturmian sequence x_n can also be defined as a $\{-1, 1\}$-sequence such that for every k the number $p(k)$ of

[2]The three gaps problem is directly connected (see Slater [12]) with the so called Steinhaus' *three steps problem* asserting: If the sequence $\{1\alpha\}, \{2\alpha\}, \dots, \{N\alpha\}$ with an irrational α is reordered with respect to the increasing magnitude of its terms, say $\{n_1\alpha\} < \{n_2\alpha\} < \dots < \{n_N\alpha\}$, then for every $i = 1, 2, \dots, N-1$ we have $\{n_{i+1}\alpha\} - \{n_i\alpha\} = d_1$ or d_2 or $d_1 + d_2$ where $d_1 = \{n_1\alpha\}$ and $d_2 = 1 - \{n_N\alpha\}$. This was proved independently by J. Surányi [16], V. T. Sós [13] and others.

different blocks of elements $x_{n+1}, \dots, x_{n+k}$, $n = 1, 2, \dots$, is equal to $k + 1$. The function $p(k)$ is called the complexity function. It is known that the sequence $x_n = \chi_I(\{n\alpha\})$ with a special choice of interval I of length $|I| = \{\alpha\}$ or $|I| = 1 - \{\alpha\}$ is Sturmian.

(iii) By the problem of bounded local discrepancy $D([1, N], I) = A([1, N]) - N|I|$, where $A([M, N]) = \{i \in \mathbb{N} : a_i \in [M, N]\}$. Here E. Hecke [2] and A. Ostrowski [10] proved (see Theorem 23) that $D([1, N], I)$ is bounded if $|I| = \{h\alpha\}$ for some $h \in \mathbb{Z}$ and later H. Kesten [7] proved that this condition is also necessary. Ostrowski also proved that $|D([1, N], I)| < |h|$.

In Section 4 (Theorem 10) we give a new proof of the three gaps problem and extend it (Proposition 18) in the sense, that for arbitrary $a \in A(\alpha, (0, |I|))$, $b \in A(\alpha, (1-|I|, 1))$ and every $n \in A(\alpha, I)$ we have either $n+a \in A(\alpha, I)$, or $n+b \in A(\alpha, I)$, or $n+a+b \in A(\alpha, I)$. In Theorem 19 we characterize all $k \in \mathbb{N}$ for which the equation $a_j - a_i = k$ is not solvable in $a_i, a_j \in A$. We also prove (Theorems 21 and 22) that the set A contains infinite double arithmetic progressions despite the fact that it does not contain an infinite arithmetic progression.

For Sturmian sequences it is true that the number of 1's in any pair of finite subsegments of the same length occurring in x_n can differ by at most one. In Section 6 we prove (Theorem 26) that if $|I| \le 1/2$ then any subsequence of (3) of the form x_{n+kK}, $k = 1, 2, \dots$, splits into consecutive blocks of 1's and blocks of -1's whose lengths also differ by at most one.

2. Basic preliminaries

Since the idea of the proof of the next Lemma will play a decisive role in what follows, we recap the basic ideas of its proof for the reader's convenience.

Lemma 1 ([14, Theorem 1, 2])**.** *Let α be an irrational number and $I \subset (0, 1)$. Then the set A contains arbitrarily long arithmetic progressions. More precisely, let $D \in \mathbb{N}$ be given, and $K_1, K_2 \in \mathbb{N}$ be such that*[3]

$$|1 - \{K_1\alpha\}| < \frac{|I|}{2D}, \tag{4}$$

$$|0 - \{K_2\alpha\}| < \frac{|I|}{2D}. \tag{5}$$

Then for every $n \in A$ either

$$\{n, n + K_1, n + 2K_1, \dots, n + DK_1\} \subset A \tag{6}$$

[3]Note that $\{n\alpha\}$, $n = 1, 2, \dots$, is dense in $[0, 1]$.

or

$$\{n, n + K_2, n + 2K_2, \ldots, n + DK_2\} \subset A. \tag{7}$$

Proof. Let $n \in A$, i.e. $\{n\alpha\} \in I$. The point $\{n\alpha\}$ splits the interval I into two subintervals, say I_1, the left one, and I_2, the right one, respectively. Translate these intervals towards the endpoints of $[0, 1]$ as follows: $I_1' = I_1 + 1 - \{n\alpha\}$ and $I_2' = I_2 - \{n\alpha\}$. Finally, let $I' = [0, 1] \setminus (I_1' \cup I_2')$. Then

$$\{(n + k)\alpha\} \in I_1 \Leftrightarrow \{k\alpha\} \in I_1', \tag{8}$$

$$\{(n + k)\alpha\} \in I_2 \Leftrightarrow \{k\alpha\} \in I_2' \tag{9}$$

Which of these two cases occurs depends on the fact whether $\{(n + k)\alpha\} = \{n\alpha\} + \{k\alpha\} - 1$, or $\{(n + k)\alpha\} = \{n\alpha\} + \{k\alpha\}$. Relations (8) and (9) imply

$$\{(k + n)\alpha\} \in I \Leftrightarrow \{k\alpha\} \notin I'. \tag{10}$$

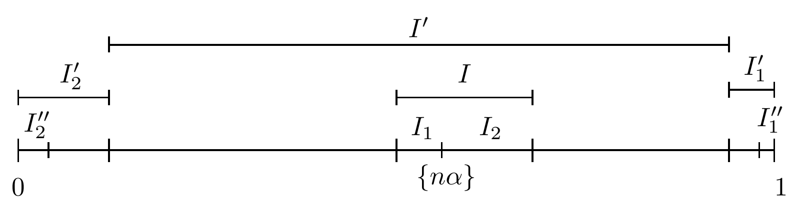

Figure 1.

Given D, let the intervals I_1'' and I_2'' of lengths $|I_1''| = |I_1'|/D$ and $|I_2''| = |I_2'|/D$, respectively, be located as depicted in the Figure 1. Then

$$\{K_1\alpha\} \in I_1'' \Rightarrow \{kK_1\alpha\} \in I_1' \quad \text{for every } k = 1, 2. \ldots, D, \tag{11}$$

$$\{K_2\alpha\} \in I_2'' \Rightarrow \{kK_2\alpha\} \in I_2' \quad \text{for every } k = 1, 2. \ldots, D. \tag{12}$$

The proofs of (11) and (12) follow from the observation that for every triple of positive integers k, K_1, and K_2 we have

$$\{k(1 - \{K_1\alpha\})\} = 1 - \{kK_1\alpha\} \quad \text{and} \quad \{k\{K_2\alpha\}\} = \{kK_2\alpha\}. \tag{13}$$

In our case, due to the choice of K_1, K_2 we have $k(1 - \{K_1\alpha\}) < k\frac{|I_1'|}{D} < |I_1'| < 1$ and $k\{K_2\alpha\} < k\frac{|I_2'|}{D} < |I_2'| < 1$ for $k = 1, 2, \ldots, D$. This together with (13) implies $\{k(1 - \{K_1\alpha\})\} = k(1 - \{K_1\alpha\}) = 1 - \{kK_1\alpha\} < |I_1'|$, and similarly $\{k\{K_2\alpha\}\} = k\{K_2\alpha\} = \{kK_2\alpha\} < |I_2'|$. This proves (11) and (12).

Consequently,

(i) if $\{K_1\alpha\} \in I_1''$, then the sequence $\{(n+K_1)\alpha\}$, $\{(n+2K_1)\alpha\}$,…,$\{(n+DK_1)\alpha\}$ lies in the interval $I_1 \subset I$, or in other words, the arithmetic progression $\{n, n+K_1, n+2K_1, \dots, n+DK_1\}$ lies completely in A,

(ii) if $\{K_2\alpha\} \in I_2''$, then the sequence $\{(n+K_2)\alpha\}$, $\{(n+2K_2)\alpha\}$,…,$\{(n+DK_2)\alpha\}$ lies in the interval $I_2 \subset I$, or that $\{n, n+K_2, n+2K_2, \dots, n+DK_2\} \subset A$,

Since $\{n\alpha\}$, $n = 1, 2, \dots$ is dense in $[0,1]$, we can find K_1 and K_2 such that

$$\{K_1\alpha\} \in \left(1 - \frac{|I|}{2D}, 1\right) \quad \text{and} \quad \{K_2\alpha\} \in \left(0, \frac{|I|}{2D}\right).$$

In addition, for every $\{n\alpha\} \in I$ either $|I_1| \geq |I|/2$ or $|I_2| \geq |I|/2$. Thus either $\{K_1\alpha\} \in I_1''$ or $\{K_2\alpha\} \in I_2''$ and by (i) or (ii) we have either (6) or (7).

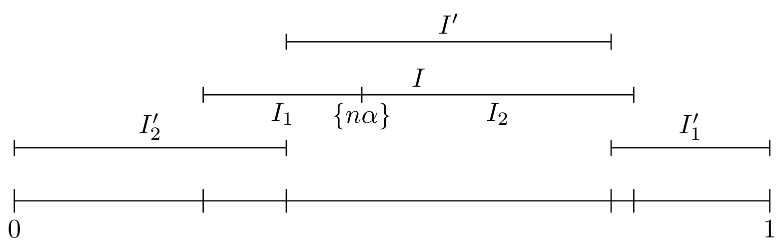

Figure 2.

Note that even if Figure 1 changes when $|I| > 1/2$, the used arguments remain valid also in this case (cf. Figure 2). $\square$

The above ideas also give the following modification of previous results.

Lemma 2. *For every $n \in A$ we have*

(j) *if $K_1 \in \mathbb{N}$ be such that $\{K_1\alpha\} \in I_1'$ and $1 - \{K_1\alpha\} < 1 - |I|$ and $D_1 = \lfloor |I_1'|/(1 - \{K_1\alpha\})\rfloor$ then $\{n, n+K_1, n+2K_1, \dots, n+D_1K_1\} \subset A$ but $n + (D_1+1)K_1 \notin A$, and similarly*

(jj) *if $\{K_2\alpha\} \in I_2'$ for some K_2 and $\{K_2\alpha\} < 1 - |I|$ then $\{n, n+K_2, n+2K_2, \dots, n+D_2K_2\} \subset A$ but $n+(D_2+1)K_2 \notin A$, where $D_2 = \lfloor |I_2'|/\{K_2\alpha\}\rfloor$.*

Proof. The definition of D_2 implies that $D_2\{K_2\alpha\} \in I_2'$ but $(D_2+1)\{K_2\alpha\} \notin I_2'$. Since $\{K_2\alpha\} < 1 - |I|$, $(D_2+1)\{K_2\alpha\} \in I'$ which by (10) implies $n + (D_2+1)K_2 \notin A$. Similarly for D_1. $\square$

Example 3. Let $\alpha = (\sqrt{5}+1)/2$ and $N = 100$ and $I = [1/2, 1]$. Here $\{\alpha\} = 0.61803\dots \in [1/2, 1]$, $I_1 = [1/2, \{\alpha\}]$, $I_2 = [\{\alpha\}, 1]$, $|I_1| = 0.11803\dots$ and $|I_2| = 0.38196\dots\dots$, i.e. $I_1' = [0.88196\dots, 1]$ and $I_2' = [0, 0.11803\dots]$. The maximum $\max_{1\le n\le N}\{n\alpha\}$ is attained for $n = 55$ and $\{55\alpha\} \doteq 0.991869\dots \in I_1$, so we can take $K_1 = 55$ and Lemma 2(j) gives that

$$D_1 = \left\lfloor \frac{|I_1|}{(1 - \max_{1\le n\le N}\{n\alpha\})} \right\rfloor = 14.$$

By Lemma 2 we have $\{1, 1+55, 1+2\cdot 55, \dots, 1+14\cdot 55\} \subset A$, but $1+15\cdot 55 \notin A$.

Similarly the minimum $\min_{1\le n\le N}\{n\alpha\}$ is attained for $n = 89$ with $\{89\alpha\} \doteq 0.005024\dots$, i.e. Lemma 2(jj) works with $K_2 = 89$ and

$$D_2 = \left\lfloor \frac{|I_2|}{\min_{1\le n\le N}\{n\alpha\}} \right\rfloor = 76.$$

Thus $\{1, 1+89, \dots, 1+76\cdot 89\} \subset A$ but $1+77\cdot 89 \notin A$. (Note also that the length of the constructed arithmetic progressions exceeds the initial segment of $N = 100$ terms.)

In Lemma 1 we proved that for fixed $n \in \mathbb{N}$, $\{n\alpha\} \in I$, the set of all $k \in \mathbb{N}$ for which $\{(n+k)\alpha\} \in I$ is given by $\{k\alpha\} \in I_1$ or $\{k\alpha\} \in I_2$. In the following lemma the $k \in \mathbb{N}$ will be fixed and we ask for all $n \in \mathbb{N}$ such that $(\{n\alpha\}, \{(n+k)\alpha\}) \in I \times I$.

Lemma 4. *Let $I \subset [0,1]$ be an interval and $k \in \mathbb{N}$. Then both numbers $\{n\alpha\}$ and $\{(n+k)\alpha\}$ lie in I if and only if the sawtooth graph of the function $y = x + \{k\alpha\} \bmod 1$ intersects the square $I \times I$ and simultaneously $\{n\alpha\}$ lies in the projection of this intersection onto the x-axis.*

Proof. Since

$$\{(n+k)\alpha\} = \begin{cases} \{n\alpha\} + \{k\alpha\}, & \text{if } \{n\alpha\} + \{k\alpha\} < 1, \\ \{n\alpha\} + \{k\alpha\} - 1, & \text{if } \{n\alpha\} + \{k\alpha\} \ge 1, \end{cases}$$

the statement $(\{n\alpha\}, \{(n+k)\alpha\}) \in I \times I$ is equivalent to the assertion of the Lemma. □

As mentioned in the introduction, the result of the previous Lemma will play an important role, therefore we introduce the following notation. Given an interval $I \subset [0,1]$ and a positive integer k, denote by X_k the intersection of the graph $\{(x, x+\{k\alpha\}) \bmod 1 : x \in [0,1]\}$ of the linear function $y = x + \{k\alpha\} \bmod 1$

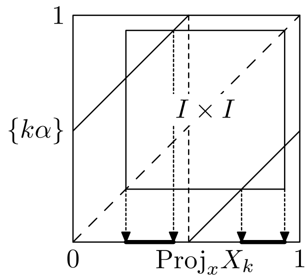

Figure 3.

with $I \times I$. The projection of X_k onto the x-axis will be denoted by $\mathrm{Proj}_x X_k$, see Figure 3.

Using this notation Lemma 4 has the form

$$(\{n\alpha\}, \{(n+k)\alpha\}) \in I \times I \Leftrightarrow \{n\alpha\} \in \mathrm{Proj}_x X_k \tag{14}$$

Note that the role of intervals I_1, I_2, I_1', I_2' in the equivalences (8) and (9) can be stated more precisely by introducing a parameter $0 < t < 1$ denoting the distance between $\{n\alpha\}$ and $\{(n+k)\alpha\}$.

Lemma 5. *For every $n, k \in \mathbb{N}$ and every $0 < t < 1$ we have*

$$0 < \{n\alpha\} - \{(n+k)\alpha\} = t \Leftrightarrow 1 - \{k\alpha\} = t, \tag{15}$$

$$0 < \{(n+k)\alpha\} - \{n\alpha\} = t \Leftrightarrow \{k\alpha\} = t \tag{16}$$

3. Lower bounds for W_M

Theorem 6. *Let $\alpha = [a_0; a_1, a_2, \dots]$ be the continued fraction expansion of a given irrational number α. Assume that $\{\alpha\} \in I$ and that it divides the interval I into two subintervals I_1 and I_2 (see Figure 1). Put*

$$M_i = q_n \left\lfloor |I_i| \, q_n \left(r_{n+1} + \frac{q_{n-1}}{q_n} \right) \right\rfloor \quad \text{for } i = 1, 2, \tag{17}$$

where $r_{n+1} = [a_{n+1}; a_{n+2}, a_{n+3} \dots]$ and $\frac{p_n}{q_n} = [a_0; a_1, a_2, \dots, a_n]$. Then

$$W_{M_i} \geq \sqrt{M_i}\sqrt{|I_i| a_{n+1}} \tag{18}$$

for odd n if $i = 1$ and for even n if $i = 2$, provided $q_{n-1} > \max(1/|I_1|, 1/|I_2|)$.

Proof. Reorder the elements of the sequence $\{1\alpha\}, \{2\alpha\}, \ldots, \{N\alpha\}$ increasingly

$$0 < \{n_1\alpha\} < \{n_2\alpha\} < \cdots < \{n_N\alpha\} < 1 \tag{19}$$

and denote

$$D_1 = \left\lfloor \frac{|I_1|}{1 - \{n_N\alpha\}} \right\rfloor \quad \text{and} \quad D_2 = \left\lfloor \frac{|I_2|}{\{n_1\alpha\}} \right\rfloor.$$

Lemma 2 shows that

each of the numbers $\{1\alpha\}, \{(1 + n_N)\alpha\}, \ldots, \{(1 + D_1 n_N)\alpha\}$ belongs to the interval I, and

that also the numbers $\{1\alpha\}, \{(1 + n_1)\alpha\}, \ldots, \{(1 + D_2 n_1)\alpha\}$ belong to the interval I.

Consequently

$$W_{M_1} \geq D_1 + 1 \quad \text{for } M_1 = 1 + D_1 n_N, \text{ and}$$
$$W_{M_2} \geq D_2 + 1 \quad \text{for } M_2 = 1 + D_2 n_1.$$

This can be simplified to the form

$$W_{M_i} \geq D_i \tag{20}$$

for $i = 1, 2$, but now with $M_1 = D_1 n_N$ and $M_2 = D_2 n_1$.

Take $N = q_n$. Using the well-known relation

$$q_n\alpha - p_n = \frac{(-1)^n}{q_n\left(r_{n+1} + \frac{q_{n-1}}{q_n}\right)}$$

and the fact that p_n/q_n is the best approximation to α of the second kind, we can derive that

$$n_1 = q_n \quad \text{for even } n, \qquad \text{and} \qquad n_N = q_n \quad \text{for odd } n.$$

Thus

$$D_i = \left\lfloor |I_i|\, q_n \left(r_{n+1} + \frac{q_{n-1}}{q_n} \right) \right\rfloor,$$

where $i = 1$ for odd n and $i = 2$ for even n and

$$M_i = D_i q_n.$$

If θ_i is the fractional part

$$\theta_i = \left\{ |I_i|\, q_n \left(r_{n+1} + \frac{q_{n-1}}{q_n} \right) \right\},$$

then

$$D_i = \sqrt{M_i}\sqrt{|I_i|\, a_{n+1}}\, C,$$

where

$$C = \sqrt{\left(\frac{r_{n+1} + \frac{q_{n-1}}{q_n}}{a_{n+1}}\right)\left(1 - \frac{\theta_i}{|I_i|\, q_n\big(r_{n+1} + \frac{q_{n-1}}{q_n}\big)}\right)}.$$

If $q_{n-1} > \max(1/|I_1|, 1/|I_2|)$, then $C > 1$ and thus C can be omitted in (20) to get (18). □

Theorem 7. *Given an irrational α and an arbitrary interval $I \subset [0,1]$, the inequality*

$$W_M \geq \frac{\max(|I|, 1 - |I|)}{2D_{\lfloor\sqrt{M}\rfloor}},$$

holds for all sufficiently large M, where $D_{\lfloor\sqrt{M}\rfloor}$ is the extremal discrepancy of the sequence $\{1\alpha\}, \{2\alpha\}, \dots, \{\lfloor\sqrt{M}\rfloor\alpha\}$.

Proof. Since the value of W_M is influenced by blocks of $+1$'s or by blocks of -1's sitting on places with indices n in an arithmetic progression, we shall proceed in two steps.

1^0. For each sufficiently large M there exists a positive integer B such that

$$\frac{|I|}{2B} > D_{\lfloor\sqrt{M}\rfloor} \geq \frac{|I|}{2(B+1)}. \tag{21}$$

The left hand side inequality of (21) shows that there exists an integer K_1, $1 \leq K_1 \leq \lfloor\sqrt{M}\rfloor$, for which $\{K_1\alpha\} \in \big(0, \frac{|I|}{2B}\big)$ and that there also exists a K_2, $1 \leq K_2 \leq \lfloor\sqrt{M}\rfloor$, with $\{K_2\alpha\} \in \big(1 - \frac{|I|}{2B}, 1\big)$. Since there exists an integer n, $1 \leq n \leq \lfloor\sqrt{M}\rfloor$, such that $\{n\alpha\} \in I$, Lemma 1 implies that each element of the finite sequence

$$\{n\alpha\}, \{(n+K)\alpha\}, \dots, \{(n+BK)\alpha\} \tag{22}$$

lies in I for a K with $1 \leq K \leq \lfloor\sqrt{M}\rfloor$. Thus

$$W_M \geq B + 1 \tag{23}$$

provided that (22) is a subsequence of

$$\{1\alpha\}, \{2\alpha\}, \dots, \{M\alpha\}.$$

This is equivalent to

$$n + BK \leq M. \tag{24}$$

Since both $n, K \le \sqrt{M}$, the inequality (24) holds if $(B+1)\sqrt{M} \le M$, i.e. if $B+1 \le \sqrt{M}$. On the other hand, (21) implies that $B < |I|/(2D_{\lfloor\sqrt{M}\rfloor})$, therefore inequality (24) holds if $\sqrt{M} - 1 > |I|/(2D_{\lfloor\sqrt{M}\rfloor})$, i.e. if

$$\frac{|I|}{2} < D_{\lfloor\sqrt{M}\rfloor}\sqrt{M} - D_{\lfloor\sqrt{M}\rfloor}. \tag{25}$$

Moreover, the extremal discrepancy D_N of an arbitrary sequence $t_1, t_2, \dots, t_N$ in $[0,1)$ satisfies (see [15, p. 1–43])

$$\frac{1}{N} \le D_N \le 1,$$

and the sequence $\{m\alpha\}$, $m = 1, 2, \dots$, is uniformly distributed. Consequently, (25) is true for all sufficiently large M. Then (23) implies that

$$W_M \ge \frac{|I|}{2D_{\lfloor\sqrt{M}\rfloor}}$$

holds for all sufficiently large M.

2^0. Now take an n with $\{n\alpha\} \in I$. For sufficiently large M there exists $n + k \le \sqrt{M}$ such that $\{(n+k)\alpha\} \notin I$, what, by Lemma 1, is equivalent to $\{k\alpha\} \in I'$, where interval I' is determined as it is shown in Figure 1. If we now take interval I' instead of I in the previous step 1^0 we get

$$W_M \ge \frac{|I'|}{2D_{\lfloor\sqrt{M}\rfloor}}.$$

Since $|I'| = 1 - |I|$, the Theorem is proved. □

Theorem 8. *If α is an irrational algebraic number and $I \subset [0,1]$ is an arbitrary interval, then*

$$W_M = M|1 - 2|I|\,| + O\big(M^{\frac{1}{2}+\varepsilon}\big). \tag{26}$$

for every $\varepsilon > 0$.

Proof. We obtain directly from the definition of $x_n = \chi_I(\{n\alpha\})$ that

$$\begin{aligned}\sum_{k=1}^{N} x_{n+kK} &= \#\{k \le N; \{(n+kK)\alpha\} \in I\} \\ &\quad - \#\{k \le N; \{(n+kK)\alpha\} \in [0,1] - I\} \\ &= N(|I| - (1 - |I|)) + O(ND_N((n+kK)\alpha)),\end{aligned}$$

where $D_N((n+kK)\alpha))$ is the discrepancy of the sequence

$$\{(n+K)\alpha)\}, \{(n+2K)\alpha)\}, \dots, \{(n+NK)\alpha)\}.$$

By Lemma 1 of [4] we have

$$D_N((n+kK)\alpha)) = D_N(kK\alpha) \le KD_N(k\alpha),$$

where $D_N(k\alpha)$ is the discrepancy of the sequence $\{1\alpha\}, \{2\alpha\}, \dots, \{N\alpha\}$. Thus

$$\left|\sum_{k=1}^{N} x_{n+kK}\right| = N|1-2|I|\,| + O(KND_N(k\alpha)). \tag{27}$$

According to the definition, W_M is the maximum of the left hand side of (27) over all possible $n, K, D \in \mathbb{N}$ such that $1 \le n+K < n+NK \le M$.

Theorem 3.2 in [3, p. 123] shows that

$$ND_N(k\alpha) = O(N^{1-\frac{1}{\eta}+\varepsilon}) \tag{28}$$

if α is of the finite type η (see Definition 3.4 in [3, p. 121]). Since an algebraic irrational α is of type $\eta = 1$ (cf. Example 3.1 [3, p. 124]), relation (28) reduces to the form

$$ND_N(k\alpha) = O(N^{\varepsilon}).$$

We distinguish two cases:

1^0. If $K > \sqrt{M}$ then

$$N \le \frac{M-n}{K} \le \sqrt{M},$$

and $\left|\sum_{k=1}^{N} x_{n+kK}\right| \le \sqrt{M}$ in this case.

2^0. Let $K \le \sqrt{M}$. Since $1 \le n+K < n+NK \le M$, the error term in (27) reduces to $O(M^{\frac{1}{2}+\varepsilon})$. Similarly, the maximal possible value of N (in $N|1-2|I|\,|$) such that $n+NK \le M$ is $N = M$ if $n = 1$ and $K = 1$. This finishes the proof of (26). □

4. Differences of consecutive terms of $A(\alpha, I)$ with arbitrary $I \subset [0,1]$

Let α be an irrational number, the interval $I \subset [0,1]$ be fixed but arbitrary, and let

$$A = A(\alpha, I) = \{n \in \mathbb{N} : \{n\alpha\} \in I\} = \{a_1 < a_2 < \dots\} \quad \text{and}$$

$$\Delta = \{a_{n+1} - a_n : n \in \mathbb{N}\}.$$

This section will be devoted to the study of the set of differences Δ. Slater [11] proved the so-called three gaps problem saying not only that Δ is finite, but in addition it has at most three distinct elements. More precisely:

Theorem 9 (Slater [11], [12])**.** *Given an interval I of the form $I = (0, t)$, $t \le 1/2$, define a and b as the least positive integers such that $\{a\alpha\} \in (0, t)$ and $\{b\alpha\} \in (1-t, 1)$. Let $\{n\alpha\} \in (0, t)$ and let k be minimal with $\{(n+k)\alpha\} \in (0, t)$. Then*

$$k = \begin{cases} a, & \text{if } 0 \le \{n\alpha\} < t - \{a\alpha\}, \\ a+b, & \text{if } t - \{a\alpha\} \le \{n\alpha\} < 1 - \{b\alpha\}, \\ b, & \text{if } 1 - \{b\alpha\} \le \{n\alpha\} < t. \end{cases} \tag{29}$$

Moreover a and b are relatively prime.

The next result gives another way how to compute the elements of Δ for an arbitrary interval $I \subset (0, 1)$. As a consequence we obtain an interesting fact that Δ depends only on t, that is it has the property to be shared by all intervals $I \subset (0, 1)$ of a given length $|I| = t$, $0 < t < 1$. Consequently also the integers a, b of Theorem 9 depend only on t.[4] Due to the importance of Slater's Theorem we shall now state our next theorem bringing Slater's result in a new connection with our geometric approach, despite the fact that the proofs will partially depend on some parts of Proposition 13 and Theorem 19 which are proved later, the proofs of corresponding used parts do not depend on the results of the following theorem, however.

Theorem 10. *Let I be an arbitrary interval of $(0, 1)$. Let N be such an integer that the sequence $\{1\alpha\}, \{2\alpha\}, \dots, \{N\alpha\}$ meets both intervals $(0, |I|/2)$, and $(1 - |I|/2, 1)$. Let the function $k(x)$ be defined for $x \in [0, |I|]$ as*[5]

$$\begin{aligned} k(x) = \min \big\{ &\min\{i \le N : \{i\alpha\} \in (0, x)\}, \\ &\min\{i \le N : \{i\alpha\} \in (1 - |I| + x, 1)\} \big\}. \end{aligned} \tag{30}$$

Then we have:

(i) *the set Δ coincides with the set $\{k(x) : x \in [0, |I|]\}$,*

[4]In the report [1] of the meeting of the Mathematical Society held on December 1, 1950 we can read that K. Florek independently claims that the minimal k mentioned in Theorem 9 can be determined in the same way for an interval $I \subset [0, 1]$ of an arbitrary length t, $t \le 1/2$. Since Florek gives no proofs, we supply here a proof for this general situation using our approach.

[5]Here we use the convention that $\min \emptyset = \infty$.

(ii) *if* $\{n\alpha\} \in I$ *and* I *is arbitrary, say* $I = (u, v)$*, then the minimal* k *with* $\{(n+k)\alpha\} \in I$ *equals* $k(\{n\alpha\} - u)$*,*

(iii) *the function* $k(x)$ *and the set* Δ *depend only on the length* $|I|$ *of the interval* I *but not on the position of* I *within* $(0, 1)$*,*

(iv) *if* $I = (0, t)$ *with* $t \leq 1/2$ *then the value* $k(\{n\alpha\})$ *coincides with* k *given by* (29)*,*

(v) *we have* $t - \{a\alpha\} \leq 1 - \{b\alpha\}$ *for every* $t \leq 1/2$ *and* a, b *defined in Theorem 9.*

Proof. Note that the choice of the number N satisfying the assumptions of the Theorem does not influence the value of function k due to the fact that we take the minima over the given sets.

(i) and (ii): Let $\{n\alpha\} \in I$. Then the minimal k such that $\{(n+k)\alpha\} \in I$ is given by

$$k = \min\left\{\min\{i \leq N : \{i\alpha\} \in I_1'\}, \min\{i \leq N : \{i\alpha\} \in I_2'\}\right\}, \tag{31}$$

where the intervals I_1', I_2' are defined in Figure 1.

To prove this, let K_1, K_2 be such that $\{K_1\alpha\} \in (1 - |I|/2, 1)$, and $\{K_2\alpha\} \in (0, |I|/2)$. Since $(0, |I|/2) \subset I_2'$ or $(1 - |I|/2, 1) \subset I_1'$, then either $\{(n+K_2)\alpha\} \in I$ or $\{(n+K_1)\alpha\} \in I$, that is $k \leq \max\{K_1, K_2\} \leq N$. On the other hand, $\{(n+k)\alpha\} \in I$ implies that either $\{k\alpha\} \in I_1'$ or $\{k\alpha\} \in I_2'$, what shows that k belongs to the sets over which the minimum is computed. Finally note that there follows from the density of $\{n\alpha\}$ that $I_2' = (0, x)$ can be arbitrarily close to $(0, |I|)$ (see also Figure 1).

(iii): This follows from the definition (30) of function $k(x)$ which depends only on the length of I and not on its position within $(0, 1)$. The invariancy of Δ with respect to a translation of I follows from (i).

(iv): This is actually Slater's result expressed in terms of our function k. In the rest of the proof we prove (iv) using our geometric approach developed in Lemmas 1 and 4, thereby giving a new proof of Slater's Theorem.

Suppose that $I = (0, t)$ and $t \leq 1/2$. Let N be the first positive integer such that the initial segment $\{1\alpha\}, \{2\alpha\}, \ldots \{N\alpha\}$ meets both intervals $(0, t)$ and $(1 - t, 1)$ and that a and b are the least positive integers with $\{a\alpha\} < t$ and $\{b\alpha\} > 1 - t$. Clearly, either $a = N$ or $b = N$. In what follows we shall suppose that

$$b = N. \tag{32}$$

for the sake of simplicity. Reorder the numbers $\{1\alpha\}, \{2\alpha\}, \ldots \{N\alpha\}$ with respect to their increasing magnitude to get, say $0 < \{n_1\alpha\} < \{n_2\alpha\} < \cdots < \{n_N\alpha\} < 1$. Clearly $n_N = b = N$. What is less obvious is that $n_1 = a$. This follows from the following first observation:

1^0. $\{a\alpha\} < \{(a+i)\alpha\}$ for $i = 1, 2, \dots, b-1$.

Suppose on the contrary, that $\{a\alpha\} > \{(a+i)\alpha\}$ for some such i, cf. Figure 4. Then Lemma 1 and (8) imply

$$\{(a+i)\alpha\} \in I_1 \Leftrightarrow \{i\alpha\} \in I_1' \subset (1-t, 1).$$

Since if $i < b$, then $\{i\alpha\} \notin (1-t, 1)$, and we have $i \geq b$.

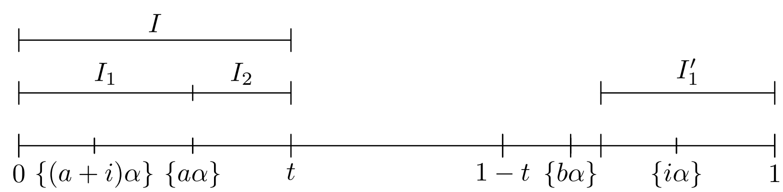

Figure 4.

We similarly have:

2^0. $\{(b+i)\alpha\} < \{b\alpha\}$ for $i = 1, 2, \dots, a-1$.

Again, if $\{(b+i)\alpha\} > \{b\alpha\}$, then (cf. Figure 5) Lemma 1 and relation (9) yield

$$\{(b+i)\alpha\} \in I_2 \Leftrightarrow \{i\alpha\} \in I_2' \subset (0, t),$$

which in turn implies that $i \geq a$ for $a = \min\{i < N : \{i\alpha\} \in (0, t)\}$.

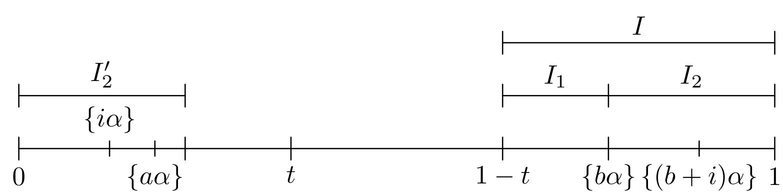

Figure 5.

Further we prove:

3^0. The numbers a and b are minimal positive integers such that if $\{n\alpha\} \in I$ and k is a minimal positive integer with $\{(n+k)\alpha\} \in I$ then either $k = a$ or $k = b$.

Suppose that $\{n\alpha\} \in I$ and $\{(n+k)\alpha\} \in I$. Then Lemma 4 implies that $\{n\alpha\} \in \operatorname{Proj}_x X_k$ (i.e. $\operatorname{Proj}_x X_k \neq \emptyset$), and that the point $(\{n\alpha\}, \{\{n\alpha\} + \{k\alpha\}\}) = (\{n\alpha\}, \{(n+k)\alpha\})$ lies on the graph of $y = x + \{k\alpha\} \bmod 1$, see Figure 3. Since

$I = (0,t)$, the projection $\mathrm{Proj}_x X_k$ is non-empty if and only if the graph of $y = x + \{k\alpha\} \bmod 1$ meets either $(0,t) \times (0,t)$ or $(0,t) \times (1-t,1)$, that is if $\{k\alpha\} \in (0,t)$ or $\{k\alpha\} \in (1-t,1)$ (see also Theorem 19). Geometrically seen, the sets X_{a+i}, $i \geq 0$, are in the "shadow" of X_a when projected from interval I upwards. This geometrical argument shows that if for $n \in A$ we have $n+a+i \in A$, then necessarily also $n+a \in A$. Along the same arguments we also can prove the implication, if $n \in A$ and $n+b+i \in A$, then $n+b \in A$, This proves the minimality of a and b in the above described way.

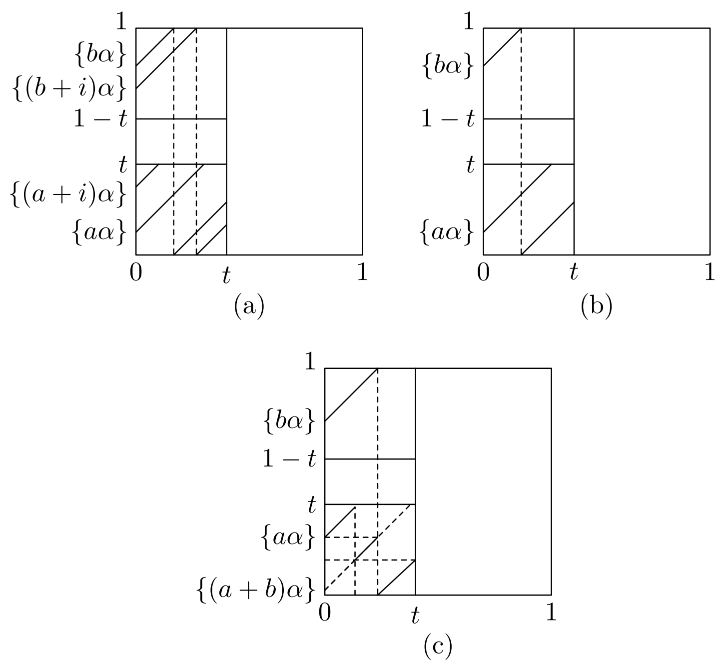

Figure 6.

The above proof shows the importance of the coverage of interval I by the union $\mathrm{Proj}_x X_a \cup \mathrm{Proj}_x X_b$. As a byproduct we obtained that if $\mathrm{Proj}_x X_a \cup \mathrm{Proj}_x X_b = I$, then the minimal k of Slater's Theorem can attain only two values a and b. In the next step we shall analyse the cases when $\mathrm{Proj}_x X_a \cup \mathrm{Proj}_x X_b$ covers $I = (0,t)$ or does not.

4^0. A condition under which $a+b$ joins the possible cases $\{a,b\}$.

Consider two cases of the relationship between $\mathrm{Proj}_x X_a \cup \mathrm{Proj}_x X_b$ and I:

4^0_1. $1 - \{b\alpha\} \le t - \{a\alpha\}$.

Then, cf. Figure 6(b), $\mathrm{Proj}_x X_a \cup \mathrm{Proj}_x X_b = (0, t)$. Consequently, for every $n \in A$ we have either $n + a \in A$ or $n + b \in A$.

4^0_2. $t - \{a\alpha\} < 1 - \{b\alpha\}$.

In other words, $t < \{a\alpha\} + 1 - \{b\alpha\}$. Then, see Figure 6(c), the segment $I \setminus (\mathrm{Proj}_x X_a \cup \mathrm{Proj}_x X_b)$ is the projection of the segment joining the points $(t - \{\alpha a\}, t - (1 - \{\alpha b\}))$ and $(1 - \{b\alpha\}, \{a\alpha\})$, that is $\mathrm{Proj}_x X_a \cup \mathrm{Proj}_x X_{a+b} \cup \mathrm{Proj}_x X_b = (0, t)$. Thus for every $n \in A$ we have either $n + a \in A$ or $n + b \in A$ or $n + a + b \in A$ in accordance with (29).

We know (cf. the previous cases 1^0 and 2^0) that the values a and b are the least possible. To prove that also $a + b$ is the least possible, we shall employ the result of Theorem 19 below. As already mentioned, the proof of this theorem does not depend on the proof of the results just proved, however. In one direction, the arguments of part 3^0 show that $\mathrm{Proj}_x X_{a+b-i} \subset \mathrm{Proj}_x X_a$ if $\{(a+b-i)\alpha\} \in (0, t)$, and similarly $\mathrm{Proj}_x X_{a+b-i} \subset \mathrm{Proj}_x X_b$ if $\{(a + b - i)\alpha\} \in (1 - t, 1)$. On the other hand, Theorem 19 shows that the case $\{(a+b-i)\alpha\} \in (t, 1-t)$ cannot be realized for $n \in A$ such that $n + a + b - i \in A$.

5^0. The constants a and b depend on the length $|I| = t$ of interval I but not on the position of I.[6]

To see this note that the lengths of the intersections of the translated square $[u, t + u) \times [u, t + u)$ with the line segments of the graph of function $y = x + \{q\alpha\} \bmod 1$, q a positive integer, are preserved, and consequently does not depend on u.

This proves the first part of the statement of Theorem 10. The second one claims that

6^0. The integers a, b are relatively prime.

Slater [12, (8), (9)] proved this through the equality $b\{a\alpha\} + a(1 - \{b\alpha\}) = 1$, which implies $a(1 + \lfloor b\alpha \rfloor) - b\lfloor a\alpha \rfloor = 1$, and the coprimness of a, b follows immediately. In what follows we use another argument to prove this.

Suppose on the contrary, that $\gcd(a, b) = d > 1$. It follows from 3^0 and 4^0 that every $n \in A(\alpha, I)$ with $I \subset [0, 1]$, $|I| = t$, has the form $n = n_1 + xa + yb$, where n_1 is the minimal element of $A(\alpha, I)$. By 5^0 the sets $A = A(\alpha, (0, t))$ and $B = A(\alpha, (1 - t, 1))$ corresponding to the translated intervals $(0, t)$ and $(1 - t, 1)$ share the same a and b. Since the first element of A is a and the first element of B is b, any element n of A or B is divisible by d. Consequently, if $d \nmid n$ then $\{n\alpha\} \in (t, 1 - t)$, and moreover $t < 1/2$ under the assumption $d > 1$. Let $t_1 = \sup t$, where the supremum is taken over those $t < 1/2$ for which every

[6] What follows is actually the second proof of this fact. The first one is given (iii).

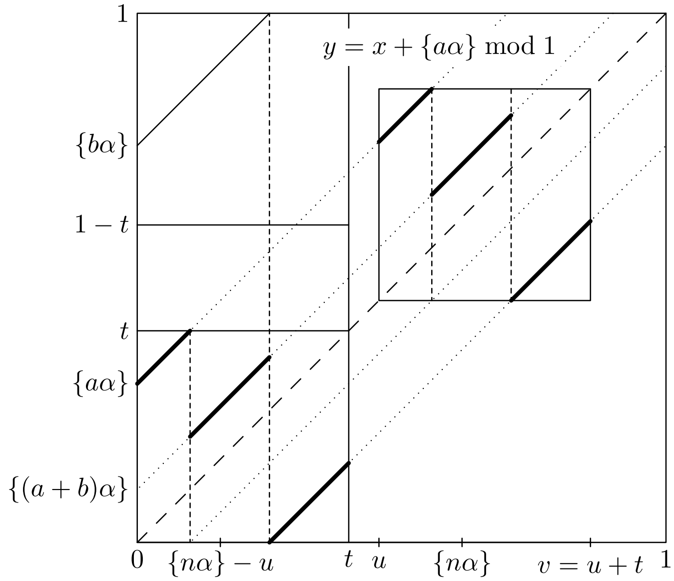

Figure 7.

element in $A(\alpha, (0, t))$ is divisible by d. Now, choose integers n and $n+k$ satisfying the conditions (see Figure 8):

(i) $d \nmid n$, e.g. $\{n\alpha\} \in [t_1, 1-t)$;

(ii) $\{(n+k)\alpha\} \in (0, t_1)$, e.g. $d|(n+k)$;

(iii) $0 < \{n\alpha\} - \{(n+k)\alpha\} < t$.

Relation (15) of Lemma 5 shows that $1 - \{k\alpha\} = \{n\alpha\} - \{(n+k)\alpha\}$, therefore $1 - \{k\alpha\} < t$. Consequently $k \in B$, and thus $d|k$, a contradiction, which proves 6^0.

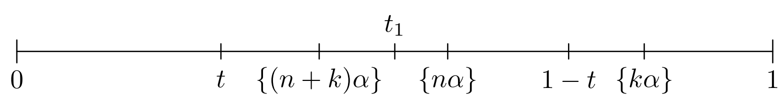

Figure 8.

Slater [12, p. 1118] writes that the difference $a + b$ appears if $t < 1 - \{b\alpha\} + \{a\alpha\}$. Using 4_1^0 and 4_2^0 we can conclude from (v) that this characterizes the appearance of this difference:

(v) This result actually says that the strict inequality $1 - \{b\alpha\} < t - \{a\alpha\}$ is not possible in the case 4_1^0.

Indeed,

(a) If $a=1$ or $b=1$, then the next Proposition 13, (i) and (ii), shows that $t-\{a\alpha\}\le 1-\{b\alpha\}$.

(b) Let $1-\{b\alpha\}<t-\{a\alpha\}$. 4_1^0 implies $\Delta=\{a,b\}$ and $a>1$ together with $b>1$ by (a). Let $a<b$. Suppose that there exists a $\Delta t>0$ such that intervals $(0,t)$ a $(0,t+\Delta t)$ have the same constants a, b. Let $A(\alpha,(0,t))=(a_1<a_2<\dots)$ and $A(\alpha,(t,t+\Delta t))=(c_1<c_2<\dots)$. Then for every c_j there exists an a_i with $a_i<c_j<a_{i+1}$. Then $a_{i+1}-c_j=a$ and also $c_j-a_i=a$, i.e. $a_{i+1}-a_i=b=2a$ what contradicts the fact that $\gcd(a,b)=1$. If $1-t=\{(b-1)\alpha\}$ then such Δt does not exist, and we can use a negative Δt in a similar way. □

Remark 11. Since $d\cdot A(d\alpha,I)\subset A(\alpha,I)$ for every positive integer d, the set $A(\alpha,I)$ has the following interesting divisibility property: For every positive integer d there exist infinitely many $a_i\in A(\alpha,I)$ such that d divides a_i.

Example 12. Let $\alpha=(\sqrt{5}+1)/2$ and $I=(1/2,1)$. The N in Theorem 10 is $N=3$, where $\{\alpha\}=0.61\dots$, $\{2\alpha\}=0.23\dots$ and $\{3\alpha\}=0.85\dots$. Then for $x=0.15$ the minimum $k(x)=\min\{\min\emptyset,\min\{3\}\}=3$, for $x=0.23$ it is $k(x)=\min\{\min\{2\},\min\{3\}\}=2$ and for $x=0.5$ it is $k(x)=\min\{\min\{2\},\min\{1,3\}\}=1$. Thus the shortest distances between $a_i,a_j\in A$ are 1, 2, or 3 only, and each of them is realized infinitely many times. This coincides with the statement of Theorem 9 where $t=1/2$, $a=2$, $b=1$, $a+b=3$, $t-\{2\alpha\}=0.27\dots<1-\{\alpha\}=0.38\dots$, and in (29) (after shifting by 0.5) we have

$$k=\begin{cases}2, & \text{if } 0.5<\{n\alpha\}<0.77\dots,\\ 3, & \text{if } 0.77\dots<\{n\alpha\}<0.88\dots,\\ 1, & \text{if } 0.88\dots<\{n\alpha\}<1.\end{cases}$$

Moreover, $b\{a\alpha\}+a(1-\{b\alpha\})=1\cdot 0.23\dots+2\cdot 0.38\dots=1$, which implies $2(1+\lfloor 1\alpha\rfloor)-1\lfloor 2\alpha\rfloor=1$ and really $2\cdot 2-1\cdot 3=1$.

In some cases we are able to give the differences a and b explicitly. For this purpose, to stress their dependence on the length t of the interval I we shall write $\Delta_t=\{a_{n+1}-a_n : n=1,2,\dots\}$ instead of the previously used notation Δ, where $A(\alpha,I)=\{a_1<a_2<\dots\}$.

Proposition 13. *If $t\le 1/2$, then*

(i) *If $\{\alpha\}\in(0,t)$ then $a=1$, $b=\left\lfloor\frac{1-t}{\{\alpha\}}\right\rfloor+1$ and*
$\Delta_t=\{1,b,1+b\}$ if $t<1-\{(b-1)\alpha\}$, but

$\Delta_t = \{1, b\}$, *and* $b > 2$, *if* $t = 1 - \{(b-1)\alpha\}$.

(ii) *If* $\{\alpha\} \in (1-t, 1)$ *then* $b = 1$, $a = \left\lfloor \frac{1-t}{1-\{\alpha\}} \right\rfloor + 1$ *and*

$\Delta_t = \{a, 1, a+1\}$ *if* $t < \{(a-1)\alpha\}$, *but*

$\Delta_t = \{a, 1\}$, $a > 2$, *if* $t = \{(a-1)\alpha\}$.

(iii) *If* $\{\alpha\} \in (t, 1-t)$ *then* $\min\{a, b\} \le \frac{1}{t} - 1$.

Proof. (i) If for some positive integer j we have $j\{\alpha\} < 1$, then $j\{\alpha\} = \{j\alpha\}$. This observation yields the expression for b in (i). Moreower $\{(b-1)\alpha\} \le 1-t < \{b\alpha\}$ due to minimality of b, and $\{b\alpha\} - \{\alpha\} \le \{(b-1)\alpha\}$. If $\{(b-1)\alpha\} < 1-t$, then $t - \{\alpha\} < 1 - \{b\alpha\}$, and Theorem 9 implies $\Delta_t = \{1, b, 1+b\}$. If $\{(b-1)\alpha\} = 1-t$, then $b > 2$ and $\Delta_t = \{1, b\}$.[7]

(ii) If $j(1-\{\alpha\}) < 1$ then $j(1-\{\alpha\}) = 1 - \{j\alpha\}$, what implies the expression for a given in (ii). Moreover, $\{a\alpha\} < t \le \{(a-1)\alpha\}$, due to minimality a. If $t < \{(a-1)\alpha\}$, then $t - \{a\alpha\} < \{(a-1)\alpha\} - \{a\alpha\} = 1 - \{\alpha\} = 1 - \{b\alpha\}$ and Theorem 9 implies $\Delta_t = \{a, 1, a+1\}$. If $t = \{(a-1)\alpha\}$, then $t - \{a\alpha\} = 1 - \{b\alpha\}$ what implies $a > 2$ and $\Delta_t = \{a, 1\}$.[8]

(iii) This result can be found in [12, p. 1119]. We reprove it now by our method.

As in our proof of Theorem 9 assume $a < b$. Then case 1^0 of the proof of Theorem 10 implies that

$$\{1\alpha\}, \{2\alpha\}, \dots, \{(a-1)\alpha\} \in (t, 1-t).$$

We claim that $|\{i\alpha\} - \{k\alpha\}| \ge t$ for every pair (i, k) with $i, k \in \{1, 2, \dots, a-1\}$ and $i \ne k$.

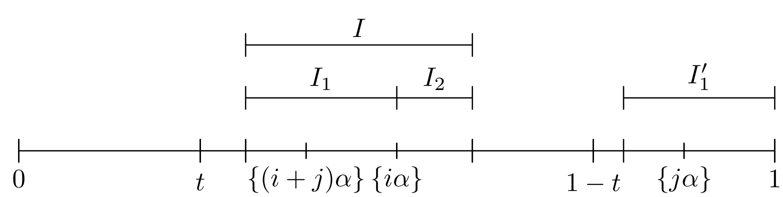

Figure 9.

To prove this, assume on the contrary that $|\{i\alpha\} - \{k\alpha\}| < t$ for some (i, k) such that $i < k = i + j$ for some j. Suppose that $\{(i+j)\alpha\}$ lies as shown on

[7]Note in connection with part (b) in proof of (v) where the stretching of the interval lead to a contradiction, in this case we have $\{(b-1)\alpha\} \in (1-t-\Delta t, 1)$ and $\Delta_{t+\Delta t} = \{1, b-1, b\}$.

[8]Again, for suitable Δt we have $\{(a-1)\alpha\} \in (0, t+\Delta t)$ and $\Delta_{t+\Delta t} = \{a-1, 1, a\}$.

Figure 9. Lemma 1 via relation (8) shows that $\{(i + j)\alpha\} \in I_1 \Leftrightarrow \{j\alpha\} \in I'_1$. Since the position of I is not relevant, we can choose I in such position that $I'_1 \subset (1-t, 1)$. Consequently $j > b$, due to minimality of b. But this is impossible because $j < a < b$.

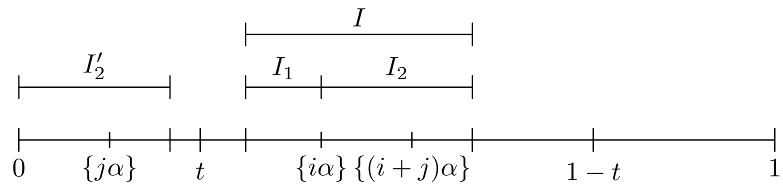

Figure 10.

If $\{(i+j)\alpha\}$ and $i\alpha$ lies as in Figure 10, then using Lemma 1 and relation (9) we have $\{(i+j)\alpha\} \in I_2 \Leftrightarrow \{j\alpha\} \in I'_2 \subset (0, t)$, and if $|\{(i+j)\}\alpha - \{i\alpha\}| < t$ then we can choose interval I in such a way that $I'_1 \subset (1 - t, 1)$ and again $j > b$, a contradiction again. $\square$

Remark 14. Parts (i) and (ii) of Proposition 13 imply that if $t \leq 1/2$ then the case $\Delta_t = \{a, b\} = \{1, 2\}$ is not possible. This can also be proved directly. Namely, if, say, $\{1\alpha\} \in (0, t)$ and $\{2\alpha\} \in (1 - t, 1)$ and if $1 - \{2\alpha\} \leq t - \{1\alpha\}$, then we get the impossible inequality $1 \leq t + \{1\alpha\}$. The case $\{1\alpha\} \in (1 - t, 1)$ can be handled in a similar way.

Note also an important fact implied by Proposition 13 (iii): if $t = 1/2$ then $\min(a, b) = 1$.

Now, we shall focus on the case $|I| \geq 1/2$. We shall determine the minimal k for which $\{(n+k)\alpha\} \in I$ provided $\{n\alpha\} \in I$. The case $|I| = 1/2$ is also covered by Slater's Theorem 9 and for the associated differences a, b we have either $a = 1$ or $b = 1$ depending on the fact whether $\{\alpha\} \in (0, 1/2)$ or $\{\alpha\} \in (1/2, 1)$ (another proof is in Remark 14 above). So we can suppose that $t > 1/2$. For the sake of simplicity of the formulation of the result and used notation we shall first analyze the case $I = (0, t)$.

Proposition 15. *Let $I = (0, t)$ with $t > 1/2$ and suppose that $\{n\alpha\} \in I$. Let k be the minimal positive integer such that $\{(n + k)\alpha\} \in I$.*
(I) *Let $\{1.\alpha\} \in (0, 1/2)$ and let b be the minimal positive integer such that $\{b\alpha\} \in (1/2, 1)$. Then for every $j = 0, 1, 2, \ldots, b - 1$ we have:*

If $t \in (1-\{(b-j)\alpha\}, 1-\{(b-j-1)\alpha\})$, *then*

$$k = \begin{cases} 1, & \text{if } 0 < \{n\alpha\} < t - \{1.\alpha\}, \\ b-j+1, & \text{if } t-\{1.\alpha\} < \{n\alpha\} < 1-\{(b-j)\alpha\}, \\ b-j, & \text{if } 1-\{(b-j)\alpha\} < \{n\alpha\} < t. \end{cases} \tag{33}$$

If $t = 1-\{(b-j-1)\alpha\}$, *then*

$$k = \begin{cases} 1, & \text{if } 0 < \{n\alpha\} < t - \{1.\alpha\} = 1-\{(b-j)\alpha\}, \\ b-j, & \text{if } 1-\{(b-j)\alpha\} < \{n\alpha\} < t. \end{cases} \tag{34}$$

(II) *Let* $\{1.\alpha\} \in (1/2, 1)$ *and let* a *be the minimal positive integer such that* $\{a\alpha\} \in (0, 1/2)$. *Then for every* $j = 0, 1, 2, \dots, a-1$ *we have:*

If $t \in (\{(a-j)\alpha\}, \{(a-j-1)\alpha\})$, *then*

$$k = \begin{cases} a-j, & \text{if } 0 < \{n\alpha\} < t - \{(a-j)\alpha\}, \\ a-j+1, & \text{if } t-\{(a-j)\alpha\} < \{n\alpha\} < 1-\{1.\alpha\}, \\ 1, & \text{if } 1-\{1.\alpha\} < \{n\alpha\} < t. \end{cases} \tag{35}$$

If $t = \{(a-j-1)\alpha\}$, *then*

$$k = \begin{cases} a-j, & \text{if } 0 < \{n\alpha\} < t - \{(a-j)\alpha\} = 1-\{1.\alpha\}, \\ 1, & \text{if } 1-\{1.\alpha\} < \{n\alpha\} < t. \end{cases} \tag{36}$$

Proof. (I) Slater's Theorem 9 implies that $\Delta_{1/2} = \{1, 1+b, b\}$. Increasing the length of interval I we add new indices into A and this consequently decreases the mutual differences, therefore for every $t > 1/2$ and $i \in \Delta_t$ we have $i \le b+1$. If we represent all graphs of functions $y = x + \{i\alpha\} \bmod 1$, $i = 1, 2, \dots, b+1$ in the unit square $(0,1)^2$, then the desired k has the following geometric representation

$$\begin{aligned} k = \min\{i \le b+1; &(\text{vertical-line } x = \{n\alpha\}) \\ &\cap (\text{graph } y = x + \{i\alpha\} \bmod 1) \cap (I \times I) \ne \emptyset\}. \end{aligned}$$

The equations (33) and (34) directly follow from Figure 11. The case (II) can be proved in a similar way. □

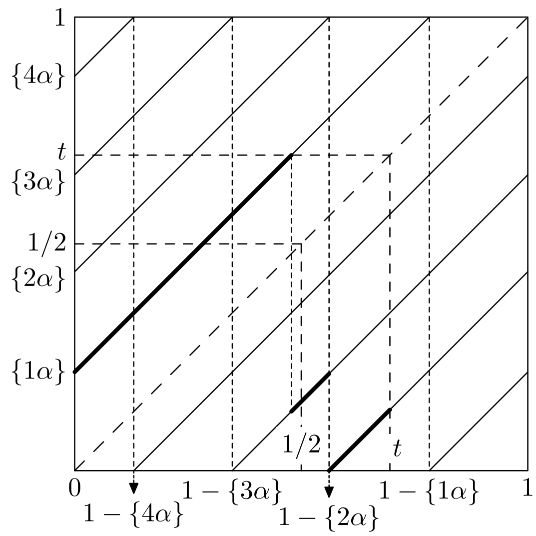

Figure 11. For $t = 1/2$ we have $a = 1, b = 3, a + b = 4$.

Proposition 16. *Let $I = (u, v) \subset (0, 1)$ be an arbitrary interval of length $|I| = t > 1/2$ and suppose that $\{n\alpha\} \in I$. Then the minimal positive integer k such that $\{(n + k)\alpha\} \in I$ can be determined by relations* (33)–(35) *with $\{n\alpha\}$ replaced by $\{n\alpha\} - u$.*

Proof. We can reduced this general case to the one considered in the previous theorem using the translation applied in the proof of Theorem 10, 5^0, which shifts $I \times I$ onto the square $(0, t) \times (0, t)$, see also Figure 7. The proof is then finished using the fact that the lengths of intersections of the line segments of graphs of functions $y = x + \{qx\} \bmod 1$ with $I \times I$ and $(0, t) \times (0, t)$ remain unchanged. $\square$

Remark 17. From Proposition 15 we see that Slater's Theorem 9 also holds in case $|I| > 1/2$ if the constants a, b are selected in the following way:
(I) If $\{\alpha\} \in (0, 1/2)$ and $\overline{b}$ is the least positive integer such that $\{\overline{b}\alpha\} \in (1/2, 1)$, then $a = 1$, and b is the least number belonging to $\{1, 2, \ldots, \overline{b} - 1\}$ such that $1 - \{b\alpha\} < t$.
(II) If $\{\alpha\} \in (1/2, 1)$ and $\overline{a}$ is again the least positive integer such that $\{\overline{a}\alpha\} \in (0, 1/2)$, then $b = 1$, and a is the least number belonging to $\{1, 2, \ldots, \overline{a} - 1\}$ such that $\{a\alpha\} < t$.

Now we shall apply our proof of Slater's Theorem to sets $C \subset \mathbb{N}$ having the property that for every $n \in A$ there exists a $c \in C$ such that $n + c \in A$, where $A = A(\alpha, (0, t))$.

Proposition 18. *For all $t \leq \frac{1}{2}$ we have:*

(i) *for every $a_{i_1}, a_{i_2}, \dots, a_{i_k}$ in A, there exists $n \in A$ such that every $n + a_{i_1}, n + a_{i_2}, \dots, n + a_{i_k}$ does not belong to A;*

(ii) *if for a finite $C \subset \mathbb{N}$ we have that for every $n \in A$ there exists $c \in C$ such that $n + c \in A$ then C must contain elements from both A and B, where $B = A(\alpha, (1 - t, 1))$;*

(iii) *for arbitrary $a \in A$, $b \in B$ we have that for every $n \in A$ either $n + a \in A$ or $n + b \in A$ or $n + a + b \in A$. Moreover:*

(iii_1) *if $t < \{a\alpha\} + 1 - \{b\alpha\}$ then all three cases of* (iii) *can occur.*

(iii_2) *If $t > \{a\alpha\} + 1 - \{b\alpha\}$ then there suffices two from the possibilities given in* (iii)*, that is, for every $n \in A$ we have either $n + a \in A$ or $n + b \in A$.*

Proof. (i) Figure 12(a) shows that $\{a_{i_2}\alpha\} < \{a_{i_1}\alpha\} \Rightarrow \operatorname{Proj}_x X_{a_{i_1}} \subset \operatorname{Proj}_x X_{a_{i_2}}$ and that

(ii) $\operatorname{Proj}_x X_{a_{i_s}} \subsetneqq (0, t)$ for every $a_{i_s} \in A$.

(iii) There follows from Figure 12(b) that in Slater's Theorem we have $\operatorname{Proj}_x X_a \cup \operatorname{Proj}_x X_{a+b} \cup \operatorname{Proj}_x X_b = (0, t)$ not only for minimal a, b, but for arbitrary $a \in A$ and $b \in B$, because

(iii_1) $\operatorname{Proj}_x X_{a+b} \neq \emptyset \Leftrightarrow t - \{a\alpha\} < 1 - \{b\alpha\}$ (see Figure 12(b)). For $x \in (t - \{a\alpha\}, 1 - \{b\alpha\})$ we have

$$x + \{(a + b)\alpha\} \bmod 1 = x + \{a\alpha\} + \{b\alpha\} - 1$$

and therefore the straight line $y = x + \{(a + b)\alpha\} \bmod 1$ joins the points $(1 - \{b\alpha\}, \{a\alpha\})$ and $(t - \{a\alpha\}, t - (1 - \{b\alpha\}))$. Thus

$$\operatorname{Proj}_x X_{a+b} \supset (t - \{a\alpha\}, 1 - \{b\alpha\})$$

for arbitrary $a \in A$ and $b \in B$ if $t < \{a\alpha\} + 1 - \{b\alpha\}$.

(iii_2) Figure 12(c) shows that $\operatorname{Proj}_x X_a \cap \operatorname{Proj}_x X_b \neq \emptyset \Leftrightarrow t - \{a\alpha\} > 1 - \{b\alpha\}$. Thus

$$\operatorname{Proj}_x X_a \cup \operatorname{Proj}_x X_b = (0, t)$$

for arbitrary $a \in A$ and $b \in B$ if $t > \{a\alpha\} + 1 - \{b\alpha\}$. □

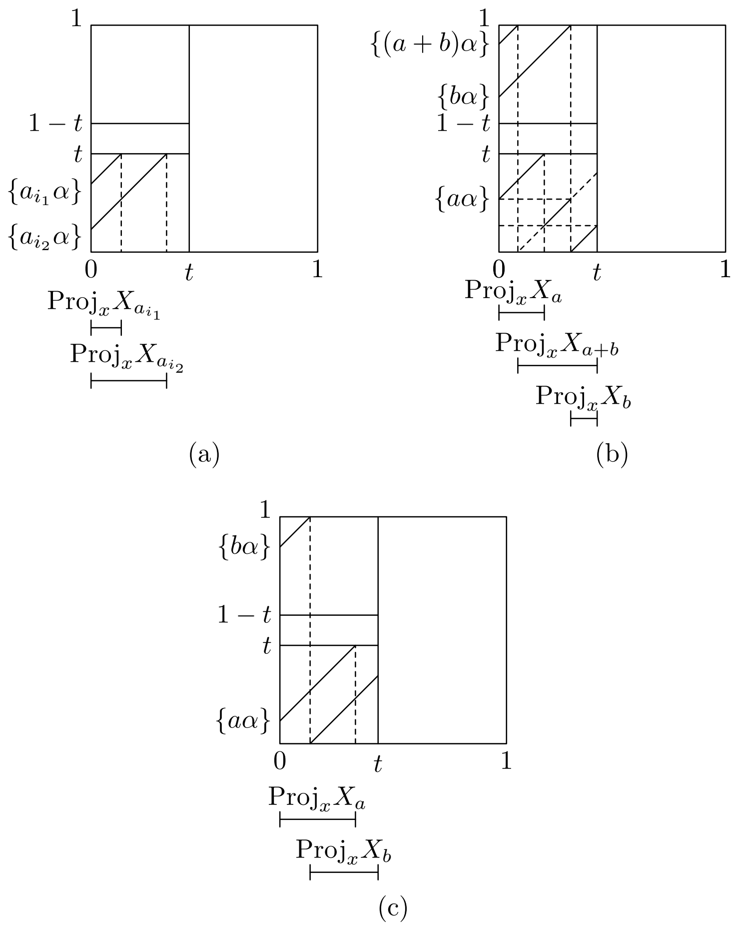

Figure 12.

Also note, that if $U = \lfloor a\alpha \rfloor$, $V = 1 + \lfloor b\alpha \rfloor$, $u = \{a\alpha\}$ and $v = 1 - \{b\alpha\}$, then Surányi [16] proved that for every irrational $\alpha' \in (U/u, V/v)$ the ordering of values

$$\{1\alpha'\}, \{2\alpha'\}, \ldots. \{N\alpha'\}$$

remains the same as in (19). This implies that for every such α' the numbers a, b from Slater's Theorem 9 are the same, and consequently the same is also the set Δ.

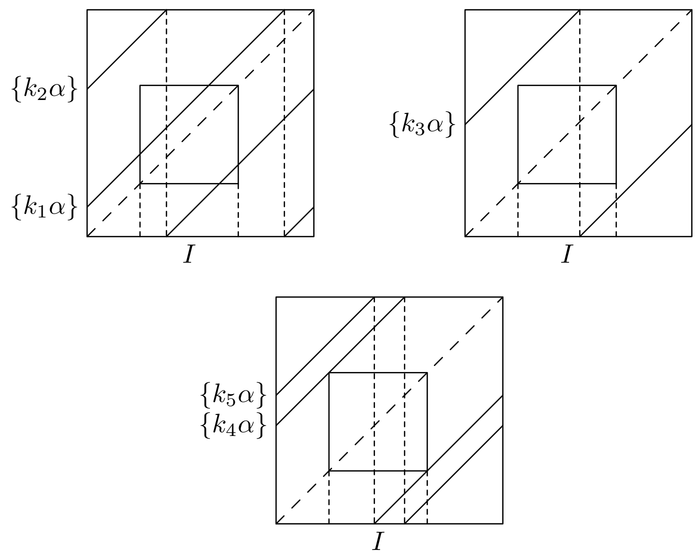

Figure 13.

Our earlier Theorem 10 can be amended in the following way:

Theorem 19. *Suppose that* $|I| \le 1/2$ *and denote* $B = \{k \in \mathbb{N} : \{k\alpha\} \in (|I|, 1-|I|)\}$. *Then*

(a) *If* $k \in B$ *then the equation* $a_j - a_i = k$ *is not solvable in* $a_i, a_j \in A$;

(b) *If* $k \notin B$ *then the equation* $a_j - a_i = k$ *has infinitely many solutions in* $a_i, a_j \in A$ *with the exception, when* $\{k\alpha\} = |I|$ *or if* $1 - \{k\alpha\} = |I|$ *and* I *is closed, in which case it possesses at most one solution.*

Proof. There follows directly from Lemma 4 that if k is fixed, then the points of the form $(\{n\alpha\}, \{(n+k)\alpha\})$ lie on the graph of the lines $y = x + \{k\alpha\} - 1$ or $y = x + \{k\alpha\}$. Therefore given an $n \in A$ we have $n + k \in A$ if and only if $(\{n\alpha\}, \{(n+k)\alpha\}) \in I \times I$ (cf. Figure 13). Moreover, the distance between the two line segments of the graph is $\sqrt{2}/2$. Consequently, if $|I| < 1/2$ the interval $I \times I$ can meet at most one of the branches.

To prove the Theorem consider the following cases:

1° Let k be in position of k_1 or k_2 of Figure 13, i.e. a segment of the graph of $y = (x + \{k_i\alpha\}) \bmod 1$, $i = 1, 2$, intersects $I \times I$. Since the points of the form $(\{n\alpha\}, \{(n+k_i)\alpha\})$, $n = 1, 2, \dots$, $i = 1, 2$, are dense in the graph, there are infinitely many of them in the intersection of $y = (x + \{k_i\alpha\}) \bmod 1$, $i = 1, 2$, with $I \times I$, i.e. $(\{n\alpha\}, \{(n+k_i)\alpha\}) \in I \times I$, $i = 1, 2$, for infinitely many n. Therefore the equation $a_j - a_i = k_1$ (and similarly the equation $a_j - a_i = k_2$) has infinitely many solutions in a_j and a_i from A.

2° If $k = k_3$ as in Figure 13 then all points of the form $(\{n\alpha\}, \{(n + k_3)\alpha\})$, $n = 1, 2, \dots$, do not lie inside or in $I \times I$, i.e.

$$\{n\alpha\} \in I \Rightarrow \{(n + k_3\alpha) \notin I\}$$

for every $n = 1, 2, \dots$. Thus the equation $a_j - a_i = k_3$ is not solvable in $a_i, a_j \in A$.

3° If $|I| < 1/2$, then for all values of k with $\{k\alpha\}$ between $\{k_4\alpha\} = |I|$ and $\{k_5\alpha\} = 1 - |I|$ the graph of $y = x + \{k_i\alpha\} \bmod 1$, $i = 4, 5$, does not intersect the interval $I \times I$, and this interval is maximal for the k's. This yields (a).

4° If $|I| = 1/2$ and I is closed then the equation $a_j - a_i = k$ has a unique solution if and only if $\{k\alpha\} = |I|$ or $1 - \{k\alpha\} = |I|$. This together with 1° implies (b). □

Example 20. Take $I = (0, 1/3)$ and $\alpha = (1 + \sqrt{5})/2$. Then the initial segment of set A is $\{2, 5, 7, 10, 13, 15, 18, 23, 26, 28, 31, 34, 36, 39, 44, 47, 49, 52, 57, 60, 62, 65, 68, 70, 73, 78, 81, 83, 86, 89, 91, 94, 96, 99, \dots\}$ and the initial segment of the set B for whose elements k_s the equation $a_j - a_i = k_s$ has no solution in A is $\{1, 4, 9, 12, 14, 17, 20, 22, 25, 30, 33, 35, 38, 41, 43, 46, 48, 51, 54, 56, 59, 64, 67, 69, 72, 75, 77, 80, 85, 88, 90, 93, 98, \dots\}$. Note that for $I = (0, 1/3)$ and any irrational α the sets A and B from Theorem 19 have the form $A = A(\alpha, I) = \{a_1 < a_2 < \dots\}$ and $B = A(\alpha, I') = \{k_1 < k_2 < \dots\}$ where $I' = [1/3, 2/3]$. By Theorem 10 these sets have the same set of differences, i.e. the same a, b, and $a + b$. Thus for any $a_j, a_i \in A$ and arbitrary $k_s \in B$ we have

$$a_j - a_i = xa + yb + z(a + b), \qquad k_s = k_1 + x'a + y'b + z'(a + b),$$

where a, y, z, x', y', z' are non-negative integers. But by Theorem 19 we have $a_j - a_i \neq k_s$ for all positive integers i, j, and s. Moreover, the set B for $A(\alpha, I')$ is the same as B for $B = A(\alpha, I')$. Thus $k_j - k_i \neq k_s$ for all i, j, and s. The set $A(\alpha, I')$ is also the B set of the set $A(\alpha, I'')$, where $I'' = [2/3, 1]$.

5. Subsequences in $A(\alpha, I)$

Since the sequence $\{(n + kK)\alpha\}$, $k = 1, 2, \dots$, is uniformly distributed in $[0, 1]$ for irrational α, the set A does not contain an infinite arithmetic progression. But Lemma 1 implies that A contains infinite double-arithmetic progressions of the types described in the next two theorems.

Theorem 21. *Let $D \in \mathbb{N}$, and K_1, K_2 satisfy conditions* (4) *and* (5), *respectively. Then for every given $n \in A$ the set A contains an infinite double arithmetic progression of the form $n, n+K_{i_1}, n+2K_{i_1}, \dots, n+DK_{i_1}, n+DK_{i_1}+K_{i_2}, n+DK_{i_1}+2K_{i_2}, \dots, n+DK_{i_1}+DK_{i_2}, n+DK_{i_1}+DK_{i_2}+K_{i_3}, \dots$, where $i_1, i_2, i_3, \dots \in \{1,2\}$, and K_{i_s}, $K_{i_{s+1}}$ need not be different.*

Proof. Iterate the process used in Lemma 1 starting with $n + DK_{i_1} \in A$ instead of $n \in A$ in the second stage, then continue with $n+DK_{i_1}+DK_{i_2} \in A$, etc. □

The above ideas with Lemma 2 also give the following modification of the previous result in which the differences K_1, K_2 really alternate due to choice of maximal possible lengths D_1, D_2, $D_3, \dots$.

Theorem 22. *Let $\{K_1\alpha\} \in [1-|I|/2, 1]$ and let $\{K_2\alpha\} \in [0, |I|/2]$ be such that $1-\{K_1\alpha\} < 1-|I|$ and $\{K_2\alpha\} < 1-|I|$. Then for every $n \in A$ there exists a sequence D_i, $i = 1, 2, \dots$, such that A contains an infinite double arithmetic progression (i.e. a progression with two alternating differences) of the form $n, n+K_1, n+2K_1, \dots, n+D_1K_1, n+D_1K_1+K_2, n+D_1K_1+2K_2, \dots, n+D_1K_1+D_2K_2, n+D_1K_1+D_2K_2+K_1, \dots, n+D_1K_1+D_2K_2+D_3K_1, n+D_1K_1+D_2K_2+D_3K_1+K_2, \dots, n+D_1K_1+D_2K_2+D_3K_1+D_4K_2, \dots$, where D_i's are maximal in the sense that $n+D_1K_1+K_1 \notin A$, $n+D_1K_1+D_2K_2+K_2 \notin A$, $n+D_1K_1+\dots+D_{2i+1}K_1+K_1 \notin A$, $n+D_1K_1+\dots+D_{2i}K_2+K_2 \notin A, \dots$, and it can be computed by* (j), (jj) *in Lemma 2.*

6. Arithmetical properties of $A(\alpha, I)$ with $|I| = \{h\alpha\}$

In this part we shall study the sets $A(\alpha, I)$ where the length $|I|$ is equal to $\{h\alpha\}$ for some integer h. Let $A((M,N]) = \#\{i \in \mathbb{N} : a_i \in (M,N]\}$ and let $D((M,N], I) = A((M,N]) - (N-M)|I|$ denote the local discrepancy.[9] The impetus to this type of results is given by the following result by E. Hecke, A. Ostrowski and H. Kesten mentioned in the introduction:

Theorem 23. *Let α be an irrational number and $I \subset [0,1]$ be an interval and $h \in \mathbb{N}$. Then the local discrepancy $D([1,N], I)$ is bounded if and only if the length $|I| = \{h\alpha\}$ for some $h \in \mathbb{Z}$ and $|D([1,N], I)| < |h|$.*

[9] The classical local discrepancy $D([1,N], I)$ is usually defined as $D([1,N], I) = \#\{n \le N : \{n\alpha\} \in I\} - N \cdot |I|$.

Note that for $D((M,N],I) = A((M,N]) - (N-M)I$ we have $|D((M,N],I)| < 2|h|$. For irrational α and $|I| = \{h\alpha\}$ the set $A(\alpha, I)$ has the following property:

Theorem 24. *If $(M,N]$ and $(M',N']$ are two arbitrary intervals such that $N' - M' = N - M$ and $|I| = \{h\alpha\}$ then*

$$\big|\#\{i \in \mathbb{N} : a_i \in (M,N]\} - \#\{i \in \mathbb{N} : a_i \in (M',N']\}\big| \leq 2|h| - 1. \tag{37}$$

Furthermore, the following well-known Proposition 25 transforms the complexity function $p(k)$ in definition of Sturmian sequence (see Part 1 (ii)) to bounded local discrepancy.

Proposition 25 ([8])**.** *A $\{1,-1\}$-sequence x_n is Sturmian if and only if*

(i) *it is non-eventually periodic,*

(ii) *the number of 1's in any pair of finite subsegments of the same length occurring in x_n can differ by at most one.*

Since if $|I| = \{1\alpha\}$ or $|I| = \{-1\alpha\} = 1 - \{\alpha\}$ then $2|h| - 1 = 1$, Theorem 24 and Proposition 25 imply that the set A and the binary $\{-1,1\}$-sequence $x_n = \chi_I(\{n\alpha\})$ are Sturmian for arbitrary such intervals. This result was known only for special cases $I = (0, \{\alpha\})$ and $I = (0, 1 - \{\alpha\})$. A similar property can also be proved for an arbitray interval $I \subset (0,1)$.

Consider an arbitrary subsequence x_{n+kK}, $k = 0, 1, 2 \ldots$, of $x_j = \chi_I(\{j\alpha\})$, $j = 1, 2, \ldots$, satisfying conditions $x_n = x_{n+K} = 1$. Split this subsequence x_{n+kK} into blocks of 1's of lengths $D_0, D_2, \ldots$, and blocks of -1's of lengths $D_1, D_3, \ldots$, that is

$$\begin{aligned}
&x_n = x_{n+K} = x_{n+2K} = \cdots = x_{n+D_0K} = 1,\\
&x_{n+(D_0+1)K} = x_{n+(D_0+2)K} = \cdots = x_{n+(D_0+D_1)K} = -1,\\
&x_{n+(D_0+D_1+1)K} = x_{n+(D_0+D_1+2)K} = \cdots = x_{n+(D_0+D_1+D_2)K} = 1,\\
&\ldots .
\end{aligned}$$

Then using Lemma 1 and 2 we can prove the following result:

Theorem 26. *Let the number $D_0, D_1, D_2, D_3, \ldots$ be defined as above.*

(i) *If $|I| \leq 1/2$, then there exist integers $D^{(1)}$ and $D^{(2)}$ such that*

$$|D_{2i-1} - D^{(1)}| \leq 1, \quad \text{and} \quad |D_{2i} - D^{(2)}| \leq 1 \tag{38}$$

for every $i = 1, 2, \ldots$.

(ii) *If* $|I| = 1/2$ *then* $D^{(1)} = D^{(2)}$.

(iii) *If* $|I| > 1/2$ *and* $\{K\alpha\} < 1 - |I|$ *then there exist integers* $D^{(1)}$ *and* $D^{(2)}$ *such that* (38) *again holds.*

PROOF. (I) For the sake of simplicity first we suppose that $|I| \leq 1/2$.

Our hypotheses $\{n\alpha\} \in I$ and $\{(n+K)\alpha\} \in I$ imply that $\{K\alpha\} \in I_1'$ or $\{K\alpha\} \in I_2'$, where the intervals I, I_1', I_2' are as in Figure 1. Consider the two alternatives separately.

1^0. Let $\{K\alpha\} \in I_2'$. Introduce new intervals I_n, $n = 0, 1, 2, \ldots$, on the positive real axis as follows: $I_0 = I_2'$, $I_1 = I'$, I_2 is the union of I_1' and the interval which we get after translating I_2' by 1 to the right, I_3 is the interval I' translated by 1 to the right, I_4 is I_2 translated by 1 to the right, etc., see Figure 14.

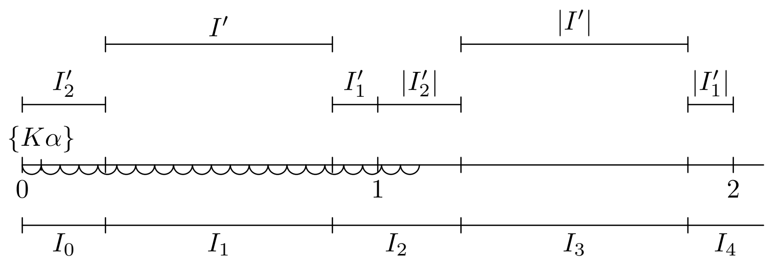

Figure 14.

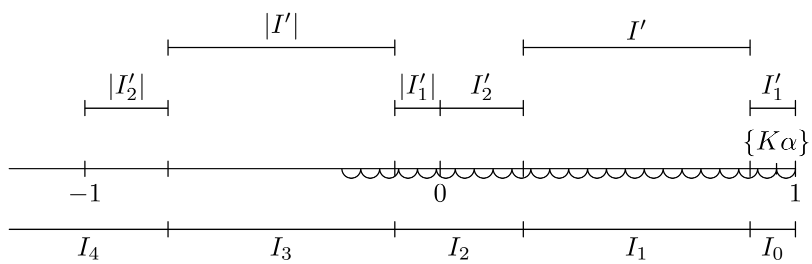

Figure 15.

Thus $|I_{2i}| = |I|$ and $|I_{2i-1}| = |I'|$, $i = 1, 2, \ldots$, where $|I'| = 1 - |I|$. Since $\{K\alpha\} < |I| \leq 1 - |I| = |I'|$, then we have

$$D_i = \#\{k \in \mathbb{N} : k\{K\alpha\} \in I_i\}$$

for $i = 0, 1, 2, \ldots$. For the numbers

$$D^{(0)} = \left\lfloor \frac{|I_2'|}{\{K\alpha\}} \right\rfloor, \quad D^{(1)} = \left\lfloor \frac{1 - |I|}{\{K\alpha\}} \right\rfloor, \quad D^{(2)} = \left\lfloor \frac{|I|}{\{K\alpha\}} \right\rfloor. \tag{39}$$

we have $|D_0 - D^{(0)}| \le 1$, $|D_{2i-1} - D^{(1)}| \le 1$ and $|D_{2i} - D^{(2)}| \le 1$ for $i = 1, 2, \ldots$.
2^0. In the case $\{K\alpha\} \in I_1'$ similar intervals I_i, $i = 0, 1, 2, \ldots$, in $(-\infty, 1]$ can be defined and (39) must be replaced by

$$D^{(0)} = \left\lfloor \frac{|I_1'|}{1 - \{K\alpha\}} \right\rfloor, \quad D^{(1)} = \left\lfloor \frac{1 - |I|}{1 - \{K\alpha\}} \right\rfloor, \quad D^{(2)} = \left\lfloor \frac{|I|}{1 - \{K\alpha\}} \right\rfloor. \tag{40}$$

(II) Let $|I| > 1/2$ and $\{K\alpha\} \in I_2'$. Then there are K's such that $\{K\alpha\} > |I'| = 1 - |I|$. In this case we have $D^{(1)} = 0$ in (39), and for some i it may happen that $\#\{k \in \mathbb{N} : k\{K\alpha\} \in I_{2i-1}\} = 0$. Thus the length of some blocks of consecutive $1'$s can be $> 2D^{(2)} - 2$, under these circumstances.
3^0. However, if $\{K\alpha\} < 1 - |I|$ then (38) again holds with $D^{(0)}, D^{(1)}$, and $D^{(2)}$ defined in (39). □

Another form of a proof of Theorem 26 we can get using the "circular" representation of the previous model applying the map $\varphi(t) = e^{2\pi i t}$ which maps interval $[0, 1]$ onto a circle.

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