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Stability of the entropy equation

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Abstract. In this paper we prove that the so-called entropy equation, i.e.,

$$H(x, y, z) = H(x + y, 0, z) + H(x, y, 0)$$

is stable in the sense of Hyers and Ulam on the positive cone of $\mathbb{R}^3$, assuming that the function H is approximatively symmetric in each variable and approximatively homogeneous of degree α, where α is an arbitrarily fixed real number.

1. Introduction and preliminaries

The stability theory of functional equation originates from a famous question of S. M. Ulam concerning the additive Cauchy equation. He asked whether is it true that the solution of the additive Cauchy equation differing slightly from a given one, must of necessity be close to the solution of this equation. In 1941 D. H. Hyers gave an affirmative answer to the previous question. Nowadays this result (see [7]) is referred to as *the stability of the Cauchy equation*. Since then the stability theory of functional equations has become a developing field of research, see e.g., [4], [5], [11] and their references.

In the theory of stability there exist several methods. From our point of view however the method of invariant means plays a key role. Concerning this topic we offer the expository paper Day [3]. Although the only result needed from [3]

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is, that on every commutative semigroup there exist an invariant mean, that is, every commutative semigroup is *amenable*.

In what follows, denote $\mathbb{R}$ the set of the real numbers, furthermore, on the symbols $\mathbb{R}_{+}$ and $\mathbb{R}_{++}$ we understand the set of the nonnegative and the positive real numbers, respectively.

The aim of this paper is to prove that *the entropy equation*, i.e., equation

$$H(x, y, z) = H(x + y, 0, z) + H(x, y, 0) \tag{1.1}$$

is stable on $\mathbb{R}^3_{++}$.

In [9] A. Kamiński and J. Mikusiński determined the continuous and 1-homogeneous solutions of equation (1.1). This result was strengthened by J. Aczél in [1], by proving the following.

Theorem 1.1. *Let*

$$D = \left\{(x, y, z) \in \mathbb{R}^3 \mid x \geq 0, y \geq 0, z \geq 0, x + y + z > 0\right\}.$$

Assume that the function $H : D \to \mathbb{R}$ is symmetric, positively homogeneous of degree 1, satisfies the functional equation

$$H(x, y, z) = H(x + y, z, 0) + H(x, y, 0)$$

on D and the map $x \longmapsto H(1 - x, x, 0)$ is either continuous at a point or bounded on an interval or integrable on the closed subintervals of $]\,0, 1\,[$ or measurable on $]\,0, 1\,[$. Then, and only then

$$H(x, y, z) = x \log(x) + y \log(y) + z \log(z) - (x + y + z) \log(x + y + z)$$

holds for all $(x, y, z) \in D$, with arbitrary basis for the logarithm and with the convention $0 \cdot \log(0) = 0$.

Using a result of Jessen–Karpf–Thorup [8], which concerns the solution of the cocycle equation, Z. Daróczy proved the following (see [2]).

Theorem 1.2. *If a function $H : D \to \mathbb{R}$ is symmetric in D and satisfies the equation (1.1) in the interior of D and the map $(x, y) \mapsto H(x, y, 0)$ is positively homogeneous (of order 1) for all $x, y \in \mathbb{R}_{++}$, then there exists a function $\varphi : \mathbb{R}_{++} \to \mathbb{R}$ such that*

$$\varphi(xy) = x\varphi(y) + y\varphi(x)$$

holds for all $x, y \in \mathbb{R}_{++}$ and

$$H(x, y, z) = \varphi(x + y + z) - \varphi(x) - \varphi(y) - \varphi(z)$$

for all $(x, y, z) \in D$.

During the proof of the main result the stability of the *cocycle equation* is needed. This theorem can be found in [12].

Theorem 1.3. *Let S be a right amenable semigroup and let $F : S \times S \to \mathbb{C}$ be a function, for which the function*

$$(x, y, z) \longmapsto F(x, y) + F(x + y, z) - F(x, y + z) - F(y, z) \tag{1.2}$$

is bounded on $S \times S \times S$. Then there exists a function $\Psi : S \times S \to \mathbb{C}$ satisfying the cocycle equation, i.e.,

$$\Psi(x, y) + \Psi(x + y, z) = \Psi(x, y + z) + \Psi(y, z) \tag{1.3}$$

for all $x, y, z \in S$ and for which the function $F - \Psi$ is bounded by the same constant as the map defined by (1.2)*.*

About the symmetric, 1-homogeneous solutions of the cocycle equation one can read in [8]. Furthermore, the symmetric and α-homogeneous solutions of equation (1.3) can be found in [10], as a consequence of Theorem 3. The general solution of the cocycle equation without symmetry and homogeneity assumptions, on cancellative abelian semigroups was determined by M. Hosszú in [6].

2. The main result

Our main result is the following.

Theorem 2.1. *Let ε_1, ε_2, ε_3 be arbitrary nonnegative real numbers, $\alpha \in \mathbb{R}$, and assume that the function $H : D \to \mathbb{R}$ satisfies the following system of inequalities.*

$$|H(x, y, z) - H(\sigma(x), \sigma(y), \sigma(z))| \leq \varepsilon_1 \tag{2.1}$$

for all $(x, y, z) \in D$ and for all $\sigma : \{x, y, z\} \mapsto \{x, y, z\}$ permutation;

$$|H(x, y, z) - H(x + y, 0, z) - H(x, y, 0)| \leq \varepsilon_2 \tag{2.2}$$

for all $(x, y, z) \in D^\circ$, where D° denotes the interior of the set D;

$$|H(tx, ty, 0) - t^\alpha H(x, y, 0)| \leq \varepsilon_3 \tag{2.3}$$

holds for all $t, x, y \in \mathbb{R}_{++}$. Then, in case $\alpha = 1$ there exists a function $\varphi : \mathbb{R}_{++} \to \mathbb{R}$ which satisfies the functional equation

$$\varphi(xy) = x\varphi(y) + y\varphi(x), \quad (x, y \in \mathbb{R}_{++})$$

and

$$|H(x,y,z)-[\varphi(x+y+z)-\varphi(x)-\varphi(y)-\varphi(z)]|\leq\varepsilon_1+\varepsilon_2 \tag{2.4}$$

holds for all $(x,y,z)\in D^{\circ}$*; in case* $\alpha=0$ *there exists a constant* $a\in\mathbb{R}$ *such that*

$$|H(x,y,z)-a|\leq 8\varepsilon_3+25\varepsilon_2+49\varepsilon_1 \tag{2.5}$$

for all $(x,y,z)\in D^{\circ}$*; finally, in all other cases there exists a constant* $c\in\mathbb{R}$ *such that*

$$|H(x,y,z)-c\left[(x+y+z)^{\alpha}-x^{\alpha}-y^{\alpha}-z^{\alpha}\right]|\leq\varepsilon_1+\varepsilon_2 \tag{2.6}$$

holds on D°*.*

Proof. Using inequality (2.3) we will show that the map $(x,y)\mapsto H(x,y,0)$ is homogeneous of degree α, in fact, assuming that $\alpha\neq 0$. Due to (2.3)

$$\left|H(x,y,0)-\frac{H(tx,ty,0)}{t^{\alpha}}\right|\leq\frac{\varepsilon_3}{t^{\alpha}}$$

holds for all $t,x,y\in\mathbb{R}_{++}$. Hence, if we define

$$t_0=\begin{cases}0 & \text{if } \alpha<0\\ +\infty & \text{if } \alpha>0,\end{cases}$$

then for all $x,y\in\mathbb{R}_{++}$

$$\lim_{t\to t_0}t^{-\alpha}H(tx,ty,0)=H(x,y,0). \tag{2.7}$$

Thus we have for arbitrary $s,x,y\in\mathbb{R}_{++}$

$$\begin{aligned}H(sx,sy,0)&=\lim_{t\to t_0}t^{-\alpha}H(tsx,tsy,0)=\lim_{t\to t_0}t^{-\alpha}s^{-\alpha}H((ts)x,(ts)y,0)s^{\alpha}\\&=s^{\alpha}\lim_{t\to t_0}(ts)^{-\alpha}H((ts)x,(ts)y,0)=s^{\alpha}H(x,y,0).\end{aligned} \tag{2.8}$$

Therefore, the map $(x,y)\mapsto H(x,y,0)$ is homogeneous of degree α, indeed.

In what follows, we will investigate inequalities (2.1) and (2.2). Interchanging x and z in (2.2), we obtain that

$$|H(z,y,x)-H(z+y,0,x)-H(z,y,0)|\leq\varepsilon_2 \tag{2.9}$$

is satisfied for all $(x,y,z)\in D^{\circ}$. Inequalities (2.1), (2.2), (2.9) and the triangle inequality imply that

$$|H(x+y,0,z)+H(x,y,0)-H(y+z,0,x)-H(z,y,0)|\leq 2\varepsilon_2+\varepsilon_1 \tag{2.10}$$

is fulfilled for all $(x, y, z) \in D^{\circ}$. Applying inequality (2.1) three times, from (2.10), we get that

$$|H(x+y, z, 0) + H(x, y, 0) - H(x, y+z, 0) - H(y, z, 0)| \leq 2\varepsilon_2 + 4\varepsilon_1 \quad (2.11)$$

holds on D°. Therefore, the function F defined by

$$F(x, y) = H(x, y, 0) \quad (x, y \in \mathbb{R}_{++}) \quad (2.12)$$

satisfies

$$|F(x, y) - F(y, x)| \leq \varepsilon_1, \quad (x, y \in \mathbb{R}_{++}) \quad (2.13)$$

due to inequality (2.1). Since, for the function H inequality (2.11) holds, we receive that

$$|F(x+y, z) + F(x, y) - F(x, y+z) - F(y, z)| \leq 2\varepsilon_2 + 4\varepsilon_1. \quad (x, y, z \in \mathbb{R}_{++}) \quad (2.14)$$

We have just proved that in case $\alpha \neq 0$, $H(x, y, 0)$ is homogeneous of degree α, therefore

$$F(tx, ty) = t^{\alpha} F(x, y) \quad (\alpha \neq 0, t, x, y \in \mathbb{R}_{++}) \quad (2.15)$$

furthermore, because of (2.12) and (2.3), we obtain that

$$|F(tx, ty) - F(x, y)| \leq \varepsilon_3, \quad (t, x, y \in \mathbb{R}_{++}) \quad (2.16)$$

in case $\alpha = 0$.

The set D° is a commutative semigroup with the usual addition. Thus it is amenable, as well. Therefore, by Theorem 1.3., there exists a function $G : \mathbb{R}^2_{++} \to \mathbb{R}$ which is a solution of the cocycle equation, and for which

$$|F(x, y) - G(x, y)| \leq 2\varepsilon_2 + 4\varepsilon_1 \quad (2.17)$$

holds for all $x, y \in \mathbb{R}_{++}$. Additionally, by a result of [6] there exist a function $f : \mathbb{R}_{++} \to \mathbb{R}$ and a function $B : \mathbb{R}^2_{++} \to \mathbb{R}$ which satisfies the following system

$$\begin{aligned} B(x+y, z) &= B(x, z) + B(y, z), \\ B(x, y) + B(y, x) &= 0, \end{aligned} \qquad (x, y, z \in \mathbb{R}_{++})$$

such that

$$G(x, y) = B(x, y) + f(x+y) - f(x) - f(y). \quad (x, y \in \mathbb{R}_{++})$$

All in all, this means that

$$|F(x,y) - (B(x,y) + f(x+y) - f(x) - f(y))| \leq 2\varepsilon_2 + 4\varepsilon_1 \tag{2.18}$$

holds for all $x, y \in \mathbb{R}_{++}$.

Using the above properties of the function B, we will show that B is identically zero on $\mathbb{R}^2_{++}$. Indeed, by reason of the triangle inequality and (2.18),

$$\begin{aligned}|2B(x,y)| = |B(x,y) - B(y,x)| &\leq |F(x,y) - (B(x,y) + f(x+y) - f(x) - f(y))| \\ &+ |F(y,x) - (B(y,x) + f(y+x) - f(y) - f(x))| \\ &+ |F(x,y) - F(y,x)| \leq (2\varepsilon_2 + 4\varepsilon_1) + (2\varepsilon_2 + 4\varepsilon_1) + \varepsilon_1 = 4\varepsilon_2 + 9\varepsilon_1\end{aligned}$$

is fulfilled for all $x, y \in \mathbb{R}_{++}$. Thus B is bounded on the set $\mathbb{R}^2_{++}$. On the other hand, B is biadditive. However, only the identically zero function has these properties. Therefore, $B \equiv 0$ on $\mathbb{R}_{++}$.

Then we get that the function $K : \mathbb{R}^2_{++} \to \mathbb{R}$ defined by

$$K(x,y) = F(x,y) - G(x,y) \quad (x, y \in \mathbb{R}_{++})$$

is bounded on $\mathbb{R}^2_{++}$ by $2\varepsilon_2 + 4\varepsilon_1$, in view of inequality (2.18). In case $\alpha \neq 0$, it can be seen that

$$K(tx, ty) = F(tx, ty) - G(tx, ty), \quad (t, x, y \in \mathbb{R}_{++})$$

furthermore, making use of (2.15)

$$K(tx, ty) = t^{\alpha} F(x,y) - G(tx, ty)$$

holds for all $t, x, y \in \mathbb{R}_{++}$. Rearranging this,

$$\frac{K(tx, ty)}{t^{\alpha}} = F(x,y) - \frac{G(tx, ty)}{t^{\alpha}}$$

for all $t, x, y \in \mathbb{R}_{++}$. Since the function K is bounded on $\mathbb{R}^2_{++}$, we receive that

$$F(x,y) = \lim_{t \to t_0} \frac{G(tx, ty)}{t^{\alpha}}. \quad (x, y \in \mathbb{R}_{++})$$

Because of the symmetry of the function G, the function F is symmetric, as well. Furthermore, G satisfies the cocycle equation on $\mathbb{R}_{++}$, that is,

$$G(x+y, z) + G(x,y) = G(x, y+z) + G(y,z), \quad (x, y, z \in \mathbb{R}_{++})$$

especially,

$$\frac{G(tx+ty,tz)}{t^{\alpha}}+\frac{G(tx,ty)}{t^{\alpha}}=\frac{G(tx,ty+tz)}{t^{\alpha}}+\frac{G(ty,tz)}{t^{\alpha}}$$

is also satisfied for all $t, x, y, z \in \mathbb{R}_{++}$. Taking the limit $t \to t_0$ we obtain that

$$F(x+y,z)+F(x,y)=F(x,y+z)+F(y,z). \quad (x,y,z\in\mathbb{R}_{++})$$

This means that also the function F satisfies the cocycle equation on $\mathbb{R}^2_{++}$. Additionally, F is homogeneous of degree α ($\alpha \neq 0$) and symmetric. Using Theorem 5. in [8], in case $\alpha = 1$, and a result of [10] in all other cases, we get that

$$F(x,y)=\begin{cases} c\left[(x+y)^{\alpha}-x^{\alpha}-y^{\alpha}\right], & \text{if} \quad \alpha\notin\{0,1\} \\ \varphi\left(x+y\right)-\varphi(x)-\varphi(y), & \text{if} \quad \alpha=1 \end{cases} \tag{2.19}$$

where the function $\varphi : \mathbb{R}_{++} \to \mathbb{R}$ satisfies the functional equation

$$\varphi\left(xy\right)=x\varphi(y)+y\varphi(x)$$

for all $x, y \in \mathbb{R}_{++}$, and $c \in \mathbb{R}$ is a constant. In view of the definition of the function F, this yields that

$$H(x,y,0)=c\left[(x+y)^{\alpha}-x^{\alpha}-y^{\alpha}\right] \tag{2.20}$$

for all $x, y \in \mathbb{R}_{++}$ in case $\alpha \notin \{0, 1\}$, and

$$H(x,y,0)=\varphi\left(x+y\right)-\varphi(x)-\varphi(y) \tag{2.21}$$

for all $x, y \in \mathbb{R}_{++}$ in case $\alpha = 1$. Finally, inequalities (2.1), (2.2) and equation (2.20) imply that

$$\begin{aligned} &|H(x,y,z)-c[(x+y+z)^{\alpha}-x^{\alpha}-y^{\alpha}-z^{\alpha}]|\leq|H(x,y,z)-H(x+y,0,z)\\ &\quad-H(x,y,0)|+|H(x+y,0,z)-H(x+y,z,0)|\\ &\quad+|H(x+y,z,0)-c[(x+y+z)^{\alpha}-(x+y)^{\alpha}-z^{\alpha}]|\\ &\quad+|H(x,y,0)-c[(x+y)^{\alpha}-x^{\alpha}-y^{\alpha}]|\leq\varepsilon_1+\varepsilon_2 \end{aligned} \tag{2.22}$$

for all $x, y, z \in \mathbb{R}_{++}$, if $\alpha \notin \{0, 1\}$, and by reason of inequalities (2.1), (2.2) and equation (2.21) we obtain that

$$\begin{aligned} &|H(x,y,z)-(\varphi\left(x+y+z\right)-\varphi(x)-\varphi(y)-\varphi(z))|\leq|H(x,y,z)-H(x+y,0,z)\\ &\quad-H(x,y,0)|+|H(x+y,0,z)-H(x+y,z,0)| \end{aligned}$$

$$+\left|H(x+y,z,0)-(\varphi(x+y+z)-\varphi(x+y)-\varphi(z))\right|$$
$$+\left|H(x,y,0)-(\varphi(x+y)-\varphi(x)-\varphi(y))\right|\leq\varepsilon_1+\varepsilon_2 \tag{2.23}$$

for all $x,y,z\in\mathbb{R}_{++}$, if $\alpha=1$.

In case $\alpha=0$, we get from (2.16), that particularly

$$|F(x,y)-F(2x,2y)|\leq\varepsilon_3. \quad (x,y\in\mathbb{R}_{++})$$

Thus inequality (2.17) implies

$$|F(x,y)-G(2x,2y)|$$
$$\leq|F(x,y)-F(2x,2y)|+|F(2x,2y)-G(2x,2y)|\leq\varepsilon_3+2\varepsilon_2+4\varepsilon_1$$

holds for all $x,y\in\mathbb{R}_{++}$. On the other hand

$$|G(2x,2y)-[2G(x,y)-F(1,1)]|\leq3(\varepsilon_3+2\varepsilon_2+4\varepsilon_1)$$

for all $x,y\in\mathbb{R}_{++}$, since,

$$|F(1,1)-G(x,x)|$$
$$\leq|F(1,1)-F(x,x)|+|F(x,x)-G(x,x)|\leq\varepsilon_3+2\varepsilon_2+4\varepsilon_1$$

where we used inequalities (2.16) and (2.18). All in all, this means that

$$|F(x,y)-F(1,1)|\leq|G(2x,2y)-F(x,y)|+|G(2x,2y)-2G(x,y)+F(1,1)|$$
$$+2|G(x,y)-F(x,y)|\leq\varepsilon_3+2\varepsilon_2+4\varepsilon_1+3(\varepsilon_3+2\varepsilon_2+4\varepsilon_1)$$
$$+2(2\varepsilon_2+4\varepsilon_1)=4\varepsilon_3+12\varepsilon_2+24\varepsilon_1 \tag{2.24}$$

is fulfilled for all $x,y\in\mathbb{R}_{++}$.

Due to the definition of the function F,

$$|H(x,y,0)-F(1,1)|\leq4\varepsilon_3+12\varepsilon_2+24\varepsilon_1 \tag{2.25}$$

for all $x,y\in\mathbb{R}_{++}$, where we used inequality (2.24). Finally, in view of (2.1), (2.2) and (2.25) we obtain that

$$|H(x,y,z)-2F(1,1)|\leq|H(x,y,z)-H(x+y,0,z)-H(x,y,0)|$$
$$+|H(x+y,0,z)-H(x+y,z,0)|+|H(x+y,z,0)-F(1,1)|$$
$$+|H(x,y,0)-F(1,1)|\leq\varepsilon_2+\varepsilon_1+(4\varepsilon_3+12\varepsilon_2+24\varepsilon_1)$$
$$+(4\varepsilon_3+12\varepsilon_2+24\varepsilon_1)=8\varepsilon_3+25\varepsilon_2+49\varepsilon_1 \tag{2.26}$$

holds for all $(x,y,z)\in D^{\circ}$. Let $a=2F(1,1)$ to get the desired inequality. $\square$

With the choice $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0$ one can recognize the solutions of equation (1.1).

Corollary 2.1. *Assume that the function $H : D \to \mathbb{R}$ is symmetric, homogeneous of degree α, where $\alpha \in \mathbb{R}$ is arbitrary but fixed. Furthermore, suppose that H satisfies equation* (1.1) *on the set D°. Then, in case $\alpha = 1$ there exists a function $\varphi : \mathbb{R}_{++} \to \mathbb{R}$ which satisfies the functional equation*

$$\varphi(xy) = x\varphi(y) + y\varphi(x), \quad (x, y \in \mathbb{R}_{++})$$

and

$$H(x,y,z) = \varphi(x+y+z) - \varphi(x) - \varphi(y) - \varphi(z) \tag{2.27}$$

holds for all $(x,y,z) \in D^\circ$; in all other cases there exists a constant $c \in \mathbb{R}$ such that

$$H(x,y,z) = c\left[(x+y+z)^\alpha - x^\alpha - y^\alpha - z^\alpha\right] \tag{2.28}$$

holds on D°.

Remark 2.1. Our theorem says that the entropy equation is stable in the sense of Hyers and Ulam.

Remark 2.2. In 2005 an article of J. Tabor and J. Tabor has appeared with exactly the same title as that of the present paper. However, in [13] the Hyers–Ulam stability of the functional equation

$$L\left(\sum_{j=1}^{3} k_j f(p_j)\right) = \sum_{j=1}^{3} k_j g(p_j)$$

is proved, for the Banach space X, $0 \le p_j \le 1$, $k_j \in \mathbb{N} \cup \{0\}$, $\sum_{j=1}^{3} k_j p_j = 1$, where $f : [0,1] \to \mathbb{R}_+$, $g : [0,1] \to X$ and $L : \mathbb{R}_+ \to X$ unknown continuous functions satisfying some additional conditions.

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