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On the advertible completion of a topological algebra

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Abstract. We present relations between topologically invertible elements, topological divisors of zero and invertible elements of unital topological algebras. We define the advertible completion of a unital topological algebra A and give an explicit description of this new algebra when A is commutative. This description is exemplified in the algebra of holomorphic functions on the complex unit closed disc endowed with an m-convex topology.

1. Introduction

First we recall some definitions. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, a *topological algebra* is a topological linear space with an associative jointly continuous multiplication making it an algebra over $\mathbb{F}$.

A *locally convex algebra* A is a topological algebra that it is a locally convex space; in this case its topology can be given by a family $\{\|\cdot\|_\alpha, \alpha \in \Lambda\}$ of seminorms such that for every $\alpha \in \Lambda$ there exists $\beta \in \Lambda$ satisfying

$$\|xy\|_\alpha \leq \|x\|_\beta \|y\|_\beta \quad \text{for all } x, y \in A. \tag{1}$$

A locally convex algebra A is said to be *multiplicatively convex* (in short form *m-convex*) if its topology is defined by a family $\{\|\cdot\|_\alpha, \alpha \in \Lambda\}$ of seminorms such that (1) can be replaced by

$$\|xy\|_\alpha \leq \|x\|_\alpha \|y\|_\alpha \quad \text{for all } \alpha \in \Lambda \text{ and all } x, y \in A.$$

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Let A be a topological algebra with a unit e. The set of invertible elements of A is denoted by $G(A)$. If $G(A)$ is open, then A is called a *Q-algebra*.

If $(A, \{\|\cdot\|_\alpha, \alpha \in \Lambda\})$ is an m-convex algebra, then for every $\alpha \in \Lambda$ we can consider the normed algebra $A/\ker(\|\cdot\|_\alpha)$, the completion A_α of this algebra and the projection π_α from A to A_α given by the composition $A \rightarrow|, A/\ker(\|\cdot\|_\alpha) \hookrightarrow A_\alpha$.

Let A be a topological algebra with unit e. A net $(a_\lambda)_{\lambda\in\Lambda}$ in A is called *right* (resp. *left*) *advertibly convergent* with respect to $a \in A$ if $aa_\lambda \to e$ (resp. $a_\lambda a \to e$), and then we also say that $(a_\lambda)_{\lambda\in\Lambda}$ is a *right* (resp. *left*) *topological inverse* of a.

An element $a \in A$ is *topologically invertible* if it has a right and left topological inverses. We denote the set of all topologically invertible elements in A by $G_t(A)$. Obviously, $G(A) \subset G_t(A)$.

If a right (resp. left) topological inverse of $a \in A$ converges, then its limit is the right (resp. left) inverse of a. Thus, an element is invertible in A iff it has a convergent right and left topological inverses in A.

The net $(a_\lambda)_{\lambda\in\Lambda}$ in A is called *advertibly convergent* (in short form *advertible*) if it is right and left advertibly convergent with respect to some $a \in A$. The topological algebra A is called *advertibly complete* if every advertible and Cauchy net in A converges there.

The advertibly complete topological algebras were introduced in [7] by S. Warner. He considered this notion in the context of the locally m-convex algebras. He realized that "advertibly complete locally m-convex algebras possess the fundamental properties of the Banach algebras".

One of the classical properties of the m-convex algebras is the following:

Proposition 1.1. *Let A be an m-convex unital algebra, then the following statements are equivalent:*

a) *A is advertibly complete.*

b) *$a \in A$ is invertible if and only if $\pi_\alpha(a)$ is invertible in A_α for every $\alpha \in \Lambda$.*

Obviously the complete algebras and the Q-algebras are advertibly complete. Nevertheless, these two first classes of algebras do not cover all the class of advertibly complete algebras as it is shown by Example 3.1 and the final part of the remark that follows the next result.

Proposition 1.2. *For any topological algebra A with a unit e, properties* (a)–(d) *below satisfy:*
a) $\implies$ b) $\implies$ c) $\implies$ d).

a) *A is a Q-algebra.*

b) *Every maximal bilateral ideal of A is closed.*

c) $G_t(A) = G(A)$.

d) *A is advertibly complete.*

Proof. a) $\implies$ b) and c) $\implies$ d) are trivial. To prove b) $\implies$ c) we assume that there exists $a \in A$ such that $a \in G_t(A) \setminus G(A)$, which implies that the ideal I generated by a is dense; this contradicts that I is contained in some closed maximal ideal. □

Remark 1.1. We have that b) $\not\Longrightarrow$ a) since in [5] a topological algebra in which all maximal ideals are closed is exhibited. Nevertheless, it is not a Q-algebra. Also c) $\not\Longrightarrow$ b) because c) is always true when A is a complete m-convex algebra, and the algebra $\mathcal{E}$ of all entire complex functions endowed with the compact-open topology is such an algebra, however $\mathcal{E}$ has dense maximal ideals of infinite codimension [8]. Finally, d) $\not\Longrightarrow$ c) because from a result given in [2] it follows that $f_0 \in G_t(A) \setminus G(A)$ if $f_0 : \mathbb{R} \to \mathbb{C}$ is a continuous bounded function that does not vanish in any point but 0 belongs to the closure $\overline{f_0(\mathbb{R})}$ of its range, and based in [4] we know that A is a complete algebra, where A is the space $C_b(\mathbb{R})$ of all the complex continuous bounded functions defined in $\mathbb{R}$ endowed with the strict topology β, i.e. the locally convex topology given by the family of seminorms $\|f\|_\varphi = \sup_{x\in\mathbb{R}} |f(x)\varphi(x)|$ for $f \in C_b(\mathbb{R})$ and φ ranging on the space of all complex continuous functions defined in $\mathbb{R}$ that vanish at infinity.

In [7] it is proved the following

Proposition 1.3. *Let A be a normed unital algebra, then properties* a)–d) *are equivalent.*

2. Invertibility and topological invertibility

We shall describe some relations between topological invertible elements, topological divisors of zero and invertible elements of a unital topological algebra.

Let A be a topological algebra. Recall that an element $a \in A$ is a *left* (*right*) *topological divisor of zero* if there is a net $\widetilde{c} = (c_\lambda)$, such that (ac_λ) (resp. $(c_\lambda a)$) converges to zero and (c_λ) does not converges to zero. The element $a \in A$ is called *bilateral topological divisor of zero* if it is a *left and right* topological divisor of zero.

Let (a_λ) be a net in a topological linear space X. Following [3], we say that (a_λ) is a *bounded net* if for every neighborhood of zero U there exist an index

λ_U and $k_U > 0$ such that $a_\lambda \in k_U U$ for every $\lambda \geq \lambda_U$. Every Cauchy net is a bounded net.

We shall generalize Theorem 3 of [2]. In order to do this we first define the topological algebra $l^\infty(A)$.

Let $\mathcal{N}$ be a fundamental system of neighborhoods of zero in A, consisting of closed and balanced sets, directed in the usual manner. Let us consider $l^\infty(A)$, the algebra of all bounded nets $\widetilde{x} = (x_\lambda)_{\lambda\in\mathcal{N}}$, with the coordinatewise operations, and the bilateral ideal $c_0(A)$ consisting of all nets $\widetilde{x}$ converging to zero.

Given a neighborhood $U \in \mathcal{N}$ and an index λ_0 we define $V_{(U,\lambda_0)}$ as the set

$$\{(a_\lambda)_{\lambda\in\mathcal{N}} \in l^\infty(A): \text{ there exists } \lambda_1 \geq \lambda_0 \text{ such that} a_\lambda \in U \text{ for all } \lambda \geq \lambda_1\}.$$

Since $\{V_{(U,\lambda_0)}\}$ is a non empty collection of non empty sets and $V_{(U_3,\lambda_3)} \subset V_{(U_1,\lambda_1)} \cap V_{(U_2,\lambda_2)}$, if $U_3 = U_1 \cap U_2$ and $\lambda_3 \geq \lambda_1, \lambda_2$, then $\{V_{(U,\lambda_0)}\}$ is a filterbase.

Lemma 2.1. *For any topological algebra A, $l^\infty(A)$ is a topological algebra under the topology τ_∞ that has $\{V_{(U,\lambda_0)}\}$ as fundamental system of neighborhoods of zero. In this algebra, $c_0(A)$ is a closed bilateral ideal.*

Proof. The sets of the family $\{V_{(U,\lambda_0)}\}$ satisfy:

i) Each $V_{(U,\lambda_0)}$ is balanced and absorbent.

ii) $V_{(W,\lambda_0)} + V_{(W,\lambda_0)} \subset V_{(U,\lambda_0)}$ if $W + W \subset U$.

iii) $V_{(W,\lambda_0)} \cdot V_{(W,\lambda_0)} \subset V_{(U,\lambda_0)}$ if $W \cdot W \subset U$.

The first assertion of the lemma follows from these facts. Now we prove that $c_0(A)$ is a closed set. Let $((a_{\lambda,\mu})_\lambda)_\mu$ be a net in $c_0(A)$ converging to $(b_\lambda) \in l^\infty(A)$. Let $U \in \mathcal{N}$ and λ_0 be an index, then there exists $W \in \mathcal{N}$ such that $W + W \subset U$. Choose μ such that $(a_{\lambda,\mu})_\lambda \in (b_\lambda)_\lambda + V_{(W,\lambda_0)}$. There exists $\lambda_1 \geq \lambda_0$ such that $a_{\lambda,\mu} - b_\lambda \in W$ for all $\lambda \geq \lambda_1$. On the other hand, since $(a_{\lambda,\mu})_\lambda \in c_0(A)$, there exists another index λ_2 such that $a_{\lambda,\mu} \in W$ for all $\lambda \geq \lambda_2$. Therefore, for all $\lambda \geq \max(\lambda_1, \lambda_2)$, we have $b_\lambda \in U$, which proves that $(b_\lambda) \in c_0(A)$. □

Henceforth we are assuming that A is a topological algebra with a unit e and A_* is its completion.

Proposition 2.1. *Let (a_λ) be a right advertibly convergent net in A with respect to $a \in A$. If (a_λ) is not a Cauchy net, then a is a left topological divisor of zero.*

Theorem 2.1. *Suppose $a \in A$ has left and right topological inverses $\widetilde{b} = (b_\lambda)_{\lambda\in\mathcal{N}}$ and $\widetilde{c} = (c_\lambda)_{\lambda\in\mathcal{N}}$, respectively. If $\widetilde{b}$ or $\widetilde{c}$ is a bounded net, then a is invertible in A_*.*

Proof. Assume that $\widetilde{b} = (b_\lambda)_{\lambda\in\mathcal{N}}$ is a bounded net in A and a is not right invertible in A_*. Therefore $\widetilde{b}$ is a bounded net in A_* and $(c_\lambda)_{\lambda\in\mathcal{N}}$ is not a Cauchy net in A. Thus, a is a left topological divisor of zero. Let $(d_\mu)_{\mu\in\mathcal{M}}$ be a net in A such that $ad_\mu \to 0$, but $d_\mu \nrightarrow 0$.

Consider the topological algebra $l^\infty(A_*)$, its closed ideal $c_0(A_*)$ and the algebra $l^\infty(A_*)/c_0(A_*)$ with the quotient topology. The class determined in this algebra by a bounded net $\widetilde{x}$ is denoted by $[\widetilde{x}]$. In particular, for $a \in A$ let $[\widetilde{a}]$ be the class of the constant net $(a)_{\lambda\in\mathcal{N}}$. Since the net $\widetilde{b}$ is bounded, we have that $[\widetilde{a}]$ is left invertible with left inverse $[\widetilde{b}]$. Also, $[\widetilde{a}]$ is a left topological divisor of zero because $[\widetilde{a}][\widetilde{d_\mu}] \to 0$, where $\widetilde{d_\mu}$ is the constant net (d_μ) for every $\mu \in \mathcal{M}$. Since $[\widetilde{b}][\widetilde{a}] = [\widetilde{e}]$ and $[\widetilde{a}]\ [\widetilde{d_\mu}] \to 0$, we have that $[\widetilde{d_\mu}] \to 0$, and this implies that $d_\mu \to 0$ in A, which is a contradiction. Therefore, $(c_\lambda)_{\lambda\in\mathcal{N}}$ is a Cauchy net in A and a is right invertible in A_*. Since $(c_\lambda)_{\lambda\in\mathcal{N}}$ is bounded we can proceed similarly to prove that a is left invertible in A_*. □

Suppose $b \in G_t(A)$ and one of its lateral topological inverses is bounded, then by Theorem 2.1 it has an inverse b^{-1} in A_*. We denote by $G_{\mathfrak{bt}}(A)$ the set of all such elements b. Clearly $G(A) \subset G_{\mathfrak{bt}}(A)$ and we also have:

$$G_{\mathfrak{bt}}(A) = \{b \in A : b^{-1} \text{ exists in } A_*\}. \tag{2}$$

Indeed, we have already noticed that $G_{\mathfrak{bt}}(A)$ is contained in the set at the right side of the above equality. On the other hand, suppose $b \in A$ has an inverse $b^{-1} \in A_*$ and let $\mathcal{N}$ be a fundamental system of neighborhoods of zero in A formed by closed sets, then the family $\mathcal{N}^- = \{\overline{V} : V \in \mathcal{N}\}$ of closures in A_* is a fundamental system of neighborhoods of zero in A_*. There exist a net (a_λ) in A that converges to b^{-1}. Thus (a_λ) is a Cauchy net in A and $a_\lambda b \to e$ in A_*, i.e. given any $\overline{V} \in \mathcal{N}^-$ there exists λ_0 such that $a_\lambda b - e \in \overline{V}$ if $\lambda \geq \lambda_0$. It follows that $a_\lambda b \to e$ in A since $a_\lambda b - e \in \overline{V} \cap A = \overline{V}^A = V$ for $\lambda \geq \lambda_0$. Therefore $b \in G_{\mathfrak{bt}}(A)$.

For the commutative case we have the following generalization.

Corollary 2.1. *Suppose A is a commutative algebra and let B be an advertibly complete algebra between A and A_*. Then an element $a \in A$ is invertible in B if and only if it has a bounded topological inverse in A. That is*

$$G_{\mathfrak{bt}}(A) = \{b \in A : b^{-1} \text{ exists in } B\}.$$

In particular, algebra A is advertibly complete is equivalent to the statement: an element $a \in A$ is invertible in A if and only if it has a bounded topological inverse in A.

We point out that Theorem 2.1 remains true if we replace A_* for any of its subalgebras B that contains A and that is what we shall call a *bilaterally advertibly complete algebra*. This means that every Cauchy net in B converges there whenever it is left or right advertibly convergent.

Obviously every bilaterally advertibly complete algebra is an advertibly complete algebra and the two notions coincide when A is commutative. We think that this coincidence does not always hold in noncommutative algebras; however, we have no example in this direction.

3. The advertible completion of a topological algebra

Let us consider the class $\{B\}$ of all advertibly complete subalgebras B of A_* that contain algebra A. We define the *advertible completion* of A as the subalgebra $\widehat{A} = \cap B$ of A_*.

Proposition 3.1. *$\widehat{A}$ is the smallest advertibly complete subalgebra of A_* that contains algebra A.*

Proof. We need only to prove that $\widehat{A}$ is advertibly complete. Let (a_λ) be an advertible and Cauchy net in $\widehat{A}$, then it is a Cauchy net in A_* and it converges to an element $b \in A_*$. Therefore (a_λ) is convergent to b in each advertibly complete subalgebra $B \subset A_*$ and then $b \in \widehat{A}$. □

Let $\langle A \rangle$ be the subalgebra of A_* generated by A and the set $G_{\mathfrak{bt}}(A)^{-1} = \{b^{-1} \in A_* : b \in G_{\mathfrak{bt}}(A)\}$.

In what follows we assume that A is also a commutative algebra. Then $G_{\mathfrak{bt}}(A)$ is closed under multiplication and

$$G_0 = \{x \in A_* : x = ab^{-1} \text{ for } a, b \in G_{\mathfrak{bt}}(A)\}$$

is a subgroup of $G(A_*)$. Moreover, we have:

$$\langle A \rangle = \{x \in A_* : x = ab^{-1} \text{ for } a \in A \text{ and } b \in G_{\mathfrak{bt}}(A)\}. \tag{3}$$

We point out that it is also true that

$$\langle A \rangle = \{x \in A_* : x = ab^{-1} \text{ for } a \in A \text{ and } b = e \text{ or } b \in G_{\mathfrak{bt}}(A) \setminus G(A)\}.$$

We summarize some of the above facts in the following.

Theorem 3.1. *For a commutative topological algebra A with unit e we have:*

$$\widehat{A} = \langle A \rangle.$$

Proof. Clearly $\langle A\rangle$ is a subalgebra of A_* that contains A and it is contained in $\widehat{A}$.

It remains only to show that $\langle A\rangle$ is advertibly complete. Suppose (x_λ) is an advertible and Cauchy net in $\langle A\rangle$. Then (x_λ) converges to some $x_0 \in A_*$ and there exists $x = ab^{-1}$ in $\langle A\rangle$ such that $xx_\lambda \to e$ in $\langle A\rangle$. Thus $ab^{-1}x_0 = e$. Therefore $a \in A$ has an inverse in A_* and it follows from (2) that $a \in G_{\mathfrak{bt}}(A)$ and we have by (3) that $x_0 = ba^{-1}$ belongs to $\langle A\rangle$. □

We do not know if the previous theorem is true in case A is not a commutative algebra.

Example 3.1. Let A be the commutative algebra $H(\overline{D})$ of all holomorphic functions in the complex closed unit disc $\overline{D}$, equipped with the pointwise operations and the compact-open topology τ_D in the open unit disc D. This topology is given by the sequence of m-convex seminorms $(\|\cdot\|_n)_{n\in\mathbb{N}}$ defined as $\|f\|_n = \sup_{|z|\le r_n} |f(z)|$, where (r_n) is an increasing sequence of positive numbers converging to one. The completion A_* of A is $(H(D), \tau_D)$, and we have that $f \in A$ is invertible if and only if $f(z) \neq 0$ for every $z \in \overline{D}$.

Let S be the boundary of D and $w \in S$. It follows from Corollary 2.1 that A is not advertibly complete since $z - w$ is a non-invertible element of A that has a bounded topological inverse $(-\sum_{i=0}^{n} \frac{z^i}{w^{i+1}})_{n=1}^{\infty}$.

The inverse of $(z-w)$ in A_* is the function $\frac{1}{z-w}$. Thus, $\prod_{w\in W}(z-w)^{n_w} \in G_{\mathfrak{bt}}(A) \setminus G(A)$ for every non empty set $W \subset S$ and any collection of natural numbers $\{n_w : w \in S\}$.

Any $g \in G_t(A) \setminus G(A)$ must vanishes in points in S and only there. Then, $g \in G_{\mathfrak{bt}}(A) \setminus G(A)$ if and only if $g(x) = g_0(x)\prod_{w\in W}(z-w)^{n_w}$ for some function $g_0 \in G(A)$, a non empty finite set $W \subset S$ and natural numbers n_w. Then

$$\widehat{A} = \left\{ \frac{f(x)}{\prod_{w\in W}(z-w)^{n_w}} : f \in H(\overline{D}) \text{ and } W \subset S \text{ is a finite set} \right\}$$

where, as usual, the product of an empty set of factors is 1. In this example we have $A \subsetneqq \widehat{A} \subsetneqq A_*$.

On the other hand, if we consider the normed algebra $H(\overline{D})$ with the norm $\|\cdot\|_\infty$, where $\|f\|_\infty = \max_{|z|\le 1} |f(z)|$, then it is a Q-algebra since $\mathfrak{M}(H(\overline{D})) = \overline{D}$ is compact. Thus, $H(\overline{D})$ is advertibly complete and therefore $H(\overline{D}) = \widehat{H(\overline{D})}$.

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