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Explicit formulas for generators of triangular norms

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Abstract. Triangular norms are associative operations which represent conjunctions in fuzzy logic. They were also studied in the context of probabilistic metric spaces. It is known that each continuous Archimedean triangular norm can be determined by additive and multiplicative generators. However, finding a generator of a given triangular norm may be a difficult task. The geometry of the generator does not seem to reflect the properties of the triangular norm in an intuitive way. We show that this need not be the case for a large class of triangular norms which allow to reconstruct the generators from partial derivatives of triangular norms. This class is broad enough to cover all continuous Archimedean triangular norms which we found in the literature.

1. Introduction

A *triangular norm* (shortly, a *t-norm*) is a function $T : [0,1]^2 \to [0,1]$ which is commutative, associative, nondecreasing in both arguments, and which satisfies $T(x,1) = x$ for every $x \in [0,1]$. T-norms appear in different contexts, e.g., in probabilistic metric spaces [29] or in aggregation of partial knowledge from different sources. In fuzzy logic, they serve as conjunctions of truth values from $[0,1]$.

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Associativity induces restrictions on possible forms of t-norms. Roughly speaking, a t-norm can be represented as addition or multiplication, up to a change of scale determined by an isotone function called the (additive or multiplicative) *generator*. (Corrections at the boundary of the domain may be necessary.) It is known that every continuous Archimedean t-norm has a generator [1], [2], [10], [17], [21]. Although the proof is constructive, it can hardly be applied in practice – it constructs a sequence of points which is then refined to a dense set and continuity ends the procedure. Despite some optimization, the proof is complicated and needs not result in an explicit formula for the generator. There are explicit formulas for generators ([3], [8], [28], see Theorem 2.2 and Theorem 2.4), but they use infima over infinite sequences. These might be difficult to compute. Also the current computer algebraic systems cannot help with this task.

Here we offer an alternative. We show that partial derivatives of t-norms admit to obtain formulas for generators in a closed form. As the partial derivatives need not exist, our approach cannot be applied to all t-norms, but it seems sufficiently general for all practical applications. An advantage of our approach is that it relates (the geometry of) the generator directly to (the geometry of) the t-norm.

Let us mention that the result of the paper also contributes to the following question: Which subsets of the domain uniquely determine an Archimedean t-norm? Sufficiency has been proved for some subsets of the unit square [6], [7], [9], [15]. Here we give a similar result, yet working with the first partial derivatives instead of function values.

2. Preliminaries

In this section we present some basic facts about t-norms. More details can be found e.g. in the books by Alsina, Frank, and Schweizer [5] or by Klement, Mesiar, and Pap [13].

A continuous t-norm T is

- *Archimedean* if $\forall x \in \,]0,1[\, : T(x,x) < x$,
- *strict* if $\forall x \in \,]0,1[\, : 0 < T(x,x) < x$,
- *nilpotent* if T is Archimedean and not strict.

E.g., the *product t-norm* $T_P(x,y) = x \cdot y$ is strict, the *Łukasiewicz t-norm* $T_L(x,y) = \max\{x+y-1, 0\}$ is nilpotent. Throughout this paper we deal only with continuous Archimedean t-norms.

The *residuated implication* of a t-norm T is the function $I_T : [0,1]^2 \to [0,1]$,

$$I_T\,(x,y) = \sup\,\{z \in [0,1] \mid T\,(x,z) \le y\}\,.$$

The residuated implication of a t-norm is often called the *residuum.* For a continuous Archimedean t-norm T, the supremum can be replaced by the maximum and the following properties hold for all $x, y \in [0,1]$:

$$T\,(x,0) = 0, \qquad I_T\,(1,y) = y, \qquad T\,(x, I_T\,(x,y)) = \min\{x,y\}.$$

The fuzzy *negation* induced by T is $N_T : [0,1] \to [0,1]$,

$$N_T(x) = I_T\,(x,0) = \sup\,\{z \in [0,1] \mid T\,(x,z) = 0\}\,.$$

If T is nilpotent, N_T is an involutive fuzzy negation. If T is strict, N_T is the *Gödel negation*,

$$N_{\mathrm{G}}(x) = \begin{cases} 0 & \text{if } \ x > 0, \\ 1 & \text{if } \ x = 0. \end{cases}$$

For every $x \in [0,1]$ and $n \in \mathbb{N} \cup \{0\}$ we define a *natural power* $x_T^{(n)}$ with respect to the t-norm T by $x_T^{(0)} = 1$ and $x_T^{(n)} = T\big(x, x_T^{(n-1)}\big)$ for $n \in \mathbb{N}$.

The *support*, $\operatorname{Supp} T$, of a t-norm T is the closure of the set

$$\big\{(x,y) \in [0,1]^2 \mid T\,(x,y) > 0\big\}\,.$$

If T is strict, $\operatorname{Supp} T = [0,1]^2$. If T is nilpotent,

$$\operatorname{Supp} T = \bigcup_{y \in [0,1]} [N_T\,(y)\,,1] \times \{y\}\,.$$

An *additive generator* of a continuous Archimedean t-norm T is a strictly decreasing continuous extended real function $t : [0,1] \to [0,\infty]$ such that $t(1) = 0$ and

$$T\,(x,y) = t^{(-1)}\,(t(x) + t\,(y))\,, \tag{1}$$

where

$$t^{(-1)}(z) = \begin{cases} 0 & \text{if } \ z > t\,(0)\,, \\ t^{-1}(z) & \text{if } \ z \le t\,(0)\,. \end{cases}$$

The value $t\,(0)$ is ∞ if T is strict, finite if T is nilpotent.

Remark 2.1. When we speak of a (real) function, we mean that it attains finite real values. We make an exception for the additive generator which is an *extended* (real) function, i.e., it may achieve also ∞ (at 0).

A *multiplicative generator* of a continuous Archimedean t-norm T is a strictly increasing continuous function $\theta : [0,1] \to [0,1]$ such that $\theta(1) = 1$ and

$$T(x,y) = \theta^{[-1]}\left(\theta(x) \cdot \theta(y)\right), \tag{2}$$

where

$$\theta^{[-1]}(z) = \begin{cases} 0 & \text{if } z < \theta(0), \\ \theta^{-1}(z) & \text{if } z \geq \theta(0). \end{cases}$$

For a continuous Archimedean t-norm T, we denote by $\mathcal{M}_T$ resp. $\mathcal{A}_T$, the set of all its multiplicative resp. additive generators. These sets are infinite; an additive generator is determined by a t-norm up to a positive multiplicative constant, a multiplicative generator up to a positive power. If θ is a multiplicative generator of T, then $t = -\ln\theta$ is an additive generator. The reverse transformation is $\theta = \mathrm{e}^{-t}$.

In the following sections we are going to derive formulas which allow to obtain generators directly from the t-norms. For a comparison, let us recall two older results. The following is by Craigen and Páles [8]:

Theorem 2.2. *Let T be a strict t-norm. Fix an arbitrary element $c \in\,]0,1[$. Then the extended function $t : [0,1] \to [0,\infty]$ defined by*

$$t(x) = \inf\left\{\frac{m-n}{k} \;\middle|\; m,n,k \in \mathbb{N} \text{ and } c_T^{(m)} < T\left(x_T^{(k)}, c_T^{(n)}\right)\right\} \tag{3}$$

is an additive generator of T.

Remark 2.3. The original assumption of Craigen and Páles [8] was that T is a binary associative, continuous, and cancellative operation on a real interval. Commutativity and monotonicity are obtained as a consequence. This approach applies to strict t-norms restricted to the interval $]0,1]$ (or strict t-conorms restricted to $[0,1[$). Formula (3) was explicitly formulated by Alsina [3], but with the reverse ordering (valid for t-conorms, not for t-norms). The correct formulation appeared in [13], [14].

The second result is by Pi-Calleja [3], [28]:

Theorem 2.4. *Let $T : [0,1]^2 \to [0,1]$ be a binary operation which is associative, strictly increasing on $]0,1]^2$, and has a neutral element 1. Suppose further* *that the function* $x \mapsto T(x,x)$ *is continuous and that there exists* $\alpha \in]0,1[$ *such that* T *is continuous on* $[0,1] \times [\alpha, 1]$. *For each pair* $(x,y) \in]0,1] \times]0,1[$ *we define* $[x|y] \in \mathbb{N} \cup \{0\}$ *as the unique number such that*

$$y_T^{([x|y]+1)} < x \leq y_T^{([x|y])}.$$

Fix an arbitrary element $c \in]0,1[$. *Then* T *is a strict t-norm and the extended function* $t : [0,1] \to [0,\infty]$ *defined by*

$$t(x) = \begin{cases} \lim_{z \to 1_-} \dfrac{[x|z]}{[c|z]} & \text{if } x > 0, \\ \infty & \text{if } x = 0 \end{cases}$$

is an additive generator of T.

Both methods above give explicit formulas for an additive generator of a t-norm. However, the need of infima, limits, or the operation $[.|.]$ allows us to find the generator in a closed form only in very special cases.

3. Relations between partial derivatives of t-norms and their generators

We shall work with derivatives of generators, using the Newton notation for derivatives of functions of one variable, e.g. t', θ'. A t-norm T is a function of two variables; we shall denote by D the operator of the partial derivative with respect to the first variable,

$$\mathrm{D}T(x,y) = \lim_{h \to 0} \frac{T(x+h,y) - T(x,y)}{h} = \lim_{z \to x} \frac{T(z,y) - T(x,y)}{z - x}.$$

(Due to commutativity of T, the partial derivative with respect to the second variable is unnecessary.)

Assumption 3.1. The partial derivative $\mathrm{D}T$ will be considered only in the support $\operatorname{Supp} T$. In particular,

$$\mathrm{D}T(1,y) = \lim_{x \to 1_-} \frac{y - T(x,y)}{1 - x}$$

is the *left* partial derivative with respect to the first variable. If T is strict, then

$$\mathrm{D}T(0,y) = \lim_{x \to 0_+} \frac{T(x,y)}{x}$$

is the *right* partial derivative. For T nilpotent, we require the second argument $y > 0$; then $\mathrm{D}T(x, y)$ is defined for all $x \in [N_T(y), 1]$, in particular,

$$\mathrm{D}T\left(N_T(y), y\right) = \lim_{z \to N_T(y)_+} \frac{T(z, y)}{z - N_T(y)} \tag{4}$$

is the *right* partial derivative. Since T is nilpotent, the negation N_T is involutive. Therefore, substituting $x = N_T(y)$, we can write (4) as

$$\mathrm{D}T\left(x, N_T(x)\right) = \lim_{z \to x_+} \frac{T(z, N_T(x))}{z - x}.$$

For T nilpotent and $y = 0$, the line $\{(x, 0) \mid x \in \mathbb{R}\}$ intersects $\operatorname{Supp} T$ only at a single point $(1, 0)$ and $\mathrm{D}T(x, 0)$ is undefined for any x.

Notice that for $(x, y) \in \operatorname{Supp} T$ pseudo-inverses coincide with inverses in (1), (2) and therefore $T(x, y) = t^{-1}(t(x) + t(y)) = \theta^{-1}(\theta(x) \cdot \theta(y))$.

We shall refer to the following lemma:

Lemma 3.2. *Let f be a real function which is differentiable on an interval I. Let f possess an inverse function g. Each point $z \in f(I)$ where g is differentiable and $f'(g(z)) \neq 0$ satisfies*

$$g'(z) = \frac{1}{f'(g(z))}.$$

Assumption 3.3. Throughout the paper, we assume the existence and finiteness of all derivatives occurring in formulas. This assumption will be stated explicitly in theorems. Unless specified otherwise, we suppose that the denominators of all fractions are non-zero.

From Lemma 3.2 we obtain the following formulas:

$$\mathrm{D}T(x, y) = \frac{t'(x)}{t'(T(x, y))}, \tag{5}$$

$$\mathrm{D}T(x, y) = \frac{\theta(y) \cdot \theta'(x)}{\theta'(T(x, y))}. \tag{6}$$

The properties $T(x, 0) = 0$ and $T(x, 1) = x$ imply the following equations:

$$\mathrm{D}T(x, 0) = 0 \qquad \text{(for } T \text{ strict)}, \qquad \mathrm{D}T(x, 1) = 1. \tag{7}$$

For $a \in \,]0, 1]$, the *a-level set* of a t-norm T is the set

$$\{(x, y) \in \operatorname{Supp} T \mid T(x, y) = a\} = \{(x, I_T(x, a)) \mid x \in [a, 1]\}.$$

For a given t-norm, it is possible to find (all) its additive and multiplicative generators, but the procedure is rather complex [3], [8], [17], [21], [28]. The proofs are constructive but the geometry of the generators is not apparently related to the geometry of the t-norm. We shall show that special instances of formulas (5), (6) admit to derive the generators in a more straightforward way. We shall obtain either a generator or its derivative. We are going to present several examples of substitutions which leave only one expression containing the generator.

Example 3.4 ([22]). Let us substitute $x = 0$ in (6):

$$\mathrm{D}T\,(0, y) = \frac{\theta\,(y) \cdot \theta'\,(0)}{\theta'\,(0)} = \theta\,(y)\,.$$

We obtained directly the multiplicative generator. Due to the requirement $(x, y) \in \operatorname{Supp} T$, this is applicable only if T is strict. For a correct cancellation, we also need $\theta'\,(0) \in \,]0, \infty[$. Such a multiplicative generator θ exists only for some strict t-norms. If it exists, it is unique as will be shown in Proposition 4.1.

Remark 3.5. If we substitute $x = 0$ in (5), we obtain

$$\mathrm{D}T\,(0, y) = \frac{t'\,(0)}{t'(0)},$$

but the right-hand side is undefined because additive generators of strict t-norms do not have finite derivatives at 0.

Example 3.6 ([22]). Let us substitute $x = 1$ in (5):

$$\mathrm{D}T\,(1, y) = \frac{t'(1)}{t'\,(y)} = \frac{b_{t,1}}{t'\,(y)}, \qquad t'\,(y) = \frac{b_{t,1}}{\mathrm{D}T\,(1, y)}.$$

The constant $b_{t,1} = t'(1)$ is not known, but it is irrelevant because an additive generator is determined up to a multiplicative constant. Substituting any finite negative value for $b_{t,1}$ gives rise to an additive generator of T (after integration with initial value $t(1) = 0$).

Remark 3.7. If we substitute $x = 1$ in (6), we obtain

$$\mathrm{D}T\,(1, y) = \frac{\theta\,(y) \cdot \theta'(1)}{\theta'\,(y)}, \qquad (\ln\theta)'\,(y) = \frac{\theta'\,(y)}{\theta\,(y)} = \frac{\theta'(1)}{\mathrm{D}T\,(1, y)}.$$

This leads to the same results as Example 3.6 for the additive generator $t = -\ln\theta$.

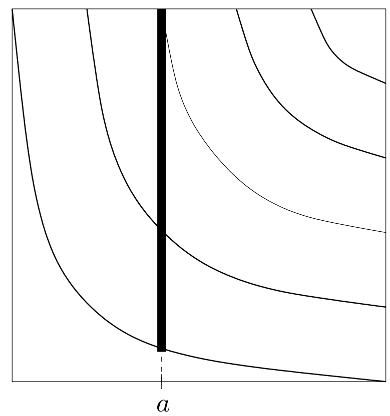

Figure 1. Example of a nilpotent t-norm expressed by its system of level sets. The bold line denotes the set of pairs (a, y) for which the derivative $\mathrm{D}T\,(a, y)$ is used in Example 3.8.

Example 3.6 gives a relation between the derivative of an additive generator and the (left) partial derivative of the t-norm on the line $\{(1, y) \mid y \in [0, 1]\}$. Now we shall generalize it to a procedure which reconstructs a part of an additive generator from the partial derivatives on a line $\{(a, y) \mid y \in [0, 1]\}$, $a \in]0, 1]$.

Example 3.8. In (5) we substitute x with a fixed constant $a \in]0, 1]$. We obtain

$$\mathrm{D}T\,(a, y) = \frac{t'(a)}{t'\,(T\,(a, y))}.$$

Using a new variable $z = T\,(a, y) \in [0, a]$, we get $y = I_T\,(a, z)$,

$$\mathrm{D}T\,(a, I_T\,(a, z)) = \mathrm{D}T\,(a, y) = \frac{t'(a)}{t'\,(T\,(a, y))} = \frac{t'(a)}{t'(z)} = \frac{b_{t,a}}{t'(z)}, \tag{8}$$

$$t'(z) = \frac{b_{t,a}}{\mathrm{D}T\,(a, I_T\,(a, z))}.$$

Figure 1 shows for which pairs (a, y) the formula (8) is used. The right-hand side is determined up to a multiplicative constant $b_{t,a} = t'(a)$. Any finite negative value of this constant gives a formula for the derivative of some additive generator at $[0, a]$.

The following example shows a relation between the derivative of an additive generator and the partial derivative of the t-norm on its level set.

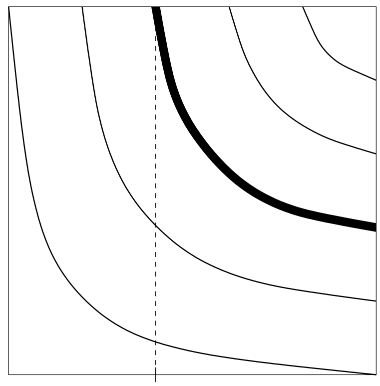

Figure 2. Example of a nilpotent t-norm expressed by its system of level sets. The bold curve denotes the set of pairs (x, y) for which the derivative $\mathrm{D}T\,(x, y)$ is used in Example 3.9.

Example 3.9. Let us substitute $y = I_T\,(x, a)$, $a \in\,]0, 1]$, in (5). For all $x \geq a$, we obtain $T\,(x, y) = a$,

$$\mathrm{D}T\,(x, I_T\,(x, a)) = \frac{t'(x)}{t'(a)} = \frac{t'(x)}{b_{t,a}}, \quad t'(x) = b_{t,a} \cdot \mathrm{D}T\,(x, I_T\,(x, a))\,.$$

Figure 2 shows all pairs $(x, y) = (x, I_T\,(x, a))$, $x \in [a, 1]$, to which the latter formula applies. The constant $b_{t,a} = t'(a)$ is irrelevant and can be replaced by any finite negative number. Due to the requirement $T\,(x, I_T\,(x, a)) = \min\{x, a\} = a$, the formula determines $t'(x)$ only for $x \in [a, 1]$.

For $a = 0$ and $t'\,(0) \in\,]-\infty, 0[$ (which is possible only if T is nilpotent), we obtain

$$t'(x) = b_{t,0} \cdot \mathrm{D}T\,(x, N_T\,(x))$$

for all $x \in\,]0, 1]$. (Recall that $\mathrm{D}T$ is the right partial derivative in this case.)

4. Reconstruction of multiplicative generators

The above ideas allow us to reconstruct the generators from partial derivatives of t-norms. Preliminary results towards this direction appeared in the previous paper [22]. We recall them here (Theorems 4.2 and 5.1) for comparison with the new results and also as an easier introduction to the new methods.

Proposition 4.1 ([22]). *Let T be a strict t-norm. If T has a multiplicative generator $\theta \in \mathcal{M}_T$ satisfying $\theta'(0) \in]0, \infty[$, then this generator is unique.*

PROOF. Any multiplicative generator of T is of the form $\sigma = \theta^p$ for some $p \in]0, \infty[$ and satisfies

$$\sigma'(0) = (\theta^p)'(0) = p \cdot \theta^{p-1}(0) \cdot \theta'(0) = \begin{cases} \infty & \text{if } p < 1, \\ \theta'(0) & \text{if } p = 1, \\ 0 & \text{if } p > 1. \end{cases} \qquad \square$$

Example 3.4 leads to the following:

Theorem 4.2 ([22]). *Let T be a strict t-norm and let θ be a multiplicative generator of T such that $\theta'(0) \in]0, \infty[$. Then*

$$\theta(y) = \mathrm{D}T(0, y) = \lim_{x \to 0_+} \frac{T(x, y)}{x}$$

for all $y \in [0, 1]$.

PROOF. The proof has been described in Example 3.4. However, Theorem 4.5 will give a more general result. □

By the symbol $\mathrm{D}_{1,2}T(0, 0)$ we denote the mixed derivative of a strict t-norm T at the point $(0, 0)$:

$$\mathrm{D}_{1,2}T(0, 0) = \lim_{x \to 0_+} \left(\frac{1}{x} \left(\lim_{y \to 0_+} \frac{T(x, y)}{y} \right) \right).$$

Proposition 4.3. *Let T be a strict t-norm with a multiplicative generator θ such that $\theta'(0) \in]0, \infty[$. Then*

$$\mathrm{D}_{1,2}T(0, 0) \in]0, \infty[\, .$$

PROOF.

$$\mathrm{D}_{1,2}T(0, 0) = \lim_{x \to 0_+} \left(\frac{1}{x} \left(\lim_{y \to 0_+} \frac{T(x, y)}{y} \right) \right) = \lim_{x \to 0_+} \frac{\theta(x)}{x} = \theta'(0) \in]0, \infty[\, . \quad \square$$

This means that, up to second order terms, T can be approximated by a non-zero multiple of the product in a neighbourhood of $(0, 0)$. As it has been shown, the condition presented by Theorem 4.2 is sufficient for a reconstruction of a multiplicative generator of a strict t-norm T. Proposition 4.3 poses a condition required by Theorem 4.2. However, we shall show that it is not necessary.

Lemma 4.4. *Let $f : [0,1] \to [0,1]$ be an increasing bijection. Suppose that the function $g : [0,1] \to [0,1]$,*

$$g(y) = \lim_{x \to 0_+} \frac{f(x \cdot y)}{f(x)},$$

is defined for all $y \in [0,1]$. Then $g(0) = 0$, $g(1) = 1$, and $g(y_1 \cdot y_2) = g(y_1) \cdot g(y_2)$ for all $y_1, y_2 \in \,]0,1[$. Thus g restricted to $]0,1[$ is either constant 0, constant 1, or it attains the form $g(y) = y^p$ for some $p \in \,]0,\infty[$.

Proof. For $y_1, y_2 \in \,]0,1[$ we have

$$\begin{aligned} g(y_1) \cdot g(y_2) &= \lim_{x \to 0_+} \frac{f(x \cdot y_1)}{f(x)} \cdot \lim_{x \to 0_+} \frac{f(x \cdot y_2)}{f(x)} \\ &= \lim_{x \to 0_+} \frac{f(x \cdot y_1 \cdot y_2)}{f(x \cdot y_2)} \cdot \lim_{x \to 0_+} \frac{f(x \cdot y_2)}{f(x)} = \lim_{x \to 0_+} \frac{f(x \cdot y_1 \cdot y_2)}{f(x)} = g(y_1 \cdot y_2). \end{aligned}$$

(The second equality follows by the substitution $x := x \cdot y_2$ in the first limit.) The only nondecreasing multiplicative functions are positive powers, i.e. functions of the type $g(y) = y^p$ for some $p \in \,]0,\infty[$, and the constants 0, resp. 1, as their limit cases for $p \to \infty$, resp. $p \to 0$. □

Theorem 4.5. *Let T be a strict t-norm. Suppose that the function $\xi : [0,1] \to [0,1]$,*

$$\xi(y) = \lim_{x \to 0_+} \frac{T(x,y)}{x},$$

is defined for all $y \in [0,1]$. Then $\xi(0) = 0$, $\xi(1) = 1$, and the restriction of ξ to $]0,1[$ is of one of the following forms:

(1) *the constant* 0,

(2) *the constant* 1,

(3) *a bijection on* $]0,1[$.

Moreover, in case (3) *the function ξ is a multiplicative generator of T.*

Proof. Since T is monotonic and upper bounded by min, the function ξ is nondecreasing and attains values in the unit interval. As T is a strict t-norm, it has a multiplicative generator; let us denote it by θ. Invoking the definition of ξ and the invertibility of θ, we obtain

$$(\xi \circ \theta^{-1})(y) = \xi(\theta^{-1}(y)) = \lim_{x \to 0_+} \frac{\theta^{-1}(\theta(x) \cdot y)}{x} = \lim_{z \to 0_+} \frac{\theta^{-1}(z \cdot y)}{\theta^{-1}(z)}$$

for all $y \in [0,1]$. Applying Lemma 4.4 to $f = \theta^{-1}$ and $g = \xi \circ \theta^{-1}$ and considering that θ^{-1} is an increasing bijection, we obtain that $\xi \circ \theta^{-1}$, as well as ξ, satisfies one of the cases (1)–(3). It remains to prove that in case (3) the mapping ξ is a multiplicative generator of T. According to Lemma 4.4, there is a $p \in]0,\infty[$ such that $\xi(\theta^{-1}(z)) = z^p$ for all $z \in [0,1]$. Using the substitution $y := \theta^{-1}(z)$, we obtain $\xi(y) = (\theta(y))^p$, thus $\xi = \theta^p$ is a multiplicative generator of T. □

Theorem 4.5 allows us to reconstruct a multiplicative generator of a strict t-norm from its first partial derivatives along the "zero border" under the condition that these derivatives give rise to a bijection. Notice that this theorem shows that, up to a division by an infinitesimally small constant, the t-norm along the "zero border" approximates the multiplicative generator.

The t-norms considered by Theorem 4.2 form a special subset of those considered by Theorem 4.5.

5. Reconstruction of additive generators

Additive generators can be derived from multiplicative generators (constructed in the latter section). However, we shall present a different technique which naturally leads to additive generators and uses partial derivatives at other points.

By an *absolutely continuous* function we mean a function which can be expressed as the integral of its derivative. (The Cantor function [32] is an example of a function which is *not* absolutely continuous.) It is worth mentioning that Theorem 4.5 does not need the assumption of absolute continuity whereas the following methods do so. Nevertheless, in Section 6 we shall show that the following methods are applicable to many t-norms which do not satisfy the assumptions of Theorem 4.5.

Example 3.6 corresponds to the following result:

Theorem 5.1 ([22]). *Let T be a continuous Archimedean t-norm and let t be an additive generator of T such that t is absolutely continuous at $]0,1]$ and $t'(1) = b_{t,1} \in]-\infty,0[$. Suppose that $\mathrm{D}T(1,y) \in]0,\infty[$ for almost all $y \in]0,1]$. Then*

$$t'(y) = \frac{b_{t,1}}{\mathrm{D}T(1,y)} \quad \text{(almost everywhere in }]0,1])$$

and

$$t(y) = \int_y^1 \frac{-b_{t,1}}{\mathrm{D}T(1,u)}\,\mathrm{d}u \tag{9}$$

for all $y \in]0,1]$.

We omit the proof; a more general result will be proved in Theorem 5.5.

Proposition 5.2. *Let T be a continuous Archimedean t-norm. If there exists an additive generator $t \in \mathcal{A}_T$ such that t' is continuous at 1 and $t'(1) \in \left]-\infty, 0\right[$, then the first partial derivatives of T at the point $(1,1)$ are continuous.*

Proof. From (7) we know that $\mathrm{D}T(1,1) = 1$. Thus the partial derivative of T with respect to the first variable is continuous at $(1,1)$ if and only if

$$\lim_{x \to 1_-, y \to 1_-} \mathrm{D}T(x,y) = 1.$$

Expressing the t-norm using its additive generator we get

$$\lim_{x \to 1_-, y \to 1_-} \mathrm{D}T(x,y) = \lim_{x \to 1_-, y \to 1_-} \frac{t'(x)}{t'\left(T(x,y)\right)} = \lim_{y \to 1_-} \frac{t'(1)}{t'(y)} = 1.$$

By commutativity, the same is obtained for the partial derivative of T with respect to the second variable. □

Remark 5.3. A t-norm satisfying the assumptions of Proposition 5.2 is differentiable at the point $(1,1)$. Up to the first order terms, it can be approximated by the Łukasiewicz t-norm in a neighbourhood of $(1,1)$.

We shall present a more general method using a combination of Examples 3.8 and 3.9. It is based on the following principle:

Lemma 5.4. *Let $x, y \in [0,1]$, $x \geq y$. Let T be a continuous Archimedean t-norm. Suppose that T has an additive generator t with finite derivatives at x, y, and $T(x,y)$. (We take the right, resp. left, derivatives at 0, resp. 1.) Suppose further that $t'(y) \neq 0$. Then*

$$\mathrm{D}T\left(x, I_T\left(x,y\right)\right) = \frac{t'(x)}{t'\left(y\right)}. \tag{10}$$

Proof. The assumptions of the lemma ensure that $\left(x, I_T\left(x,y\right)\right) \in \operatorname{Supp} T$ and $T\left(x, I_T\left(x,y\right)\right) = y$. Using Lemma 3.2 with $f = t$, $g = t^{-1}$, $z = t(x) + t\left(y\right)$, we get

$$\mathrm{D}T\left(x, I_T(x,y)\right) = \frac{t'(x)}{t'\left(T\left(x, I_T(x,y)\right)\right)} = \frac{t'(x)}{t'\left(y\right)}.$$

□

Theorem 5.5. *Let T be a continuous Archimedean t-norm. Suppose that T has an absolutely continuous additive generator with a non-zero finite derivative at some point $a \in \left]0,1\right]$. (We take the left derivative at 1.) Let $\mathrm{D}T$ be the*

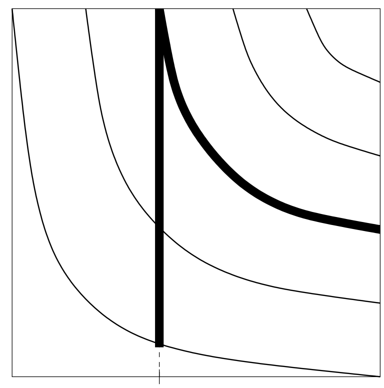

Figure 3. Example of a nilpotent t-norm expressed by its system of level sets. The bold curve denotes the set of pairs (x, y) for which the derivative $\mathrm{D}T\,(x, y)$ is used in Theorem 5.5.

partial derivative of T *with respect to the first variable in the support* $\operatorname{Supp} T$ *(cf. Assumption 3.1). Suppose that* $\mathrm{D}T\,(z, I_T(z, a))$ *exists for almost all* $z \in [a, 1]$. *Suppose further that* $\mathrm{D}T\,(a, I_T\,(a, z))$ *exists and is in* $]0, \infty[$ *for almost all* $z \in [0, a[$. *Then* T *has an additive generator*

$$t^*(x) = \int_x^1 v(z)\,\mathrm{d}z,$$

where

$$v(z) = \begin{cases} \mathrm{D}T\,(z, I_T(z, a)) & \text{if } z \geq a, \\ \dfrac{1}{\mathrm{D}T\,(a, I_T\,(a, z))} & \text{if } z < a \end{cases} \tag{11}$$

for almost all $z \in [0, 1]$. *Explicitly,*

$$t^*(x) = \begin{cases} \int_x^1 \mathrm{D}T\,(z, I_T(z, a))\,\mathrm{d}z & \text{if } x \geq a, \\ \int_x^a \dfrac{1}{\mathrm{D}T\,(a, I_T\,(a, z))}\mathrm{d}z + \int_a^1 \mathrm{D}T\,(z, I_T(z, a))\,\mathrm{d}z & \text{if } x < a. \end{cases}$$

(Figure 3 shows in which points of the domain the derivatives $\mathrm{D}T(z, I_T(z, a))$ and $\mathrm{D}T(a, I_T(a, z))$ are computed.)

Proof. If one additive generator of the t-norm T satisfies the assumptions of Theorem 5.5 for some $a \in [0, 1]$, all additive generators of T satisfy the assumptions as well. Considering that all the additive generators of T differ only by a real multiplicative constant, there exists an additive generator t of T such that $t'(a) = -1$. For $z \geq a$, we use (10) with substitutions $x = z$, $y = a$. We obtain the formulas

$$\mathrm{D}T\,(z, I_T(z,a)) = \frac{t'(z)}{t'(a)} = -t'(z), \qquad t'(z) = -\mathrm{D}T\,(z, I_T(z,a))$$

(almost everywhere) as in Example 3.9. For $z < a$, we use (10) with substitutions $x = a, y = z$. Almost everywhere we have

$$\mathrm{D}T\,(a, I_T\,(a,z)) = \frac{t'(a)}{t'(z)} = \frac{-1}{t'(z)}, \qquad t'\,(z) = \frac{-1}{\mathrm{D}T\,(a, I_T\,(a,z))} \tag{12}$$

as in Example 3.8, thus

$$t'(z) = \begin{cases} -\mathrm{D}T\,(z, I_T(z,a)) & \text{if } \; z \geq a, \\ \dfrac{-1}{\mathrm{D}T\,(a, I_T\,(a,z))} & \text{if } \; z < a \end{cases}$$

(almost everywhere). Integrating this equality, we obtain an additive generator t such that $t = t^*$. □

Corollary 5.6. *As a limit case of Theorem 5.5, we obtain the formula described in Theorem 5.1:*

$$t'\,(y) = \frac{b_{t,1}}{\mathrm{D}T\,(1,y)}.$$

Proof. Let $a = 1$. For $z \in [0,1[$, we use (12) of Theorem 5.5, since $z < a$. This leads to

$$\mathrm{D}T\,(1, I_T(1,z)) = \frac{t'(1)}{t'(z)}, \qquad t'(z) = \frac{t'(1)}{\mathrm{D}T\,(1, I_T(1,z))} = \frac{b_{t,1}}{\mathrm{D}T(1,z)}$$

for almost all $z \in [0,1]$. □

Theorem 5.5 allows us to reconstruct an additive generator when a non-negative constant $a \in\,]0,1]$ is given. The following theorem shows that even $a = 0$ can be used. However, this works for nilpotent t-norms only.

Theorem 5.7. *Let T be a nilpotent t-norm. Suppose that T has an absolutely continuous additive generator with a non-zero finite (right) derivative at the point* 0. *Let* $\mathrm{D}T$ *be the right partial derivative of T with respect to the first variable in the support* $\operatorname{Supp} T$ *(cf. Assumption 3.1). Suppose that* $\mathrm{D}T\,(z, N_T(z))$ *exists for almost all $z \in [0,1]$. Then T has an additive generator*

$$t^*(x) = \int_x^1 \mathrm{D}T\,(z, N_T\,(z))\;\mathrm{d}z.$$

Proof. If one additive generator of the t-norm T satisfies the assumptions of Theorem 5.7, all additive generators of T satisfy the assumptions as well. Considering that all the additive generators of T differ only by a positive multiplicative constant, there exists an additive generator t of T such that $t'(0) = -1$. According to Lemma 5.4:

$$\mathrm{D}T\,(z, N_T(z)) = \mathrm{D}T\,(z, I_T(z,0)) = \frac{t'(z)}{t'(0)} = -t'(z),$$

$$t'(z) = -\mathrm{D}T\,(z, N_T(z))$$

almost everywhere. Integrating this equality, we obtain an additive generator t such that $t = t^*$. □

6. Example

In this section we demonstrate the presented methods. We refer to specific families of t-norms whose definitions can be found in the monographs [5], [13]. All the methods mentioned here can be applied, e.g., to the Frank family of t-norms. Both the methods presented in the previous work [22] (Theorems 4.2 and 5.1 here) fail in some cases, while Theorem 5.5 is still applicable. This happens both for strict t-norms (e.g., the Aczél–Alsina family) and for nilpotent t-norms (e.g., the Yager family). All these examples are demonstrated in detail in the thesis of one of the authors [26].

In such examples, we obtain the same formula from two different expressions for the additive generator in (11). This is due to the simple form of the additive generators. The following example shows that this cannot be expected in general.

Example 6.1. Let $T : [0,1]^2 \to [0,1]$ be defined as:

$$T\,(x,y) = \begin{cases} \dfrac{1}{\frac{1}{x} + \frac{1}{y} - 1} & \text{if } x \le 1/2 \text{ and } y \le 1/2, \\ \dfrac{1}{\frac{1}{x} + 4\,(1-y)^2} & \text{if } x \le 1/2 \text{ and } y > 1/2, \\ \dfrac{1}{\frac{1}{y} + 4(1-x)^2} & \text{if } x > 1/2 \text{ and } y \le 1/2, \\ 1 - \sqrt{(1-x)^2 + (1-y)^2} & \text{if } (1-x)^2 + (1-y)^2 \le 1/4, \\ \dfrac{1}{1 + 4(1-x)^2 + 4(1-y)^2} & \text{otherwise.} \end{cases} \tag{13}$$

(The conditions are not mutually exclusive, but when the ranges overlap, we get the same results.) The operation T is a strict t-norm. Its partial derivative is

$$\mathrm{D}T\left(x,y\right)=\begin{cases}\dfrac{1}{x^2\left(\frac{1}{x}+\frac{1}{y}-1\right)^2} & \text{if } x\le 1/2 \text{ and } y\le 1/2,\\ \dfrac{1}{x^2\left(\frac{1}{x}+4(1-y)^2\right)^2} & \text{if } x\le 1/2 \text{ and } y> 1/2,\\ \dfrac{8(1-x)}{\left(\frac{1}{y}+4(1-x)^2\right)^2} & \text{if } x> 1/2 \text{ and } y\le 1/2,\\ \dfrac{1-x}{\sqrt{(1-x)^2+(1-y)^2}} & \text{if } \left(1-x\right)^2+\left(1-y\right)^2\le 1/4,\\ \dfrac{8(1-x)}{(1+4(1-x)^2+4(1-y)^2)^2} & \text{otherwise.}\end{cases} \tag{14}$$

It is continuous in the interior of its domain (the unit square).

Theorems 4.5 and 5.1 are not applicable to T. Let us now show the application of Theorem 5.5 for $a=1/2$. To optimize computation, we shall determine the residuum only in those points which we need. Namely, for each $z\in\,]0,1[$, we take in mind $w=I_T\left(z,1/2\right)$ and $q=I_T\left(1/2,z\right)$. We distinguish several cases.

Case 1. Assume that $z>1/2$. Then the fourth option in (13) applies and

$$\frac{1}{2}=T\left(z,I_T\left(z,\frac{1}{2}\right)\right)=T\left(z,w\right)=1-\sqrt{\left(1-z\right)^2+\left(1-w\right)^2}.$$

From this we derive

$$I_T\left(z,\frac{1}{2}\right)=w=1-\sqrt{\frac{1}{4}-\left(1-z\right)^2}.$$

In order to determine the first partial derivative in (z,w), the fourth option in (14) applies and

$$\begin{aligned}\mathrm{D}T\left(z,I_T\left(z,\frac{1}{2}\right)\right)&=\mathrm{D}T\left(z,w\right)=\frac{1-z}{\sqrt{\left(1-z\right)^2+\left(1-w\right)^2}}\\&=\frac{1-z}{\sqrt{\left(1-z\right)^2+\frac{1}{4}-\left(1-z\right)^2}}=2\left(1-z\right).\end{aligned}$$

Case 2. Assume that $0 < z \leq 1/2$ and $q \leq 1/2$. Then the first option in (13) applies and

$$z = T\left(\frac{1}{2}, I_T\left(\frac{1}{2}, z\right)\right) = T\left(\frac{1}{2}, q\right) = \frac{1}{2 + \frac{1}{q} - 1} = \frac{1}{\frac{1}{q} + 1}.$$

We derive

$$I_T\left(\frac{1}{2}, z\right) = q = \frac{1}{\frac{1}{z} - 1}.$$

The first option in (14) gives the first partial derivative at $(1/2, q)$:

$$\mathrm{D}T\left(\frac{1}{2}, I_T\left(\frac{1}{2}, z\right)\right) = \mathrm{D}T\left(\frac{1}{2}, q\right) = \frac{4}{\left(2 + \frac{1}{q} - 1\right)^2} = \frac{4}{\left(\frac{1}{q} + 1\right)^2} = 4\,z^2.$$

Case 3. Assume finally that $0 < z \leq 1/2$ and $q > 1/2$. Then the second option in (13) applies and

$$z = T\left(\frac{1}{2}, I_T\left(\frac{1}{2}, z\right)\right) = T\left(\frac{1}{2}, q\right) = \frac{1}{2 + 4\,(1-q)^2}.$$

We derive

$$I_T\left(\frac{1}{2}, z\right) = q = 1 - \sqrt{\frac{1}{4\,z} - \frac{1}{2}}.$$

The first partial derivative at $(1/2, q)$ is given by the second option in (14):

$$\begin{aligned} \mathrm{D}T\left(\frac{1}{2}, I_T\left(\frac{1}{2}, z\right)\right) = \mathrm{D}T\left(\frac{1}{2}, q\right) &= \frac{4}{\left(2 + 4\,(1-q)^2\right)^2} \\ &= \frac{4}{\left(2 + 4\left(\frac{1}{4\,z} - \frac{1}{2}\right)\right)^2} = 4\,z^2. \end{aligned}$$

The latter two cases gave the same result, thus we do not need to distinguish the conditions put on q. Now we apply Theorem 5.5 and obtain

$$v(z) = \begin{cases} \dfrac{1}{4\,z^2} & \text{if } z \leq 1/2, \\ 2\,(1-z) & \text{if } z > 1/2. \end{cases}$$

Integration gives an additive generator

$$u(x) = \begin{cases} \dfrac{1}{4\,x} - \dfrac{1}{4} & \text{if } x \leq 1/2, \\ (1-x)^2 & \text{if } x > 1/2. \end{cases}$$

7. Discussion

We have found a method which allows to construct generators of continuous Archimedean t-norms in a way much more straightforward than that of the methods presented so far [3], [8], [17], [21], [28]. Instead of constructing a dense set of values or computing limits of infinite sequences, we derive the values of generators from partial derivatives of t-norms (if they exist). Moreover, we obtain *explicit formulas* for generators. The method is not applicable to all continuous Archimedean t-norms, but it is general enough to cover all continuous Archimedean t-norms which we found in the literature [5], [13], [18].

By duality, these results can also be extended to triangular conorms. Generalization to other fuzzy logical operations (uninorms, triangular subnorms, etc.) could be also considered. Here we concentrated only on continuous Archimedean t-norms; further generalizations were left for future research.

The only possible difficulty in application of Theorem 5.5 (when its assumptions are satisfied) is the computation of the residuum. It requires to solve the equation $T(x, y) = z$ with respect to y. It may be impossible to express the exact solution analytically. Such examples obviously exist, but were not encountered in the literature on t-norms. Analogous problems apply also to the previous approaches. They require to solve the equation $y_T^{(n)} = x$ with respect to y or even with respect to n. This task also need not be solvable analytically.

It is desirable to find properties of a t-norm which ensure the applicability of Theorem 5.5. So far, its assumptions refer to the existence of an additive generator with certain properties (and then an explicit formula for this generator is found). It would be useful to check in advance whether a given t-norm is of this type. We derived necessary, but not sufficient conditions.

The results shed light on the open problem which has been studied intensively in the last years [11], [20], [23], [24], [30], [31] and which has been stated, e.g., in the list of open problems by Alsina, Frank, and Schweizer [4]:

Problem 7.1. *Can the arithmetic mean (or any non-trivial convex combination) of two distinct t-norms be a t-norm?*

We know that this can happen neither for nilpotent t-norms [27] nor for strict t-norms with smoothly differentiable additive generators satisfying an additional constraint [30]. For general strict t-norms, the question is still open. However, a rather restrictive constraint can be given also for strict t-norms satisfying the assumptions of Theorem 4.5 and Theorem 5.1 [25].

Finally, we remark that the presented methods could be also used as an "associativity checker" of certain classes of binary operations. The procedure would be based on reconstructing a generator, constructing the binary operation, and comparing it with the original operation. We postpone a full formulation of this method for a future paper.

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