Publ. Math. Debrecen
77/1-2 (2010), 211–224

Fibonacci numbers which are sums of three factorials

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Abstract. In this paper, we prove that $F_7 = 13 = 1! + 3! + 3!$ is the largest Fibonacci number expressible as a sum of three factorials.

1. Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. In [5], it is shown that if $k \geq 1$ is any fixed positive integer, then the Diophantine equation

$$F_n = m_1! + m_2! + \cdots + m_k! \tag{1}$$

has at most finitely many positive integer solutions $(n, m_1, \ldots, m_k)$ which are all effectively computable. When $k = 1$, it is an easy consequence of the Primitive Divisor theorem [3] that the largest such solution is $F_3 = 2!$ (see [6] and [8] for more general variants of this Diophantine equation). When $k = 2$, the largest such solution is $F_{12} = 4! + 5!$ (see [5]). Some variants of this problem appear in [1], where for the case $k = 3$ it was shown that $n < \mathrm{e}^{53}$. Here, we find all solutions of equation (1) when $k = 3$.

Theorem 1. *The only solutions of the Diophantine equation*

$$F_n = m_1! + m_2! + m_3!, \quad 1 \leq m_1 \leq m_2 \leq m_3, \tag{2}$$

Mathematics Subject Classification: 11D61.
Key words and phrases: Fibonacci numbers, primitive divisors, factorials, applications of linear forms in logarithms.
F. L. was supported in part by Grants SEP-CONACyT 79685 and PAPIIT 100508.

are

$$F_4 = 1! + 1! + 1!, \quad F_5 = 1! + 2! + 2!, \quad F_6 = 1! + 1! + 3!, \quad F_7 = 1! + 3! + 3!.$$

We point out that with the rôles of the Fibonacci numbers and the factorials interchanged, it was shown in [7] that

$$6! = F_{15} + F_{11} + F_9 = F_{15} + F_{10} + F_{10}$$

give the largest positive integer solutions (n, m_1, m_2, m_3) of the Diophantine equation

$$n! = F_{m_1} + F_{m_2} + F_{m_3}.$$

Our argument is based on elementary properties of the Fibonacci sequence combined with some basic facts about biquadratic fields and with a 2-adic linear form in two logarithms due to Bugeaud and Laurent [2]. For technical reasons, we shall split the argument into two parts, according to whether $m_1 = 1, 2$, or $m_1 \geq 3$, where the second case is computationally harder. We start with the 2-adic argument.

Before proceeding to the proofs, we recall a few known facts about the Fibonacci sequence. Binet's formula says that

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{3}$$

holds for all $n \geq 0$, where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$ are the two roots of the characteristic equation $x^2 - x - 1 = 0$ of the Fibonacci sequence. The sequence of Lucas numbers $(L_n)_{n \geq 0}$ starts with $L_0 = 2$, $L_1 = 1$, and obeys the same recurrence relation $L_{n+2} = L_{n+1} + L_n$ for all $n \geq 0$ as the Fibonacci sequence. Its Binet formula is

$$L_n = \alpha^n + \beta^n \quad \text{for all } n \geq 0. \tag{4}$$

There are many formulas linking the Fibonacci and Lucas numbers such as $F_{2n} = F_n L_n$ and $L_n^2 - 5F_n^2 = 4(-1)^n$ valid for all $n \geq 0$. We shall freely use such formulas throughout the paper whenever needed.

2. A linear form in logarithms to the rescue

The following lemma will be useful in what follows. For a prime ideal π in a number field $\mathbb{L}$ and an algebraic integer m in $\mathbb{L}$ we write $\nu_\pi(m)$ for the exact order of π in the factorization of the principal fractional ideal generated by m in $\mathbb{L}$. When π is a prime integer, we understand that the underlying field $\mathbb{L}$ is the field $\mathbb{Q}$ of rational numbers.

Lemma 1. *Let N be a positive integer not of the form F_m for some positive integer m. Then for all positive integers $n \geq 3$ one has*

$$\nu_2(F_n - N) < 1730 \log(6N^2) \max\{10, \log n\}^2. \tag{5}$$

Proof. We use formula (3). Since $\beta = -\alpha^{-1}$, it follows that $\beta^n = \varepsilon\alpha^{-n}$, where $\varepsilon = (-1)^n \in \{\pm 1\}$. Then

$$F_n - N = \frac{\alpha^n - \varepsilon\alpha^{-n}}{\sqrt{5}} - N = \frac{\alpha^{-n}}{\sqrt{5}}\left((\alpha^n)^2 - \sqrt{5}N\alpha^n - \varepsilon\right) = \frac{\alpha^{-n}}{\sqrt{5}}(\alpha^n - z_1)(\alpha^n - z_2),$$

where

$$z_{1,2} = \frac{\sqrt{5}N \pm \sqrt{\Delta}}{2} \quad \text{with} \quad \Delta = 5N^2 + 4\varepsilon.$$

Write $\Delta = du^2$, where d is squarefree. Note that $d > 1$, since if not then $5N^2 + 4\varepsilon = u^2$, therefore $u^2 - 5N^2 = \pm 4$. However, it is well-known that all positive integer solutions (u, N) of the above Diophantine equation are of the form $(u, N) = (L_m, F_m)$ for some positive integer m, and by hypothesis N is a positive integer which is not a Fibonacci number.

Let $\mathbb{K}_1 = \mathbb{Q}[\sqrt{5}\,]$, $\mathbb{K}_2 = \mathbb{Q}[\sqrt{d}\,]$, $\mathbb{K}_3 = \mathbb{Q}[\sqrt{5d}\,]$ and $\mathbb{L} = \mathbb{K}_1\mathbb{K}_2$. Note that $\mathbb{L} = \mathbb{Q}[z_1, z_2] = \mathbb{Q}[\sqrt{5}, \sqrt{d}\,]$. Since $d > 1$ is coprime to 5 (because $\Delta \equiv \pm 4 \pmod 5$), it follows that $\mathbb{L}$ is of degree 4. The prime 2 is inert in $\mathbb{K}_1$, because the discriminant of $\mathbb{K}_1$ is 5 (so, congruent to 5 (mod 8)), but it cannot be inert in $\mathbb{L}$ since when d is odd, one of the numbers d or $5d$ is congruent to $\pm 1 \pmod 8$. Thus, in $\mathbb{L}$, we either have $2 = \pi_1\pi_2$, where π_1 and π_2 are distinct primes, or $2 = \pi^2$, according to whether d is odd or even, respectively.

Now we let π be any prime ideal dividing 2 in $\mathbb{L}$. As we have seen, it has $N_{\mathbb{L}/\mathbb{Q}}(\pi) = 4 = 2^f$ (so, $f = 2$), and if $\pi^e \| 2$, then $e \in \{1, 2\}$. Then

$$\nu_2(F_n - N) = \frac{\nu_\pi(F_n - N)}{e} = \frac{1}{e}\left(\nu_\pi(\alpha^n - z_1) + \nu_\pi(\alpha^n - z_2)\right). \tag{6}$$

Next, let a be maximal such that $\pi^a \mid \gcd_{\mathbb{L}}(\alpha^n - z_1, \alpha^n - z_2)$. Then

$$\pi^a \mid (z_1 - z_2) = \sqrt{\Delta}, \quad \text{so} \quad \pi^{2a} \mid \Delta. \tag{7}$$

Observe that if N is odd, then so is Δ. If $4 \mid N$, then $4 \| \Delta$. Finally, if $N = 2N_0$, where N_0 is odd, then

$$\Delta = 4(5N_0^2 \pm 1),$$

and $5N_0^2 \pm 1 \equiv 4, 6 \pmod 8$. Hence, in all cases we have that $\nu_2(\Delta) \le 4$. Now, write $\Delta = 2^{\nu_2(\Delta)}\ell = \pi^{e\nu_2(\Delta)}\gamma$, where γ is an ideal in $\mathbb{L}$ coprime to π. Then the divisibility relation (7) implies that $2a \le e\nu_2(\Delta)$, yielding $2a \le 4e \le 8$, therefore $a \le 4$.

Hence, the above arguments show that

$$\nu_2(F_n - N) \le \frac{1}{e}\left(\max\{\nu_\pi(\alpha^n - z_1), \nu_\pi(\alpha^n - z_2)\} + 4\right). \tag{8}$$

Now let $i = 1, 2$, and let us find an upper bound on

$$\nu_\pi(\alpha^n - z_i).$$

For this, we apply Corollary 1 on Page 315 in [2]. We take $\alpha_1 = \alpha$, $\alpha_2 = z_i$, $b_1 = n$, $b_2 = 1$, $p = 2$. To see that α_1 and α_2 are multiplicatively independent, assume that this is not so. Then $\alpha_1^u = \alpha_2^v$ holds for some integers u and v not both zero. We may assume (by squaring the above relation if necessary), that u and v are both even. But note that $\alpha_2^v = (z_i^2)^{v/2}$ belongs to $\mathbb{Q}[\sqrt{5d}\,]$, while $\alpha_1^u \in \mathbb{Q}[\sqrt{5}\,]$. Since they are both units (the inverse of z_1 is $-\varepsilon z_2$), it follows that α_1^u is a unit which belongs to both $\mathbb{Q}[\sqrt{5}\,]$ and $\mathbb{Q}[\sqrt{5d}\,]$ and since it is positive, we get that $\alpha_1^u = 1$. Hence, $\alpha_1^u = \alpha_2^v = 1$, leading to $u = v = 0$, which is false. With the notations from [2], we have that we can take $f = 2$, $g \le p^f - 1 = 3$, $D = [\mathbb{L} : \mathbb{Q}]/f = 2$, and A_1 and A_2 to be two positive real numbers such that

$$\log A_i \ge \max\left\{h(\alpha_i), \frac{\log 2}{2}\right\}, \qquad \text{for both } i = 1, 2.$$

Here, $h(\bullet)$ is the logarithmic height. Note that

$$h(\alpha_1) = h(\alpha) = \frac{\log((1 + \sqrt{5}\,)/2)}{2},$$

so we can take $\log A_1 = (\log 2)/2$. Furthermore, note that the conjugates of z_i are the four numbers

$$\frac{\pm\sqrt{5}N \pm \sqrt{5N^2 + 4\varepsilon}}{2},$$

of which two are of absolute values

$$\left|\frac{\sqrt{5}N - \sqrt{5N^2 + 4\varepsilon}}{2}\right| = \frac{4}{2(\sqrt{5}N + \sqrt{5N^2 + 4\varepsilon}\,)} < \frac{2}{\sqrt{5}N} < 1,$$

while the other two are of absolute values

$$\left|\frac{\sqrt{5}N+\sqrt{5N^2+4\varepsilon}}{2}\right| < \sqrt{5N^2+4} < \sqrt{6N^2},$$

so we can take

$$\log A_2 = \frac{2\log(\sqrt{6N^2}\,)}{4} = \frac{\log(6N^2)}{4}.$$

Finally, we take

$$b' = \frac{b_1}{D\log A_2} + \frac{b_2}{D\log A_1} = \frac{2n}{\log(6N^2)} + \frac{1}{\log 2}.$$

Then, Corollary 1 in [2] shows that

$$\begin{aligned}\left(\frac{f}{4}\right)\nu_\pi(\alpha^n - z_i) &\le \frac{24\cdot 2\cdot 3\cdot 2^4}{(\log 2)^4}\max\{10, \log b' + \log\log 2 + 0.4\}^2 \log A_1 \log A_2\\ &\le \frac{3^2\cdot 2^5}{(\log 2)^3}\log(6N^2)\max\{10, \log b' + 0.034\}^2.\end{aligned}$$

The factor $f/4$ above, not present in [2], arises for us because in the statements of [2] all valuations are normalized, so in particular the upper bounds from there apply to the normalized valuation $(f/4)\nu_\pi(\bullet)$. Note that

$$\begin{aligned}\log b' + 0.034 &= \log\left(\frac{2\mathrm{e}^{0.034}n}{\log(6N^2)} + \frac{\mathrm{e}^{0.034}}{\log 2}\right) < \log\left(\frac{2.1n}{\log(6N^2)} + 1.5\right)\\ &< \log\left(\frac{2.1n}{4.5} + 1.5\right) < \log n,\end{aligned}$$

where the above inequalities hold because $N \ge 4$ (N is not a Fibonacci number), and $n \ge 3$. Since $3^2\cdot 2^5/(\log 2)^3 < 864.9$ and $f = 2$, we get that

$$\frac{\nu_\pi(\alpha^n - z_i)}{2} < 864.9\log(6N^2)\max\{10, \log n\}^2.$$

The above inequality together with inequality (8) gives us that

$$\nu_2(F_n - N) \le \max\{\nu_\pi(\alpha^n - z_1), \nu_\pi(\alpha^n - z_2)\} + 4 < 1730\log(6N^2)\max\{10, \log n\}^2,$$

which is what we wanted. □

From now on, we distinguish two cases according to whether $m_1 = 1, 2$, or $m_1 \ge 3$. We first ran a short calculation with Mathematica which shows that if $n \le 100$, then the only solutions are the ones appearing in the statement of Theorem 1. From now on, we assume that $n > 100$ and our goal is to prove that there are no such solutions.

We continue with some elementary considerations about the situation when $m_1 \in \{3, 4\}$.

3. The case $m_1 \in \{3,4\}$

Assume first that $m_1 \geq 3$. Then $6 \mid F_n$ and in particular $12 \mid n$, therefore $144 = F_{12} \mid F_n$. This shows, for example, that $m_1 = 3$ and $m_2 \geq 4$ is impossible, for then 8 divides both F_n and $m_2! + m_3!$ but not $m_1!$. Similarly, if $m_1 = m_2 = 3$, then either $m_3 \geq 4$, which is impossible since then 8 divides both F_n and $m_3!$ but not $m_1! + m_2! = 12$, while if $m_1 = m_2 = m_3 = 3$, then the right hand side of equation (2) is 18 which is not a Fibonacci number.

Thus, $m_1 \geq 4$. The case $m_1 = 4$ and $m_2 \geq 6$ is impossible since then 9 divides both F_n and $m_2! + m_3!$ but not $m_1! = 24$. When $m_1 = m_2 = 4$, then the case $m_3 \geq 6$ leads again to a contradiction modulo 9, while when $m_3 = 4, 5$, one gets that the right hand side of equation (2) is either 72 or 168 and none of these is a Fibonacci number. When $m_1 = 4$, $m_2 = 5$, then equation (2) becomes

$$F_n - F_{12} = m_3! \tag{9}$$

Since $12 \mid n$, one checks that the left hand side of equation (9) above can be factored as $F_{(n+12)/2}L_{(n-12)/2}$. Since $n > 100 > 12$, we have that $(n+12)/2 > 12$, therefore the number $F_{(n+12)/2}$ has a primitive prime factor p. Recall that a primitive prime factor of F_m (or L_m) is a prime divisor of F_m (or L_m) which does not divide F_ℓ (or L_ℓ) for all $1 \leq \ell < m$. For technical reasons, such a prime is taken to be different from 5. Whenever it exists, it has the property that it is congruent to ± 1 modulo m. The fact that it exists for all $m > 12$ is a result of Carmichael [3] of 1913. Returning to our problem, we get that $F_{(n+12)/2}$ has a prime factor p such that $p \equiv \pm 1 \pmod{(n+12)/2}$. In particular, $p \geq (n+12)/2 - 1 = (n+10)/2$. Since $p \mid m_3!$, we get that $m_3 \geq (n+10)/2$. Thus,

$$\begin{aligned}\alpha^n > F_n > F_n - F_{12} = F_{(n-12)/2}L_{(n+12)/2} = m_3! \geq p! \geq \left(\frac{p}{\mathrm{e}}\right)^p \\ > \left(\frac{n+10}{2\mathrm{e}}\right)^{(n+10)/2} > \left(\frac{n+10}{2\mathrm{e}}\right)^{n/2}.\end{aligned}$$

In the above calculation we used the well-known inequality $m! > (m/\mathrm{e})^m$, which holds for all $m \geq 1$. We thus get that

$$n + 10 < 2\mathrm{e}\alpha^2 < 15,$$

which is false because $n > 100$. Thus, we have just showed that if $m_1 \geq 3$, then $m_1 \geq 5$. In particular, $5 \mid F_n$, therefore $5 \mid n$. Hence, $60 \mid n$.

4. A bound on n when $m_1 \geq 3$

Up to now, we have seen that $m_1 \geq 5$ and that $60 \mid n$. We show that $n < e^{10}$. Assume, on the contrary, that $n > e^{10}$.

Let $s = \nu_2(m_1!)$. It is known that

$$s = \left\lfloor \frac{m_1}{2} \right\rfloor + \left\lfloor \frac{m_1}{4} \right\rfloor + \cdots \geq \frac{m_1}{2}, \tag{10}$$

since $m_1 \geq 5$. Since $s \geq 3$ and $2^s \mid F_n$, we get that $3 \cdot 2^{s-2} \mid n$. Since also $5 \mid n$, we get that

$$2^{s-2} \leq \frac{n}{15},$$

therefore

$$s \leq \frac{\log(4n/15)}{\log 2} < \frac{\log n}{\log 2}. \tag{11}$$

Comparing estimates (10) and (11), we get that

$$m_1 \leq \frac{2 \log n}{\log 2}. \tag{12}$$

Next we bound m_2. Let $N = m_1!$. The largest Fibonacci number which is a factorial is $2! = F_3$ (see [6]). Thus, N is not a Fibonacci number. Since $m_1 \geq 5$, we have

$$6N^2 = 6(m_1!)^2 < (3m_1!)^2 = (3 \cdot 2 \cdot 3 \cdots m_1)^2 < \left(\frac{3 \cdot 2 \cdot 3 \cdot 4}{5^4} \right)^2 m_1^{2m_1} < m_1^{2m_1}$$

(because $72/625 < 1$), we get that

$$\log(6N^2) < 2m_1 \log m_1 < \frac{4}{\log 2} \log n \log \left(\frac{2 \log n}{\log 2} \right).$$

Lemma 1 (note that $n > e^{10}$, so $\log n > 10$) now shows that

$$\nu_2(F_n - m_1!) < \frac{1730 \cdot 4}{\log 2} (\log n)^3 \log \left(\frac{2 \log n}{\log 2} \right) < 10^4 (\log n)^3 \log \left(\frac{2 \log n}{\log 2} \right).$$

Since $\nu_2(F_n - m_1!) = \nu_2(m_2!) \geq m_2/2$, we get that

$$m_2 \leq 2 \cdot 10^4 (\log n)^3 \log \left(\frac{2 \log n}{\log 2} \right).$$

Next take $N = m_1! + m_2! \le 2m_2!$. The largest Fibonacci number which is a sum of two factorials is $F_{12} = 4! + 5!$ (see [5]). Since $m_2 \ge m_1 \ge 5$, it follows that N is not a Fibonacci number. Furthermore, again as in the previous case,

$$6N^2 \le 24(m_2!)^2 < (5m_2!)^2 = (5 \cdot 2 \cdot 3 \cdots m_2)^2$$
$$< \left(\frac{2 \cdot 3 \cdot 4}{5^3}\right)^2 m_2^{2m_2} < m_2^{2m_2},$$

(because $24/125 < 1$), therefore

$$\log(6N^2) < 2m_2 \log m_2 < 4 \cdot 10^4 (\log n)^3 \log\left(\frac{2\log n}{\log 2}\right) \log m_2.$$

Let us next observe that $\log m_2 < 8 \log\log n + 1$. Indeed, to see why this is so observe that since $\log n > 10$, we have that $n \ge 5$ and for such positive integers n we know that $2^n > n^2$. Thus, it follows that

$$\log\left(\frac{2\log n}{\log 2}\right) < \log n,$$

so, in particular,

$$m_2 < 2 \cdot 10^4 (\log n)^4 < 2(\log n)^8,$$

Hence, indeed

$$\log m_2 < 8 \log\log n + 1.$$

Thus,

$$\log(6N^2) < 4 \cdot 10^4 (\log n)^3 \log\left(\frac{2\log n}{\log 2}\right)(8 \log\log n + 1)$$
$$< 32 \cdot 10^4 (\log n)^3 (\log\log n + 1.1)^2,$$

where we used the fact that $\log(2/\log 2) < 1.1$. Now Lemma 1 shows that

$$\nu_2(F_n - m_1! - m_2!) \le 1730 \cdot 32 \cdot 10^4 \cdot (\log n)^5 \cdot (\log\log n + 1.1)^2$$
$$< 6 \cdot 10^8 (\log n)^5 (\log\log n + 1.1)^2.$$

Clearly,

$$\nu_2(F_n - m_1! - m_2!) = \nu_2(m_3!) = m_3 - \sigma_2(m_3) \ge m_3 - \frac{\log(m_3+1)}{\log 2},$$

where we used $\sigma_2(m)$ for the sum of the binary digits of m (see, for example, Lemma 2.2 in [4]). Thus,

$$m_3 - \frac{\log(m_3+1)}{\log 2} < 6 \cdot 10^8 (\log n)^5 (\log\log n + 1.1)^2.$$

On the other hand,

$$m_3^{m_3} > m_3! \geq \frac{F_n}{3} > \alpha^{n-6},$$

so

$$m_3 \log m_3 > (n-6)\log\alpha,$$

which implies that

$$m_3 > \frac{(n-6)\log\alpha}{\log((n-6)\log\alpha)}. \tag{13}$$

Since the function $x \mapsto x - \log(x+1)/\log 2$ is increasing for $x > 1$, we get that

$$\frac{(n-6)\log\alpha}{\log((n-6)\log\alpha)} - \frac{1}{\log 2}\log\left(\frac{(n-6)\log\alpha}{\log((n-6)\log\alpha)} + 1\right)$$
$$< 6\cdot 10^8(\log n)^5(\log\log n + 1.1)^2,$$

giving

$$n < 2\cdot 10^{29}.$$

We now immediately get that $m_1 \leq 35$. Indeed, assume that $m_1 \geq 36$. Since

$$2^{34}\cdot 3^{17}\cdot 5^8\cdot 7^5\cdot 11^3\cdot 13^2\cdot 17^2 \mid 36!,$$

it follows that

$$2^{32}\cdot 3^{16}\cdot 5^8\cdot 7^4\cdot 11^2\cdot 13\cdot 17 \mid n,$$

but this is impossible since the number on the left above is $> 2\cdot 10^{29}$, whereas the number on the right is $< 2\cdot 10^{29}$. Thus, $m_1 \leq 35$. Next, a quick computation revealed that for each $m_1 \in [5, 35]$ there is a prime $p \in [m_1+1, 61]$ such that the congruence

$$F_x \equiv m_1! \pmod p$$

has no integer solution x. This shows that $m_2 \leq 60$. Thus,

$$N \leq 26! + 60! < 10^{82},$$

giving that $\log(6N^2) < 380$. Hence, we get that

$$m_3 - \frac{\log(m_3+1)}{\log 2} \leq \nu_2(m_3!) = \nu_2(F_n - N) \leq 1730\cdot 380(\log n)^2$$
$$< 6.6\cdot 10^5(\log n)^3.$$

Combining this with the lower bound (13) on m_3, we get

$$\frac{(n-6)\log\alpha}{\log((n-6)\log\alpha)} - \frac{1}{\log 2}\log\left(\frac{(n-6)\log\alpha}{\log((n-6)\log\alpha)} + 1\right) < 6.6\cdot 10^5(\log n)^2,$$

giving $n < 2\cdot 10^{10}$. We now get that $m_1 \le 19$, since if $m_1 \ge 20$, then since

$$2^{18}\cdot 3^8\cdot 5^4\cdot 7^2 \mid 20!$$

we get that

$$2^{16}\cdot 3^7\cdot 5^4\cdot 7 \mid n,$$

which is impossible since the number on the left above is $> 2\cdot 10^{10}$. Hence, $m_1 \le 19$ and also $m_2 \le 60$. A quick computation with Mathematica shows that for each $5 \le m_1 \le m_2$ with $m_1 \le 19$ and $m_2 \le 60$, there exists a prime p in the interval $[61, 859]$ such that the congruence

$$F_x \equiv m_1! + m_2! \pmod{p}$$

has no integer solution x. This shows that $m_3 \le 858$, therefore

$$F_n \le 19! + 60! + 858!,$$

leading to $n < 5000$, which is a contradiction. Hence, $n < \mathrm{e}^{10}$.

5. A bound on n when $m_1 \in \{1, 2\}$

Assume first that $m_1 = m_2 = 1$. Since $n > 100$, we have that m_3 is very large and in particular $F_n \equiv 2 \pmod{8}$, which implies that $n \equiv \pm 3 \pmod{12}$. We now rewrite our equation (2) as

$$F_n - F_3 = m_3!. \tag{14}$$

Since n is odd, we get that the left hand side above is $F_{(n\pm 3)/2}L_{(n\mp 3)/2}$ according to whether $n \equiv 1, 3 \pmod{4}$ (see Lemma 2 in [9]). Since $n > 100$ is large, it follows that $(n \pm 3)/2 > 12$, therefore both $F_{(n\pm 3)/2}$ and $L_{(n\mp 3)/2}$ have primitive prime factors. Thus, there is a prime $p \equiv \pm 1 \pmod{(n+3)/2}$ which divides the left hand side of equation (14) leading to the fact that $m_3 \ge p \ge (n+3)/2 - 1 = (n+1)/2$. Hence, we get that

$$\alpha^n > F_n > F_n - F_3 = m_3! \ge p! \ge \left(\frac{p}{\mathrm{e}}\right)^p > \left(\frac{n+1}{2\mathrm{e}}\right)^{n/2},$$

leading to $n + 1 < 2\mathrm{e}\alpha^2 < 15$, contradicting the fact that $n > 100$. This shows that in our range for n it is not possible that $m_1 = m_2 = 1$.

Assume still that $m_1 = 1$ but that $m_2 = 2$. Then, since $m_3 > 3$, we get that

$3 \mid F_n$, therefore $4 \mid n$. Hence, equation (2) is

$$F_n - F_4 = m_3! \tag{15}$$

Since $4 \mid n$, the left hand side above factors as $F_{(n-4)/2}L_{(n+4)/2}$. Since $(n+4)/2 > 12$, it follows that the left hand side of equation (15) has a prime factor $p \equiv \pm 1 \pmod{(n+4)/2}$. Hence, $m_3 \geq p \geq (n+4)/2 - 1 = (n+2)/2$. We thus get that

$$\alpha^n > F_n > F_n - F_4 = m_3! \geq p! \geq \left(\frac{p}{\mathrm{e}}\right)^p \geq \left(\frac{n+2}{2\mathrm{e}}\right)^{n/2},$$

leading to $n + 2 < 2\mathrm{e}\alpha^2 < 15$, which is again a contradiction.

If $m_1 = 1$ and $m_2 = 3$, then since m_3 is large, we get that $F_n \equiv 4 \pmod 8$, which is a contradiction. From now on, we assume that $m_2 \geq 4$ whenever $m_1 = 1$.

If $m_1 = m_2 = 2$, then since $m_3 > 4$, we get that $F_n \equiv 4 \pmod 8$, which is impossible. If $m_1 = 2$ and $m_2 = 3$, then since $m_3 \geq 4$, we get that $8 \mid F_n$, therefore $6 \mid n$. We thus get that equation (2) is

$$F_n - F_6 = m_3!, \tag{16}$$

where n is even. In particular, the left hand side of equation (16) above is of the form $F_{(n\pm 6)/2}L_{(n\mp 6)/2}$ according to whether $n \equiv 0, 2 \pmod 4$. Since $n > 100$ is large, we have that $(n+6)/2 > 12$, therefore the left hand side of equation (16) is divisible by a prime $p \equiv \pm 1 \pmod{(n+6)/2}$. Hence, $m_3 \geq p \geq (n+6)/2 - 1 \geq (n+4)/2$, and, as before, we reach the contradiction $n + 4 < 2\mathrm{e}\alpha^2$. From now on, we assume that $m_2 \geq 4$ when $m_1 = 2$.

Next we shall show that $n < \mathrm{e}^{10}$. Assume that this is not so.

Since $m_1 = 1, 2$, we get that $m_1! = F_t$ for some $t \in \{1, 2, 3\}$. Furthermore, when $m_2 = 2$, then $F_n \equiv 2 \pmod 8$, therefore $n \equiv \pm 3 \pmod{12}$, and, in particular, n is odd, so $n \equiv t \pmod 2$ in this case. Thus, in all these cases we have that

$$F_n - m_1! = F_n - F_t = F_{(n\pm t)/2}L_{(n\pm t)/2}, \quad n \equiv \pm t \pmod 4 \text{ and } t \in \{1, 2, 3\}.$$

We now bound m_2. Since $m_2 \geq 4$, it follows that by putting $s = \nu_2(m_2!)$, we have $s \geq m_2/2$. Note also that $s \geq 3$. Thus,

$$\frac{m_2}{2} \leq s = \nu_2(m_2!) \leq \nu_2(F_n - m_1!) = \nu_2(F_{(n\pm t)/2}L_{(n\pm t)/2}).$$

It is known that L_m is never a multiple of 8. Thus, $2^{s-2} \mid F_{(n\pm t)/2}$, leading to the conclusion that either $s = 3,\ 4$, or

$$3 \cdot 2^{s-4} \mid (n \pm t)/2.$$

Hence,

$$s \le \frac{\log(8(n+3)/3)}{\log 2} < \frac{\log(3n)}{\log 2},$$

therefore

$$m_2 < \frac{2\log(3n)}{\log 2}. \tag{17}$$

Put $N = m_1! + m_2! \le 2 + m_2!$. Then

$$6N^2 \le 6(m_2!+2)^2 < (3m_2!)^2 = (3\cdot 2\cdot 3\cdots m_2)^2 < \left(\frac{3\cdot 2\cdot 3}{4^3}\right)^2 m_2^{2m_2} < m_2^{2m_2},$$

(here, we used the fact that $m_2 \ge 4$ and $18/64 < 1$), so

$$\log(6N^2) < 2m_2 \log m_2 < \frac{4\log(3n)}{\log 2}\log\left(\frac{2\log(3n)}{\log 2}\right).$$

We are now ready to apply again Lemma 1 observing that for $m_1 = 1, 2$ and $m_2 \ge 4$, the number N is not a Fibonacci number by the results from [5]. Thus, by Lemma 1, we get that

$$\nu_2(m_3!) = \nu_2(F_n - N) < \frac{1730\cdot 4}{\log 2}(\log n)^2 \log(3n)\log\left(\frac{2\log(3n)}{\log 2}\right). \tag{18}$$

On the other hand,

$$\begin{aligned}\nu_2(m_3!) \ge m_3 - \frac{\log(m_3+1)}{\log 2} &\ge \frac{(n-6)\log\alpha}{\log((n-6)\log\alpha)} \\ &- \frac{1}{\log 2}\log\left(\frac{(n-6)\log\alpha}{\log((n-6)\log\alpha)}+1\right)\end{aligned} \tag{19}$$

(see inequality (13), for example). Combining inequalities (18) and (19) and using the fact that $1730\cdot(4/\log 2) < 10^4$, we get

$$\begin{aligned}&\frac{(n-6)\log\alpha}{\log((n-6)\log\alpha)} - \frac{1}{\log 2}\log\left(\frac{(n-6)\log\alpha}{\log((n-6)\log\alpha)}+1\right) \\ &\qquad \le 10^4(\log n)^2\log(3n)\log\left(\frac{2\log(3n)}{\log 2}\right),\end{aligned}$$

yielding $n < 4\cdot 10^{10}$. Combining this with (17), we get that $m_2 \le 73$. A short computation with Mathematica showed that for each $m_1 \in \{1, 2\}$ and $m_2 \in [4, 73]$, there exists a prime $p \in [79, 863]$ such that the congruence

$$F_x \equiv m_1! + m_2! \pmod p$$

has no positive integer solution x. Thus, $m_3 \le 863$, therefore $F_n \le 2! + 73! + 863!$, so $n < 11000$, contradicting the fact that $n > \mathrm{e}^{10}$. Hence, $n < \mathrm{e}^{10}$.

6. The final calculation

Let us assume that $n < e^{10}$. If $m_1 \geq 10$, then

$$2^8 \cdot 3^4 \cdot 5^2 \mid 10! \mid F_n,$$

leading to

$$2^6 \cdot 3^3 \cdot 5^2 \mid n,$$

which is impossible because the number on the left above is $43200 > e^{10}$. Thus, $m_1 \leq 9$. When $m_1 = 1$, or 2, inequality (17) shows that $m_2 \leq 32$. When $m_1 \in [3,9]$, a short computation with Mathematica revealed that for each $m_1 \in [3,9]$, there is a prime $p \in [11,37]$ such that the congruence $F_x \equiv m_1! \pmod{p}$ has no positive integer solution x. Thus, $m_2 \leq 36$. A short computation with Mathematica revealed that for all pairs (m_1, m_2) with $m_1 \in [1,9]$ and $m_2 \in [m_1, 36]$ except for $(m_1, m_2) = (1,1), (1,2), (2,3), (4,5)$, there is a prime $p \in [41, 523]$, such that the congruence $F_x \equiv m_1! + m_2! \pmod{p}$ has no positive integer solution x. Since the cases $(m_1, m_2) = (1,1)$, $(1,2)$, $(2,3)$, $(4,5)$ have already been treated, it follows that $m_3 \leq 522$, therefore $F_n \leq 9! + 36! + 522!$, leading to $n < 6000$. A short computation with Mathematica revealed that there are no numbers which are both of the form F_n for some $100 < n < 6000$ and $m_1! + m_2! + m_3!$ with $m_1 \leq 9$, $m_1 \leq m_2 \leq 36$ and $m_2 \leq m_3 \leq 522$, which finishes the proof of Theorem 1.

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(Received August 8, 2009; revised August 19, 2009)