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Derivations in commutators with power central values in rings

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Abstract. Let R be a prime ring of characteristic different from 2 and I a nonzero ideal of R, d a nonzero derivation of R such that $[d(x^k), x^k]^n$ is central, for all $x \in I$ where k, n are fixed positive integers. Then R satisfies s_4, the standard identity in 4 variables.

1. Introduction

Throughout this article, R is always a prime ring with center Z. For any $x, y \in R$, we set $[x, y]_1 = [x, y] = xy - yx$ and $[x, y]_n = [[x, y]_{n-1}, y]$ where $n > 1$ is a positive integer. By s_4 we denote the standard identity in 4 variables. By d we denote a nonzero derivation of R.

A well-known result proved by Posner [14] states that R must be commutative if $[d(x), x] \in Z$ for all $x \in R$. In [12] Lee and Lee generalized Posner's result by showing that if $\mathrm{char}(R) \neq 2$ and $[d(x), x] \in Z$ for all x in a noncentral Lie ideal of R, then R is commutative. As to the case when $\mathrm{char}(R) = 2$, Lanski obtained the same conclusion except when R satisfies s_4 (see [9]). In [2] Carini and De Filippis studied the situation when $[d(x), x]^n \in Z$ for all x in a noncommutative Lie ideal of R with $\mathrm{char}(R) \neq 2$. In [16] the second author and You removed the assumption of $\mathrm{char}(R) \neq 2$.

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In [6] FELZENSZWALB proved that R is commutative if $d(x^k) = 0$ for all $x \in R$, where k is a fixed positive integer. A significant extension of [6] shows that R is commutative if $[d(x^k), x^k]_n = 0$ for all x in a nonzero left ideal of R (see [11, Theorem 1]). In [15] SHIUE discussed the situation when $a[d(x^k), x^k]_n = 0$ for all x in an one-sided ideal of R, where $0 \neq a \in R$.

The purpose of this paper is to investigate the situation when $[d(x^k), x^k]^n \in Z$ for all x in a nonzero ideal of R. The main result is the following

Theorem 1. *Let R be a prime ring of characteristic different from 2 with its center Z, I a nonzero ideal of R, and d a nonzero derivation of R such that $[d(x^k), x^k]^n \in Z$ for all $x \in I$ where k, n are fixed positive integers. Then R satisfies s_4, the standard identity in 4 variables.*

The following counterexample shows that Theorem 1 is not valid on some one-sided ideals.

Example 1. Let F be a field and $R = M_m(F)$, the ring of all $m \times m$ matrix algebra over F with $m > 2$. Let e_{ij} be the matrix unit with 1 in (i, j)-entry and zero elsewhere. It is easy to check that $([e_{11}, x^k]_2)^n = 0$ for all $x \in Re_{22}$ (or, $e_{22}R$), where $n > 1$.

2. The proof of Theorem 1

By Q we denote the Martindale quotient ring of R and C the extended centroid R. The definitions and properties of these objects can be found in [1, Chapter 2].

We begin with the following easy result.

Lemma 1. *Let $R = M_2(F)$, the ring of all 2×2 matrics over a field F with $\mathrm{char}(F) \neq 2$. If a is a nonzero element of R such that $([a, x^k]_2)^n = 0$ for all $x \in R$, then $a \in F \cdot I_2$.*

PROOF. Let $a = \sum_{i,j} a_{ij} e_{ij}$ with $a_{ij} \in F$. We first claim that a is a diagonal matrix. By assumption we get

$$0 = ([a, e_{11}]_2)^{2n} = (a_{12}a_{21})^n e_{11} + (a_{12}a_{21})^n e_{22},$$

thus $a_{12}a_{21} = 0$. Without loss of generality we may assume that $a_{21} = 0$. Let $\varphi \in Aut_F(M_2(F))$ such that $\varphi(x) = (1 + e_{21})x(1 - e_{21})$. In particular, we have

$$\varphi(a) = (a_{11} - a_{12})e_{11} + a_{12}e_{12} + (a_{11} - a_{12} - a_{22})e_{21} + (a_{12} + a_{22})e_{22}.$$

Since $([\varphi(a), x^k]_2)^n = 0$ for all $x \in R$, as above we can get that $a_{12}(a_{11} - a_{12} - a_{22}) = 0$. That is, either $a_{12} = 0$ or $a_{11} - a_{12} - a_{22} = 0$. If $a_{11} - a_{12} - a_{22} = 0$, then

$$[a, e_{11} + e_{21}]_2 = -a_{12}e_{11} + a_{12}e_{12} - 3a_{12}e_{21} + a_{12}e_{22}.$$

By assumption we get

$$0 = ([a, e_{11} + e_{21}]_2)^{2n} = (-2a_{12}^2)^n e_{11} + (-2a_{12}^2)^n e_{22}$$

and so $a_{12} = 0$, this implies that a is a diagonal matrix.

Write $a = \sum_{i=1}^{2} a_{ii}e_{ii}$, we see as above that $\varphi(a) = \sum_{i=1}^{2} a_{ii}e_{ii} + (a_{11} - a_{22})e_{21}$ is also a diagonal matrix. Therefore $a_{11} = a_{22}$ and so $a \in F \cdot I_2$ as desired. ☐

If $([a, x^k]_2)^n \in F \cdot I_2$ for all $x \in M_2(F)$, one can not expect to obtain that $a \in F \cdot I_2$. For example, it is easy to check that $([e_{11}, x]_2)^2 \in F \cdot I_2$ for all $x \in M_2(F)$.

Lemma 2. *Let $R = M_m(F)$, the ring of all $m \times m$ matrices over a field F with $\mathrm{char}(F) \neq 2$. If a is a noncentral element of R such that $([a, x^k]_2)^n \in F \cdot I_m$ for all $x \in R$, then $m \leq 2$.*

Proof. Suppose on the contrary that $m > 2$. Let $a = \sum a_{ij}e_{ij}$ with $a_{ij} \in F$. Write $a = \left(\begin{smallmatrix} a_{11} & A \\ B & C \end{smallmatrix}\right)$, where $A = (a_{12}, \dots, a_{1m})$, $B = (a_{21}, \dots, a_{m1})^T$, and $C = (a_{ij})$ with $2 \leq i, j \leq m$. Since $[a, e_{11}]_2 = \left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right)$, by assumption we have

$$([a, e_{11}]_2)^{2n} = \begin{pmatrix} (AB)^n & 0 \\ 0 & (BA)^n \end{pmatrix} \in F \cdot I_m.$$

Set $\alpha = AB \in F$. Then $\left(\begin{smallmatrix} \alpha^n & 0 \\ 0 & \alpha^{n-1}BA \end{smallmatrix}\right) \in F \cdot I_m$. If $\alpha \neq 0$, then

$$\begin{pmatrix} \alpha & 0 \\ 0 & BA \end{pmatrix} \in F \cdot I_m.$$

Thus, $\alpha = a_{21}a_{12} = a_{31}a_{13}$ and $a_{21}a_{13} = 0$. Thus $\alpha = 0$, a contradiction. Hence $AB = 0$.

Let φ_{ij} be an inner automorphism of R given by $\varphi_{ij}(x) = (1 + e_{ij})x(1 - e_{ij})$ for all $x \in R$. Write $1 + e_{21} = \left(\begin{smallmatrix} 1 & 0 \\ E_2 & I_{m-1} \end{smallmatrix}\right)$, where $E_2 = (1, 0, \dots, 0)^T$ and I_{m-1} is the $(m-1)$-identity matrix. So

$$\varphi_{21}(a) = \begin{pmatrix} a_{11} - a_{12} & A \\ a_{11}E_2 - a_{12}E_2 + B - CE_2 & E_2A + C \end{pmatrix}.$$

Since $([\varphi_{21}(a), x^k]_2)^n \in F \cdot I_m$ for all $x \in R$, as above we have

$$A(a_{11}E_2 - a_{12}E_2 + B + CE_2) = 0.$$

Recalling $AB = 0$ we get from the last relation that $a_{11}a_{12} - a_{12}^2 - ACE_2 = 0$.

Since $[a, e_{11} + e_{21}]_2 = \left(\begin{smallmatrix} -a_{12} & A \\ D & E_2A \end{smallmatrix}\right)$, where $D = B + CE_2 - (a_{11} + 2a_{12})E_2$, we get

$$([a, e_{11} + e_{21}]_2)^2 = \begin{pmatrix} a_{12}^2 + AD & 0 \\ -a_{12}D + E_2AD & DA + a_{12}E_2A \end{pmatrix}.$$

Making use of both $AB = 0$ and $a_{11}a_{12} - a_{12}^2 - ACE_2 = 0$ we get $AD = -3a_{12}^2$. Thus

$$([a, e_{11} + e_{21}]_2)^2 = \begin{pmatrix} -2a_{12}^2 & 0 \\ -a_{12}D - 3a_{12}^2E_2 & DA + a_{12}E_2A \end{pmatrix},$$

Therefore

$$([a, e_{11} + e_{21}]_2)^{2n} = \begin{pmatrix} (-2a_{12}^2)^n & 0 \\ U & (DA + a_{12}E_2A)^n \end{pmatrix} \in F \cdot I_m$$

where U is a $(m-1) \times 1$ matrix. Since $\text{rank}((DA + a_{12}E_2A)^n) \leq \text{rank}(A) \leq 1$ and $m > 2$, we infer that $(-2a_{12}^2)^n = 0$ and so $a_{12} = 0$.

Now we claim that a is a diagonal matrix. Since $([\varphi_{j2}(a), x^k]_2)^n \in F \cdot I_m$ for all $x \in R$, where $j > 2$, as above we have that $-a_{1j} = \varphi_{j1}(a)_{12} = 0$. So $a_{1j} = 0$ for $j > 1$. For $1 < j < t \leq m$, as above we get from $([\varphi_{1j}(a), x^k]_2)^n \in F \cdot I_m$ for all $x \in R$, that $a_{jt} = \varphi_{1j}(a)_{1t} = 0$. This shows that a must be lower triangular. Since the transpose of a satisfies the same condition, a is indeed diagonal.

We have showed that $a = \sum_{i=1}^m a_{ii}e_{ii}$ with $a_{ii} \in F$. For $1 \leq i \neq j \leq m$, as above we get that $\varphi_{ij}(a)$ is a diagonal matrix. On the other hand $\varphi(a) = a + (a_{jj} - a_{ii})e_{ij}$, we infer that $a_{jj} = a_{ii}$ and so a is central in R, which is a contradiction. The proof is thereby complete. □

The following result is a special case of Theorem 1, which is of independent interest.

Lemma 3. *Let R be a prime ring with $\text{char}(R) \neq 2$ and I a nonzero ideal of R, d a nonzero derivation of R such that $[d(x^k), x^k]^n = 0$ for all $x \in I$ where k, n are fixed positive integers. Then R is commutative.*

Proof. By assumption we see that I satisfies the differential identity

$$\left[\sum_{i=0}^{k-1} x^i d(x) x^{k-i-1}, x^k\right]^n = 0.$$

If d is not Q-inner, by KHARCHENKO's theorem [7], I satisfies the polynomial identity $\big[\sum_{i=0}^{k-1} x^i y x^{k-i-1}, x^k\big]^n = 0$ and so for R too. It is well known that there exists a field F such that R and F_m satisfy the same polynomial identities [8, p. 57 and p. 89]. Suppose that $m \geq 2$. If we choose $x = e_{11}, y = e_{12} + e_{21}$, then we get a contradiction as follows

$$0 = \left[\sum_{i=0}^{k-1} e_{11}^i (e_{12} + e_{21}) e_{11}^{k-i-1}, e_{11}\right]^{2n} = [e_{12} + e_{21}, e_{11}]^{2n} = (-1)^n (e_{11} + e_{22}) \neq 0.$$

Thus $m = 1$ and so R is commutative.

Assume next that d is Q-inner, that is, $d(x) = [a, x]$ for all $x \in R$, where a is a noncentral element in Q. By assumption we get $([a, x^k]_2)^n = 0$ for all $x \in I$. By a theorem of CHUANG [4, Theorem 2], $([a, x^k]_2)^n = 0$ for all $x \in Q$. In case C is infinite, we have $([a, x^k]_2)^n = 0$ for all $x \in Q \otimes_C \bar{C}$, where $\bar{C}$ is the algebraic closure of C. Since both Q and $Q \otimes_C \bar{C}$ are centrally closed [5, Theorems 2.5 and 3.5], we may replace R by Q or $Q \otimes \bar{C}$ according as C is finite or infinite. Thus we may assume that R is centrally closed over C which is either finite or algebraically closed and $([a, x^k]_2)^n = 0$ for all $x \in R$. By MARTINDALE's theorem [13], R is a primitive ring and so isomorphic to a dense subring of linear transformations on a vector space V over C.

If V is infinite dimensional over C, for any given $v \in V$ we claim that v and va are C-dependent. Suppose on the contrary that v and va are C-independent. We choose $v_1, \ldots, v_{2k-1}$ such that $v, va, v_1, \ldots, v_{2k-1}$ are C-independent. By the density of R on ${}_C V$, there exists $x \in R$ such that $vx = 0$, $vax = v_1$, $v_i x = v_{i+1}$, $v_{2k-1} x = v$, where $i = 1, \ldots, 2k-2$. Thus

$$v[a, x^k]_2 = vax^{2k} = v_1 x^{2k-1} = \cdots = v_{2k-1} x = v$$

and so $0 = v([a, x^k]_2)^n = v$, a contradiction. Therefore v and va are C-dependent for any $v \in V$. A standard argument shows that $a \in C$, a contradiction. So V must be of finite dimension. That is, $R \cong M_s(C)$ for some s. In view of both Lemma 1 and Lemma 2 we get that $a \in C$, a contradiction. The proof is now complete. □

THE PROOF OF THEOREM 1. Suppose on the contrary that $\dim_C RC > 4$. By assumption we have $[d(x^k), x^k]^n \in Z$ for all $x \in I$, that is, I satisfies the following differential identity

$$\left[\left[\sum_{i=0}^{k-1} x^i d(x) x^{k-i-1}, x^k\right]^n, y\right] = 0. \tag{1}$$

If $[d(x^k), x^k]^n = 0$ for all $x \in I$, the result follows from Lemma 3. Otherwise, there exists $r \in I$ such that $[d(r^k), r^k]^n \neq 0$. Thus I satisfies the central differential identity $[d(x^k), x^k]^n$. By [3, Theorem 1] we get that R is a PI-prime ring and so is Q.

If d is not Q-inner, applying Kharchenko's theorem to (1) we get that I satisfies the polynomial identity $\big[\big[\sum_{i=0}^{k-1} x^i y x^{k-i-1}, x^k\big]^n, r\big] = 0$ and so for R too. It is well known that there exists a field F such that R and F_m satisfy the same polynomial identities. Thus $\big[\sum_{i=0}^{k-1} x^i y x^{k-i-1}, x^k\big]^n \in F \cdot I_m$. Note that $m > 2$. If we choose $x = e_{11}, y = e_{12} + e_{21}$, then

$$\left[\sum_{i=0}^{k-1} e_{11}^i (e_{12} + e_{21}) e_{11}^{k-i-1}, e_{11}^k\right]^{2n} = (-1)^n (e_{11} + e_{22}) \in F \cdot I_m.$$

This is a contradiction.

We next assume that d is an Q-inner derivation induced by a noncentral element $b \in Q$. It follows from (1) that

$$\left[\left[\sum_{i=0}^{k-1} x^i [b, x] x^{k-i-1}, x^k\right]^n, y\right] = 0 \quad \text{for all } x, y \in I. \tag{2}$$

In view of [4, Theorem 2] we have

$$\left[\left[\sum_{i=0}^{k-1} x^i [b, x] x^{k-i-1}, x^k\right]^n, y\right] = 0 \quad \text{for all } x, y \in Q. \tag{3}$$

Since there exists $r \in R$ such that $[d(r^k), r^k]^n \neq 0$, we see that (3) is a non trivial generalized polynomial identity on Q. By Martindale's theorem [13] Q is primitive ring. Since Q is a PI-ring, by the famous Kaplanksy's theorem [1, Theorem 6.1.10] we see that Q is a finite dimensional central simple algebra over C. It follows from [10, Lemma 2] that there exists a suitable field F of $\mathrm{char}(F) \neq 2$ such that $Q \subseteq M_m(F)$ and moreover $M_m(F)$ satisfies the same generalized polynomial identity (3). Then Lemma 2 tells us that $m \leq 2$, which is a contradiction. The proof is thereby complete. □

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