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Quasi-central elements and p-nilpotence of finite groups

By HUAQUAN WEI (Nanning), YANMING WANG (Guangzhou)
and GUOHUA QIAN (Changshu)

Abstract. Let G be a finite group and let P be a Sylow p-subgroup of G. An element x of G is called quasi-central in G if $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$ for each $y \in G$. In this paper, it is proved that G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent and, for all $x \in G\backslash N_G(P)$, one of the following conditions holds: (a) every element of $P\cap P^x\cap G^{\mathcal{N}_p}$ of order p or 4 is quasi-central in P; (b) every element of $P \cap P^x \cap G^{\mathcal{N}_p}$ of order p is quasi-central in P and, if $p = 2$, $P \cap P^x \cap G^{\mathcal{N}_p}$ is quaternion-free; (c) every element of $P \cap P^x \cap G^{\mathcal{N}_p}$ of order p is quasi-central in P and, if $p = 2$, $[\Omega_2(P \cap P^x \cap G^{\mathcal{N}_p}), P] \leq Z(P\cap G^{\mathcal{N}_p})$; (d) every element of $P\cap G^{\mathcal{N}_p}$ of order p is quasi-central in P and, if $p = 2$, $[\Omega_2(P \cap P^x \cap G^{\mathcal{N}_p}), P] \leq \Omega_1(P\cap G^{\mathcal{N}_p})$; (e) $|\Omega_1(P \cap P^x \cap G^{\mathcal{N}_p})| \leq p^{p-1}$ and, if $p = 2$, $P \cap P^x \cap G^{\mathcal{N}_p}$ is quaternion-free; (f) $|\Omega(P \cap P^x \cap G^{\mathcal{N}_p})| \leq p^{p-1}$. That will extend and improve some known related results.

1. Introduction

All groups considered will be finite. If P is a p-group, we denote $\Omega(P) = \Omega_1(P)$ if $p > 2$ and $\Omega(P) = \langle\Omega_1(P), \Omega_2(P)\rangle$ if $p = 2$, where $\Omega_i(P) = \langle x \in P \mid \circ(x) = p^i\rangle$. For a formation $\mathcal{F}$ and a group G, there exists a smallest normal subgroup of G, called the $\mathcal{F}$-residual of G and denoted by $G^{\mathcal{F}}$, such that $G/G^{\mathcal{F}} \in \mathcal{F}$ (refer [1]). Throughout this paper, $\mathcal{N}$ and $\mathcal{N}_p$ will denote the classes of nilpotent groups and p-nilpotent groups, respectively. A 2-group is called quaternion-free if it has no section isomorphic to the quaternion group of order 8.

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A group is said to be p-nilpotent if it has a normal p-complement. In general, a group with a p-nilpotent normalizer of the Sylow p-subgroup need not be a p-nilpotent group, for example, S_4 is a counter-example for $p = 2$. However, if one adds some embedding properties on the Sylow p-subgroup, one may obtain the desired result. For instance, Wielandt proved that a group G is p-nilpotent if it has a regular Sylow p-subgroup whose G-normalizer is p-nilpotent [2]. Ballester-Bolinches and Esteban-Romero showed that a group G is p-nilpotent if it has a modular Sylow p-subgroup whose G-normalizer is p-nilpotent [3]. Moreover, Guo and Shum obtained a similar result by use of the permutability of some minimal subgroups of Sylow p-subgroups [4].

Let G be a group. Recall that an element x of G is called quasi-central in G if $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$ for each $y \in G$. It is clear that centrality implies quasi-centrality. But the converse is not true. For example, let G be the quaternion group of order 8. Then any element of G is quasi-central and the center of G has order 2.

In this paper, we will push further the studies and obtain the following main theorem:

Theorem 1.1. *Let P be a Sylow p-subgroup of a group G. Then G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent and, for all $x \in G\backslash N_G(P)$, one of the following conditions holds:*

(a) *Every element of $P \cap P^x \cap G^{\mathcal{N}_p}$ of order p or 4 is quasi-central in P;*

(b) *Every element of $P \cap P^x \cap G^{\mathcal{N}_p}$ of order p is quasi-central in P and, if $p = 2$, $P \cap P^x \cap G^{\mathcal{N}_p}$ is quaternion-free;*

(c) *Every element of $P \cap P^x \cap G^{\mathcal{N}_p}$ of order p is quasi-central in P and, if $p = 2$, $[\Omega_2(P \cap P^x \cap G^{\mathcal{N}_p}), P] \le Z(P \cap G^{\mathcal{N}_p})$;*

(d) *Every element of $P \cap G^{\mathcal{N}_p}$ of order p is quasi-central in P and, when $p = 2$, $[\Omega_2(P \cap P^x \cap G^{\mathcal{N}_p}), P] \le \Omega_1(P \cap G^{\mathcal{N}_p})$;*

(e) *$|\Omega_1(P \cap P^x \cap G^{\mathcal{N}_p})| \le p^{p-1}$ and, if $p = 2$, $P \cap P^x \cap G^{\mathcal{N}_p}$ is quaternion-free;*

(f) *$|\Omega(P \cap P^x \cap G^{\mathcal{N}_p})| \le p^{p-1}$.*

As an application of Theorem 1.1, we give the following Theorem 1.2:

Theorem 1.2. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Then G is p-nilpotent if and only if one of the following conditions holds:*

(a) *Every element of $P \cap G^{\mathcal{N}_p}$ of order p or 4 is quasi-central in $N_G(P)$;*

(b) *Every element of $P \cap G^{\mathcal{N}_p}$ of order p is quasi-central in $N_G(P)$ and, if $p = 2$, $P \cap G^{\mathcal{N}_p}$ is quaternion-free.*

The conditions presented above are necessary and sufficient and hence are sharp. Furthermore, since $P\cap G^{\mathcal{N}_p} \leq P\cap G^{\mathcal{N}} \leq P\cap O^p(G) \leq P\cap G'$, our results can be adapted to yield the following theorems of Li, Ballester-Bolinches, Guo, Shum and Asaad:

Theorem 1.3 ([5, Theorem 1]). *Let P be a Sylow p-subgroup of a group G. If one of the following conditions holds, then G is p-nilpotent:*

(a) *If p is odd and every minimal subgroup of P lies in the center of $N_G(P)$;*

(b) *If $p=2$ and every cyclic subgroup of P of order 2 or 4 is permutable in $N_G(P)$.*

Theorem 1.4 ([6, Theorem 1]). *Let P be a Sylow p-subgroup of a group G, where p is the smallest prime divisor of $|G|$. If every element of P of order p or 4 (if $p=2$) is quasi-central in $N_G(P)$, then G is p-nilpotent*

Theorem 1.5 ([7, Theorem 1]). *Let P be a Sylow p-subgroup of a group G. If $\Omega(P\cap G') \leq Z(N_G(P))$, then G is p-nilpotent.*

Theorem 1.6 ([7, Theorem 2]). *Let P be a Sylow 2-subgroup of a group G. Suppose that $\Omega_1(P\cap G') \leq Z(P)$. If P is quaternion-free and $N_G(P)$ is 2-nilpotent, then G is p-nilpotent.*

Theorem 1.7 ([8, Main Theorem]). *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1)=1$. If every minimal subgroup of $P\cap G^{\mathcal{N}}$ is permutable in $N_G(P)$ and, when $p=2$, either every cyclic subgroup of $P\cap G^{\mathcal{N}}$ with order 4 is permutable in $N_G(P)$ or P is quaternion-free, then G is p-nilpotent.*

Theorem 1.8 ([4, Main Theorem]). *Let P be a Sylow p-subgroup of a group G. Assume that every minimal subgroup of $P\cap O^p(G)$ is permutable in P and $N_G(P)$ is p-nilpotent. Assume that, in addition, when $p=2$ then either P is quaternion-free or $[\Omega_2(P\cap O^p(G)), P] \leq \Omega_1(P\cap O^p(G))$. Then G is p-nilpotent.*

Theorem 1.9 ([9, Theorem 1]). *Let P be a Sylow p-subgroup of a group G. If $p=2$, suppose that P is quaternion-free. Then the following statements are equivalent:*

(a) *G is p-nilpotent;*

(b) *$N_G(P)$ is p-nilpotent and $\Omega_1(P\cap P^x\cap G^{\mathcal{N}}) \leq Z(P)$ for all $x\in G\backslash N_G(P)$;*

(c) *$N_G(P)$ is p-nilpotent and $|\Omega_1(P\cap P^x\cap G^{\mathcal{N}})| \leq p^{p-1}$ for all $x\in G\backslash N_G(P)$;*

(d) *$\Omega_1(P\cap G^{\mathcal{N}}) \leq Z(N_G(P))$.*

We remark that the quaternion-free hypothesis can not be removed. For example, if we take $G = GL(2,3)$ then we see that the elements

$$a = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

generate $GL(2,3)$, and the following relations hold:

$$a^8 = b^2 = c^3 = 1, \quad b^{-1}ab = a^3, \quad c^{-1}a^2c = ab, \quad c^{-1}abc = aba^2, \quad b^{-1}cb = c^2.$$

Also we see that $P = \langle a, b\rangle$ is a Sylow 2-subgroup of $GL(2,3)$ and a semidihedral group of order 16. Furthermore, $G' = O^p(G) = SL(2,3)$ and therefore

$$P \cap G^{\mathcal{N}_2} = P \cap G^{\mathcal{N}} = P \cap O^p(G) = P \cap G' = \langle a^2, ab\rangle$$

is a quaternion group of order 8. It is easily seen that $\Omega_1(P \cap G^{\mathcal{N}_2}) \le Z(P) = \langle a^4\rangle$ and $N_G(P) = P$, but G itself is not 2-nilpotent (refer [10]).

2. Preliminaries

We begin by giving some lemmas, which will be needed in our proofs.

Lemma 2.1. *Let c be an element of a group G of order p, where p is a prime divisor of $|G|$. If c is quasi-central in G, then c is centralized by every element of G of order p or 4 (if $p = 2$).*

Proof. Let x be an element of G of order p or 4 (if $p = 2$). By the hypothesis, $H = \langle x\rangle\langle c\rangle$ is a group. It is clear that c is centralized by x if x is of order p. Now assume that $p = 2$ and x is of order 4. If $[c, x] \ne 1$ and $|H| = 8$, then $c^{-1}xc = x^{-1}$ and H is isomorphic to the dihedral group of order 8. It is clear that $\langle xc\rangle\langle c\rangle \ne \langle c\rangle\langle xc\rangle$. This is contrary to the quasi-centrality of c. Hence we must have $[c, x] = 1$. We are done. □

Lemma 2.2. *Let the p'-group H act on the p-group P. If H acts trivially on $\Omega_1(P)$ and P is quaternion-free if $p = 2$, then H acts trivially on P.*

Proof. The case p odd is a direct consequence of Theorem 5.3.10 of [11] and the case p even is Lemma 2.15 of [12]. □

Lemma 2.3 ([13, Lemma 2.8(1)]). *Let M be a maximal subgroup of a group G and let P be a normal p-subgroup of G such that $G = PM$, where p is a prime. Then $P \cap M$ is a normal subgroup of G.*

Lemma 2.4 ([14, Lemma 2]). *Let $\mathcal{F}$ be a saturated formation. Assume that G is a non-$\mathcal{F}$-group and there exists a maximal subgroup M of G such that $M \in \mathcal{F}$ and $G = F(G)M$, where $F(G)$ is the Fitting subgroup of G. Then*

(1) *$G^{\mathcal{F}}/(G^{\mathcal{F}})'$ is a chief factor of G;*

(2) *$G^{\mathcal{F}}$ is a p-group for some prime p;*

(3) *$G^{\mathcal{F}}$ has exponent p if $p > 2$ and exponent at most 4 if $p = 2$;*

(4) *$G^{\mathcal{F}}$ is either an elementary abelian group or $(G^{\mathcal{F}})' = Z(G^{\mathcal{F}}) = \Phi(G^{\mathcal{F}})$ is an elementary abelian group.*

Lemma 2.5. *Let G be a group and let p be a prime number dividing $|G|$ with $(|G|, p-1) = 1$. Then*

(1) *If N is normal in G of order p, then N lies in $Z(G)$;*

(2) *If G has cyclic Sylow p-subgroups, then G is p-nilpotent;*

(3) *If M is a subgroup of G of index p, then M is normal in G.*

Proof. (1) Since $G/C_G(N)$ is isomorphic to a subgroup of $\mathrm{Aut}(N)$ and $|\mathrm{Aut}(N)| = p-1$, $|G/C_G(N)|$ must divide $(|G|, p-1) = 1$. It follows that $G = C_G(N)$ and $N \leq Z(G)$.

(2) Let $P \in \mathrm{Syl}_p(G)$ and $|P| = p^n$. Since P is cyclic, we have $|\mathrm{Aut}(P)| = p^{n-1}(p-1)$. Again, $N_G(P)/C_G(P)$ is isomorphic to a subgroup of $\mathrm{Aut}(P)$, so $|N_G(P)/C_G(P)|$ must divide $(|G|, p-1) = 1$. Thus $N_G(P) = C_G(P)$, and (2) follows by the well-known Burnside theorem.

(3) Obviously we can assume that $M \neq 1$. Moveover the result is well-known if $p = 2$. So we can also assume that p is odd. This implies that $|G|$ is odd and therefore G is solvable, by the Odd Order Theorem. If $M_G = 1$, then M is a core-free maximal subgroup of G and $|G : M| = p$. Now G is a solvable primitive group and there exists a self-centralizing minimal normal subgroup N of order p, such that $G = MN$. This implies that M is isomorphic to a nontrival subgroup of $\mathrm{Aut}(C_p) \cong C_{p-1}$ and then $|M|$ divides $p-1$. This is not possible. Therefore $M_G \neq 1$. By induction, M/M_G is normal in G/M_G and then $M = M_G$ is normal in G. □

We remark that the hypothesis $(|G|, p-1) = 1$ always holds when p is the smallest prime divisor of $|G|$, hence Lemma 2.5(3) extends a result of Frobenius (refer [15, Theorem 20]).

3. Proofs of theorems

PROOF OF THEOREM 1.1. If G is p-nilpotent, then $G^{\mathcal{N}_p} = 1$. Therefore necessity holds.

Conversely, we shall prove that each of the conditions (a)–(f) is sufficient to guarantee that G is p-nilpotent. Let G be a minimal counterexample. Then we have the following claims:

(1) M is p-nilpotent whenever $P \leq M < G$.

Since $N_M(P) \leq N_G(P)$, $N_M(P)$ is p-nilpotent. Let x be an element of $M \backslash N_M(P)$. First assume that G satisfies (a), (b) or (c). Since $P \cap M^{\mathcal{N}_p} \leq P \cap G^{\mathcal{N}_p}$, every element of $P \cap P^x \cap M^{\mathcal{N}_p}$ of order p is quasi-central in P. Now it is easy to see that M satisfies (a) or (b). If G satisfies (c), then

$$\big[\Omega_2\big(P \cap P^x \cap M^{\mathcal{N}_p}\big), P\big] \leq Z\big(P \cap G^{\mathcal{N}_p}\big) \cap \big(P \cap M^{\mathcal{N}_p}\big) \leq Z\big(P \cap M^{\mathcal{N}_p}\big),$$

hence M satisfies (c) too. If G satisfies (d), every element of $P \cap M^{\mathcal{N}_p}$ of order p is quasi-central in P. Moreover, since $\Omega_1(P \cap G^{\mathcal{N}_p})$ is an elementary abelian p-group by Lemma 2.1, we have

$$\big[\Omega_2\big(P \cap P^x \cap M^{\mathcal{N}_p}\big), P\big] \leq \Omega_1\big(P \cap G^{\mathcal{N}_p}\big) \cap \big(P \cap M^{\mathcal{N}_p}\big) = \Omega_1\big(P \cap M^{\mathcal{N}_p}\big).$$

Thus M satisfies (d). If G satisfies (e) or (f) then so does M as is easy to see. In other words, M satisfies the hypotheses of the theorem. The choice of G implies that M is p-nilpotent.

(2) $O_{p'}(G) = 1$.

If not, consider $\overline{G} = G/N$, where $N = O_{p'}(G)$. Clearly $N_{\overline{G}}(\overline{P}) = N_G(P)N/N$ is p-nilpotent, where $\overline{P} = PN/N$. For every $xN \in \overline{G} \backslash N_{\overline{G}}(\overline{P})$, since $\overline{G}^{\mathcal{N}_p} = G^{\mathcal{N}_p}N/N$ and $P \cap P^xN = P \cap P^{xn}$ for some $n \in N$, we have

$$\overline{P} \cap \overline{P}^{xN} \cap \overline{G}^{\mathcal{N}_p} = \big(P \cap P^{xn} \cap G^{\mathcal{N}_p}N\big)N/N = \big(P \cap P^{xn} \cap G^{\mathcal{N}_p}\big)N/N.$$

Because $xN \in \overline{G} \backslash N_{\overline{G}}(\overline{P})$, we get $xn \in G \backslash N_G(P)$. Now it is clear that every element of $\overline{P} \cap \overline{P}^{xN} \cap \overline{G}^{\mathcal{N}_p}$ of order p is quasi-central in $\overline{P}$ if G satisfies (a), (b) or (c). Moreover, if (c) is satisfied, then we have

$$\big[\Omega_2\big(\overline{P} \cap \overline{P}^{xN} \cap \overline{G}^{\mathcal{N}_p}\big), \overline{P}\,\big] = \big[\Omega_2\big(P \cap P^{xn} \cap G^{\mathcal{N}_p}\big), P\big]N/N \leq Z\big(P \cap G^{\mathcal{N}_p}\big)N/N,$$

that is,

$$\big[\Omega_2\big(\overline{P} \cap \overline{P}^{xN} \cap \overline{G}^{\mathcal{N}_p}\big), \overline{P}\,\big] \leq Z\big(\overline{P} \cap \overline{G}^{\mathcal{N}_p}\big).$$

If (d) is satisfied, every element of $\overline{P} \cap \overline{G}^{\mathcal{N}_p}$ of order p is quasi-central in $\overline{P}$. Besides,

$$\big[\Omega_2\big(\overline{P} \cap \overline{P}^{xN} \cap \overline{G}^{\mathcal{N}_p}\big), \overline{P}\,\big] = \big[\Omega_2\big(P \cap P^{xn} \cap G^{\mathcal{N}_p}\big), P\big]N/N \le \Omega_1\big(P \cap G^{\mathcal{N}_p}\big)N/N,$$

namely

$$\big[\Omega_2\big(\overline{P} \cap \overline{P}^{xN} \cap \overline{G}^{\mathcal{N}_p}\big), \overline{P}\,\big] \le \Omega_1\big(\overline{P} \cap \overline{G}^{\mathcal{N}_p}\big).$$

Now we see easily that $\overline{G}$ satisfies all the hypotheses of the theorem. The minimality of G implies that $\overline{G}$ is p-nilpotent and so is G, a contradiction.

(3) $G/O_p(G)$ is p-nilpotent and $C_G(O_p(G)) \le O_p(G)$.

Suppose that $G/O_p(G)$ is not p-nilpotent. Then, by Frobenius' theorem (refer [16, Theorem 10.3.2]), there exists a subgroup of P properly containing $O_p(G)$ such that its G-normalizer is not p-nilpotent. Since $N_G(P)$ is p-nilpotent, we may choice a subgroup P_1 of P such that $O_p(G) < P_1 < P$ and $N_G(P_1)$ is not p-nilpotent but $N_G(P_2)$ is p-nilpotent whenever $P_1 < P_2 \le P$. Denote $H = N_G(P_1)$. It is obvious that $P_1 < P_0 \le P$ for some Sylow p-subgroup P_0 of H. The choice of P_1 implies that $N_G(P_0)$ is p-nilpotent, hence $N_H(P_0)$ is also p-nilpotent. Let x be an element of $H \backslash N_H(P_0)$. Since $P_0 = P \cap H$, we have $x \in G \backslash N_G(P)$. Again, $P_0 \cap H^{\mathcal{N}_p} \le P \cap G^{\mathcal{N}_p}$, so every element of $P_0 \cap P_0^x \cap H^{\mathcal{N}_p}$ of order p is quasi-central in P_0 if G satisfies (a), (b) or (c) and every element of $P_0 \cap H^{\mathcal{N}_p}$ of order p is quasi-central in P_0 if G satisfies (d). Furthermore, if G satisfies (c), then

$$\big[\Omega_2\big(P_0 \cap P_0^x \cap H^{\mathcal{N}_p}\big), P_0\big] \le Z\big(P \cap G^{\mathcal{N}_p}\big) \cap \big(P_0 \cap H^{\mathcal{N}_p}\big) \le Z\big(P_0 \cap H^{\mathcal{N}_p}\big).$$

If G satisfies (d), as $\Omega_1(P \cap G^{\mathcal{N}_p})$ is an elementary abelian p-group, we have

$$\big[\Omega_2\big(P_0 \cap P_0^x \cap H^{\mathcal{N}_p}\big), P_0\big] \le \Omega_1\big(P \cap G^{\mathcal{N}_p}\big) \cap \big(P_0 \cap H^{\mathcal{N}_p}\big) = \Omega_1\big(P_0 \cap H^{\mathcal{N}_p}\big).$$

Now it is easily seen that H satisfies the hypotheses of the theorem. The minimality of G allows us to conclude that H is p-nilpotent, which is contrary to the choice of P_1. Hence $G/O_p(G)$ is p-nilpotent and G is p-solvable with $O_{p'}(G) = 1$. Consequently, we obtain $C_G(O_p(G)) \le O_p(G)$ (refer [11, Theorem 6.3.2]).

(4) $G = PQ$, where Q is an elementary abelian Sylow q-subgroup of G for a prime $q \neq p$. Moreover, P is maximal in G and $QO_p(G)/O_p(G)$ is minimal normal in $G/O_p(G)$.

For any prime divisor q of $|G|$ with $q \neq p$, since G is p-solvable, there exists a Sylow q-subgroup Q of G such that $G_0 = PQ$ is a subgroup of G ([11, Theorem 6.3.5]). If $G_0 < G$, then, by (1), G_0 is p-nilpotent. This leads to $Q \leq C_G(O_p(G)) \leq O_p(G)$, a contradiction. Thus $G = PQ$ and so G is solvable. Now let $T/O_p(G)$ be a minimal normal subgroup of $G/O_p(G)$ contained in $O_{pp'}(G)/O_p(G)$. Then $T = O_p(G)(T \cap Q)$. If $T \cap Q < Q$, then $PT < G$ and therefore PT is p-nilpotent by (1). It follows that

$$1 < T \cap Q \leq C_G(O_p(G)) \leq O_p(G),$$

which is impossible. Hence $T = O_{pp'}(G)$ and $QO_p(G)/O_p(G)$ is an elementary abelian q-group complementing $P/O_p(G)$. This yields that P is maximal in G.

(5) $|P : O_p(G)| = p$.

Clearly, $O_p(G) < P$. Let P_0 be a maximal subgroup of P containing $O_p(G)$ and let $G_0 = P_0 O_{pp'}(G)$. Then P_0 is a Sylow p-subgroup of G_0. The maximality of P in G implies that either $N_G(P_0) = G$ or $N_G(P_0) = P$. If the latter holds, then $N_{G_0}(P_0) = P_0$. On the other hand, $G^{\mathcal{N}_p} \leq O_p(G)$ by (3), hence $P \cap P^x \cap G^{\mathcal{N}_p} = G^{\mathcal{N}_p}$ for every $x \in G$. Now it is easy to check that G_0 satisfies the hypotheses of the theorem. Therefore G_0 is p-nilpotent and $Q \leq C_G(O_p(G)) \leq O_p(G)$, a contradiction. Thus $N_G(P_0) = G$ and $P_0 = O_p(G)$.

(6) $G = G^{\mathcal{N}_p} L$, where $L = \langle a \rangle [Q]$ is a non-abelian split extension of Q by a cyclic p-subgroup $\langle a \rangle$, $a^p \in Z(L)$ and the action of a (by conjugation) on Q is irreducible.

Write $T = G^{\mathcal{N}_p} Q$. Then $T \lhd G$ as $G/G^{\mathcal{N}_p}$ is p-nilpotent. Let P_0 be a maximal subgroup of P containing $G^{\mathcal{N}_p}$. Then, by the maximality of P, either $N_G(P_0) = P$ or $N_G(P_0) = G$. If $N_G(P_0) = P$, then $N_M(P_0) = P_0$, where $M = P_0 T = P_0 Q$. By (3), $G^{\mathcal{N}_p} \leq O_p(G)$, so $P \cap P^x \cap G^{\mathcal{N}_p} = G^{\mathcal{N}_p}$ for any $x \in G$. Evidently, $P_0 \cap P_0^y \cap M^{\mathcal{N}_p} \leq G^{\mathcal{N}_p}$ for all $y \in M \backslash N_M(P_0)$, hence M satisfies the hypotheses of the theorem. By the minimality of G, M is p-nilpotent. It follows that $T = G^{\mathcal{N}_p} Q = G^{\mathcal{N}_p} \times Q$ and so $Q \lhd G$, a contradiction. Thereby $N_G(P_0) = G$ and $P_0 \leq O_p(G)$. This yields from (5) that $O_p(G) = P_0$ and hence $P/G^{\mathcal{N}_p}$ is a cyclic group. Now applying the Frattini argument we have $G = G^{\mathcal{N}_p} N_G(Q)$. Therefore we may assume that $G = G^{\mathcal{N}_p} L$, where $L = \langle a \rangle [Q]$ is a non-abelian split extension of a normal Sylow q-subgroup Q by a cyclic p-group $\langle a \rangle$. Now that $|P : O_p(G)| = p$ and $O_p(G) \cap N_G(Q) \lhd N_G(Q)$, we have $a^p \in Z(L)$. Also since P is maximal in G, $G^{\mathcal{N}_p} Q / G^{\mathcal{N}_p}$ is minimal normal in $G/G^{\mathcal{N}_p}$ and consequently a acts irreducibly on Q.

(7) $G^{\mathcal{N}_p}$ has exponent p if $p > 2$ and exponent at most 4 if $p = 2$.

By Lemma 2.4 it will suffice to show that there exists a p-nilpotent maximal subgroup M of G such that $G = G^{\mathcal{N}_p} M$. In fact, let M be a maximal subgroup of G containing L. Then $M = L(M \cap G^{\mathcal{N}_p})$ and $G = G^{\mathcal{N}_p} M$. By Lemma 2.3,

$M \cap G^{\mathcal{N}_p} \lhd G$, hence $M = (\langle a\rangle(M \cap G^{\mathcal{N}_p}))Q$. Write $P_0 = \langle a\rangle(M \cap G^{\mathcal{N}_p})$ and let M_0 be a maximal subgroup of M containing P_0. Then $M_0 = P_0(M_0 \cap Q)$ and $G^{\mathcal{N}_p}M_0 < G$. By applying (1) we see that $G^{\mathcal{N}_p}M_0$ is p-nilpotent, therefore

$$M_0 \cap Q \leq C_G(O_p(G)) \leq O_p(G).$$

It follows that $M_0 \cap Q = 1$ and so P_0 is maximal in M. In this case, if $P_0 \lhd M$, then $\langle a\rangle = P_0 \cap L \lhd L$, which is contrary to (6). Hence $N_M(P_0) = P_0$ and M satisfies the hypotheses of the theorem. The choice of G implies that M is p-nilpotent, as desired.

Without losing generality, we assume in the following that $P = G^{\mathcal{N}_p}\langle a\rangle$.

(8) The exponent of $G^{\mathcal{N}_p}$ is not p.

If not, $G^{\mathcal{N}_p}$ has exponent p. First assume that G satisfies one of the conditions (a), (b), (c) and (d). Denote $G^{\mathcal{N}_p} \cap \langle a\rangle = \langle c\rangle$. Then $c^p = 1$ and $(G^{\mathcal{N}_p}/\langle c\rangle) \cap (\langle a\rangle/\langle c\rangle) = 1$. Noticing that $G^{\mathcal{N}_p} \cap \langle a\rangle < \langle a\rangle$, we have $c \in \langle a^p\rangle \leq Z(L)$. By Lemma 2.1, $G^{\mathcal{N}_p}$ is an elementary abelian p-group, hence $c \in Z(G)$. Now we consider $G/\langle c\rangle$. Let $y\langle c\rangle$ be an element of $G^{\mathcal{N}_p}/\langle c\rangle$, where $y \in G^{\mathcal{N}_p}$. By the hypotheses, $\langle y\rangle\langle a\rangle = \langle a\rangle\langle y\rangle$, hence

$$(\langle y\rangle\langle c\rangle/\langle c\rangle)(\langle a\rangle/\langle c\rangle) = (\langle a\rangle/\langle c\rangle)(\langle y\rangle\langle c\rangle/\langle c\rangle).$$

It follows that

$$(y\langle c\rangle)^{a\langle c\rangle} \in (G^{\mathcal{N}_p}/\langle c\rangle) \cap (\langle y\rangle\langle c\rangle/\langle c\rangle)(\langle a\rangle/\langle c\rangle) = \langle y\rangle\langle c\rangle/\langle c\rangle.$$

This indicates that $a\langle c\rangle$ induces a power automorphism of p-power order in the elementary abelian p-group $G^{\mathcal{N}_p}/\langle c\rangle$. Therefore $[G^{\mathcal{N}_p}/\langle c\rangle, a\langle c\rangle] = 1$ and $G^{\mathcal{N}_p}/\langle c\rangle$ is centralized by $P/\langle c\rangle$. If we write $C_{G/\langle c\rangle}(G^{\mathcal{N}_p}/\langle c\rangle) = K/\langle c\rangle$, then $P \leq K \lhd G$. By the maximality of P, either $P = K$ or $K = G$. If $P = K$ then $N_G(P) = G$ is p-nilpotent, contradicting to the choice of G. Hence $K = G$ and $[G^{\mathcal{N}_p}, Q] \leq \langle c\rangle$. This means that Q stabilizes the chain of subgroups $1 \leq \langle c\rangle \leq G^{\mathcal{N}_p}$. It follows from [11, Theorem 5.3.2] that $[G^{\mathcal{N}_p}, Q] = 1$ and Q is normal in G, a contradiction.

Now assume that G satisfies (e) or (f). Let N be a minimal normal subgroup of G contained in $O_p(G)$. Then $N_{G/N}(P/N) = N_G(P)/N$ is p-nilpotent as $N_G(P)$ is. Moreover, since $(G/N)^{\mathcal{N}_p} = G^{\mathcal{N}_p}N/N$ and $G^{\mathcal{N}_p}$ has exponent p, we obtain

$$|\Omega_1((G/N)^{\mathcal{N}_p})| = |\Omega((G/N)^{\mathcal{N}_p})| = |G^{\mathcal{N}_p}N/N| \leq |G^{\mathcal{N}_p}| \leq p^{p-1}.$$

This proves that G/N satisfies (e) or (f). Hence G/N is p-nilpotent by the choice of G. Since the class of p-nilpotent groups is a saturated formation, we may assume that N is the unique minimal normal subgroup of G contained in $O_p(G)$

and $\Phi(G) = 1$ as $\Phi(G) \le F(G) = O_p(G)$. Furthermore, $O_p(G)$ is the direct product of minimal normal subgroups of G by [2, III, Satz 4.5], thus $O_p(G) = N = G^{\mathcal{N}_p}$. Using (5), we obtain $|P| \le p^p$. It follows from [2, III, Satz 10.2(b)] that P is regular and, by Wielandt's theorem [2, IV, Satz 8.1], G is p-nilpotent, also a contradiction.

(9) The final contradiction.

From (7) and (8) we see that $p = 2$ and the exponent of $G^{\mathcal{N}_2}$ is 4. By applying Lemma 2.4, $(G^{\mathcal{N}_2})' = Z(G^{\mathcal{N}_2}) = \Phi(G^{\mathcal{N}_2})$ is an elementary abelian 2-group, it follows that $\Phi(G^{\mathcal{N}_2}) \le \Omega_1(G^{\mathcal{N}_2})$. First assume that G satisfies one of (a), (b), (c) and (d). Since $\Omega_1(G^{\mathcal{N}_2})$ is an elementary abelian 2-group, we have $\Omega_1(G^{\mathcal{N}_2}) < G^{\mathcal{N}_2}$. However, $G^{\mathcal{N}_2}/\Phi(G^{\mathcal{N}_2})$ is a chief factor of G by Lemma 2.4, so $Z(G^{\mathcal{N}_2}) = \Phi(G^{\mathcal{N}_2}) = \Omega_1(G^{\mathcal{N}_2})$. Now assume that G satisfies (a). If $\Phi(G^{\mathcal{N}_2}) \cap \langle a\rangle \ne 1$ then there exists an element c in $\Phi(G^{\mathcal{N}_2}) \cap \langle a\rangle$ such that $\circ(c) = 2$. As $\Phi(G^{\mathcal{N}_2}) \cap \langle a\rangle < \langle a\rangle$, we have $c \in \langle a^2\rangle \le Z(L)$. So $c \in Z(G)$. If $\Phi(G^{\mathcal{N}_2}) \cap \langle a\rangle = 1$ then a induces a power automorphism of 2-power order in the elementary abelian 2-group $\Phi(G^{\mathcal{N}_2})$, hence $[\Phi(G^{\mathcal{N}_2}), a] = 1$. In view of Lemma 2.1, $\Phi(G^{\mathcal{N}_2})$ is also centralized by $G^{\mathcal{N}_2}$, thereby we get $\Phi(G^{\mathcal{N}_2}) \le Z(P)$. Furthermore, by the Frattini argument,

$$G = N_G(\Phi(G^{\mathcal{N}_2})) = C_G(\Phi(G^{\mathcal{N}_2}))N_G(P).$$

Noticing that $N_G(P) = P$ and $P \le C_G(\Phi(G^{\mathcal{N}_2}))$, we obtain $C_G(\Phi(G^{\mathcal{N}_2})) = G$, i.e., $\Phi(G^{\mathcal{N}_2}) \le Z(G)$. Thus we can also take an element c in $\Phi(G^{\mathcal{N}_2})$ such that $\circ(c) = 2$ and $c \in Z(G)$. Denote $N = \langle c\rangle$ and consider $\overline{G} = G/N$. For any $y \in G^{\mathcal{N}_2}$, since y is quasi-central in P, yN is quasi-central in $\overline{P} = P/N$. This shows that $\overline{G}$ satisfies (a). The minimality of G implies that $\overline{G}$ is 2-nilpotent and so is G, a contradiction. Now assume that G satisfies (b), (c) or (d). Let M be a maximal subgroup of G containing L. Then M is 2-nilpotent by the proof of (7), hence $\Phi(G^{\mathcal{N}_2})Q$ is 2-nilpotent and $[\Phi(G^{\mathcal{N}_2}), Q] = 1$. In this case, if G satisfies (b), then Q acts trivially on $G^{\mathcal{N}_2}$ by Lemma 2.2, thus Q is normal in G, a contradiction. Assume that G satisfies (c) or (d). Denote $K = C_G(G^{\mathcal{N}_2}/\Phi(G^{\mathcal{N}_2}))$. Then, by the hypotheses, $P \le K \lhd G$. The maximality of P yields that $P = K$ or $K = G$. If the former holds, then $G = N_G(P)$ is 2-nilpotent, a contradiction. If the latter holds, then $[G^{\mathcal{N}_2}, Q] \le \Phi(G^{\mathcal{N}_2})$. Therefore Q stabilizes the chain of subgroups $1 \le \Phi(G^{\mathcal{N}_2}) \le G^{\mathcal{N}_2}$. It follows from [11, Theorem 5.3.2] that $[G^{\mathcal{N}_2}, Q] = 1$ and Q is normal in G, which is impossible.

Finally we assume that G satisfies (e). In this case, $\Omega_1(G^{\mathcal{N}_2})$ is a cyclic subgroup of order 2, of course, Q acts trivially on $\Omega_1(G^{\mathcal{N}_2})$. Consequently, Q acts trivially on $G^{\mathcal{N}_2}$ by Lemma 2.2 and Q is normal in G, a contradiction. Similarly, we can get a final contradiction if G satisfies (f). This completes our proof. □

Proof of Theorem 1.2. By Theorem 1.1, we only need to prove $N_G(P)$ is p-nilpotent under the sufficient conditions.

If $N_G(P)$ is not p-nilpotent, then $N_G(P)$ has a minimal non-p-nilpotent subgroup H. By results of Itô ([2, IV, 5.4]) and Schmidt ([2, III, 5.2]), H has a normal Sylow p-subgroup H_p and a cyclic Sylow q-subgroup H_q such that $H = [H_p]H_q$. Moreover, H_p is of exponent p if $p > 2$ and of exponent at most 4 if $p = 2$. On the other hand, the minimality of H implies that $H^{\mathcal{N}_p} = H_p$. Let x be an element of H_p of order p. Then, by the hypotheses, $\langle x\rangle H_q$ is a subgroup of H. If $\langle x\rangle H_q = H$, then $H_p = \langle x\rangle$ is cyclic and H is p-nilpotent by Lemma 2.5, a contradiction. Hence $\langle x\rangle H_q < H$ and $\langle x\rangle H_q = \langle x\rangle \times H_q$. Thus $\Omega_1(H_p)$ is centralized by H_q. Also, by Lemma 2.1, $\Omega_1(H_p)$ is centralized by H_p, so $\Omega_1(H_p) \leq Z(H)$. If H_p is of exponent p or H_p is quaternion-free, then H_q acts trivially on H_p by Lemma 2.2, that is, H_q is normal in H, a contradiction. Thus $p = 2$ and H_2 is of exponent 4. Applying Lemma 2.4, $Z(H_2)$ is an elementary abelian 2-group, hence $\Omega_1(H_2) = Z(H_2)$. Assume that (a) is satisfied. Let y be an element of H_2 of order 4. Since $\langle y\rangle$ is quasi-central in $N_G(P)$, we obtain $\langle y\rangle H_q = H_q\langle y\rangle$. If $\langle y\rangle H_q = H$ then $\langle y\rangle = H_2$ is cyclic and H is 2-nilpotent, a contradiction. So $\langle y\rangle H_q < H$ and $\langle y\rangle H_q$ is nilpotent and, consequently, $\langle y\rangle H_q = \langle y\rangle \times H_q$. Furthermore, $H_2 = \Omega_2(H_2)$ centralizes H_q and H_q is normal in H, a contradiction. The proof is complete. □

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HUAQUAN WEI
SCHOOL OF MATHEMATICS SCIENCE
GUANGXI TEACHERS EDUCATION UNIVERSITY
NANNING, 530001
CHINA

E-mail: weihuaquan@163.com

YANMING WANG
LINGNAN COLLEGE
ZHONGSHAN UNIVERSITY
GUANGZHOU, 510275
CHINA

GUOHUA QIAN
DEPARTMENT OF MATHEMATICS
CHANGSHU INSTITUTE OF TECHNOLOGY
CHANGSHU, 215500
CHINA

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