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Finsler space connected by angle in two dimensions. Regular case

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Abstract. We show that the metrical connection can be introduced in the two-dimensional Finsler space such that entailed parallel transports along curves joining points of the underlying manifold keep the two-vector angle as well as the length of the tangent vector, thereby realizing isometries of tangent spaces under the parallel transports. The curvature tensor is found. In case of the Finsleroid-regular space, constructions possess the C^{∞}-regular status globally regarding the dependence on tangent vectors. Many involved and important relations are explicitly derived.

1. Motivation and description

During all the history of development of the Finsler geometry the notion of connection was attracted sincere and great attention of investigators devoted to general theory as well as to specialized applications. The methods of construction of connection are founded upon setting forth a convenient system of axioms. Various standpoints were taken to get deeper insights into the notion (see [1]–[5]).

The general idea underlining the present work is to set forth the requirement that the connection be compatible with the preservation of the two-vector angle under the parallel transports of vectors.

The notion of angle is of key significance in geometry. In the field of two-dimensional Finsler spaces the angle between two vectors of a given tangent space can naturally be measured by the area of the domain bounded by the vectors

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and the indicatrix arc. The theorem can be proved which states that a diffeomorphism between two Finsler spaces is an isometry iff it keeps the angles thus appeared. This fundamental TAMÁSSY's theorem [6], which explains us that the angle structure fixes the metric structure in the Finsler space, gives rise to the following important question: *Does the angle structure also generate the connection?* The present work proposes a positive and explicit answer, confining the Finslerian consideration to the two-dimensional case.

Let M be a C^∞-differentiable 2-dimensional manifold, T_xM denote the tangent space supported by the point $x \in M$, and $y \in T_xM\backslash 0$ mean tangent vectors. Given a Finsler metric function $F = F(x,y)$, we obtain the two-dimensional Finsler space $F_2 = (M, F)$.

We shall use the standard Finslerian notation for local components $l^k = y^k/F$, $y_k = F\partial F/\partial y^k \equiv g_{kn}y^n$, $g_{ij} = \partial y_i/\partial y^j$ of the unit vector, the covariant tangent vector, and the Finsler metric tensor, respectively. The covariant components $l_k = g_{kn}l^n$ can be obtained from $l_k = \partial F/\partial y^k$. By means of the contravariant components g^{ij} the Cartan tensor $C_{ijk} = (1/2)\partial g_{ij}/\partial y^k$ can be contracted to yield the vector $C_k = g^{ij}C_{ijk}$. It is convenient to use the tensor $A_{ijk} = FC_{ijk}$ and the vector $A_k = FC_k = g^{ij}A_{ijk}$. The indices $i, j, \ldots$ are specified over the range $(1, 2)$. The square root $\surd$ stands always in the positive sense. It is often convenient to apply the expansion $A_{ijk} = Im_im_jm_k$ in terms of the m_i obtainable from $g_{ij} = l_il_j + m_im_j$, where I thus appeared is the so-called main scalar. Our consideration will be of local nature, unless otherwise is stated explicitly.

To each point $x \in M$, the Finsler space F_2 associates the tangent Riemannian space, to be denoted by $R_{\{x\}} := \{T_xM, g_{ij}(x,y)\}$, in which x is treated fixed and $y \in T_xM$ is variable. In the Riemannian space the $R_{\{x\}}$ reduces to the tangent Euclidean space. The remarkable and well-known property of the Riemannian Levi Civita connection is that the entailed parallel transports along curves drawn on the underlined manifold keep the length of the tangent vectors and produce the isometric mapping of the tangent Euclidean spaces.

We show that these two fundamental Riemannian properties can successfully be extended to operate in the Finsler space F_2. Namely, if sufficient smoothness holds then it proves possible to introduce the respective connection coefficients $\{N^k_i(x,y), D^k{}_{in}(x,y)\}$ in a simple and explicit way. The coefficients $N^k_i(x,y)$ are required to construct the conventional operator $d_n = \partial_{x^n} + N^k_n(x,y)\partial_{y^k}$, where $\partial_{x^n} = \partial/\partial x^n$ and $\partial_{y^m} = \partial/\partial y^m$. In modern geometrical language, the local covariant vector fields $\partial_{x^n} + N^k_n(x,y)\partial_{y^k}$ are the horizontal lifts of the ordinary gradient fields ∂_{x^n} to the tangent bundle TM. The keeping of the Finsler length of the tangent vectors means $d_n F = 0$. Let us attract also the angle function $\theta = \theta(x,y)$ (to measure the length ds of infinitesimal arc on the indicatrix by $d\theta$) and raise forth the requirement that $d_n\theta = k_n$ with a covariant vector field $k_n = k_n(x)$. If this is fulfilled, then for pairs $\{\theta_1 = \theta(x,y_1), \theta_2 = \theta(x,y_2)\}$ we obtain the nullification $d_n(\theta_2 - \theta_1) = 0$ which tells us that the *preservation of the two-vector angle* $\theta_2 - \theta_1$ holds true under the parallel transports initiated by the coefficients $N^k_i(x,y)$.

There arise the coefficients $D^k{}_{in}(x,y) = -N^k{}_{in}(x,y)$ with $N^k{}_{in}(x,y) = \partial N^k_i(x,y)/\partial y^n$. A careful analysis has shown that a simple and attractive proposal of the coefficients $N^k_i(x,y)$ (namely, (2.6) of Section 2) can be made such that nullifications $d_i F = 0$ and $d_i\theta - k_i = 0$ are simultaneously satisfied, and also the vanishing $l_k N^k{}_{nmi} = 0$ holds fine because of the representation $FN^k{}_{nmi} = -A^k{}_{mi} d_n \ln|I|$ entailed (see (2.14)), where $N^k{}_{nmi} = \partial N^k{}_{nm}/\partial y^i$, so that the action of the arisen covariant derivative on the involved Finsler metric tensor yields just the zero. The coefficients $D^k{}_{in}$ are not symmetric in the subscripts i, n.

Having realized this program, we feel sure that the arisen mappings of the space $R_{\{x\}}$ under the respective parallel transports along the curves running on M are isometries.

The coefficients $\{N^k_i(x,y), D^k{}_{in}(x,y)\}$ obtained in this way are *not* constructed from the Finsler metric tensor and derivatives of the tensor. This circumstance may be estimated to be a cardinal distinction of the Finsler connection induced by the angle structure from the conventional Riemannian precursor which exploits the Riemannian Christoffel symbols to be the coefficients $D^k{}_{in}$. The structure of the coefficients N^k_n involves the derivative $\partial\theta/\partial x^n$ on the equal footing with the derivative $\partial F/\partial x^n$ (see (2.6)).

The involved vector field $k_i = k_i(x)$ may be taken arbitrary. However, the field can be specified if the Riemannian limit of the connection proposed is attentively considered. Indeed, in the Riemannian limit the connection coefficients $D^k{}_{nh}(x,y)$ reduce to the coefficients $\bar{L}^k{}_{nh} = -L^k{}_{nh} = \bar{L}^k{}_{nh}(x)$ which are *not symmetric* with respect to the subscripts (see (4.6)). If we want to obtain the torsionless coefficients, like to the Riemannian geometry proper, we must make the choice $k_n = -n^h\nabla_n\widetilde{b}_h$ in accordance with (4.16), where ∇_n is the Riemannian covariant derivative taken with the Christoffel symbols $a^k{}_{nh}$. The $\widetilde{b}_h = \widetilde{b}_h(x)$ is a vector field chosen to fulfill $\theta(x,\widetilde{b}(x)) = 0$, and the pair $n_i, \widetilde{b}_i$ is orthonormal. With the choice we obtain $\bar{L}^k{}_{nh} = a^k{}_{nh}$, thereby completely specifying the coefficients $\{N^k_i, D^k{}_{in}\}$.

It is appropriate to construct the *osculating* Riemannian metric tensor along the vector field $\widetilde{b}^i = \widetilde{b}^i(x)$ and introduce the *θ-associated Riemannian space* to compare the Finsler properties of the space F_2 with the properties of the Riemannian precursor.

In Section 2 the required coefficients N^k_n are proposed and nearest implications are indicated. By the help of the identities $\partial m^k/\partial y^m = -(Im^k + l^k)m_m/F$ and $\partial m_m/\partial y^i = (Im_m - l_m)m_i/F$, the validity of the vanishing $l_k N^k{}_{nmi} = 0$ can readily be verified. The angle function θ introduced does measure the length of the indicatrix arc according to $ds = d\theta$ (see (2.23)). We derive also the equality $\sum_{\{y_1,y_2\}} = (1/2)(\theta_2 - \theta_1)$, where the left-hand side is the area of the sector bounded by the vectors y_1, y_2 and the indicatrix arc (see (2.27)). The equality demonstrates clearly that, in context of the two-dimensional theory to which our treatment is restricted, the method of introduction of the angle by the help of the function θ is equivalent to the method founded in [6] on the notion of area. We are entitled therefore to raise the thesis that, in such a context, the angle-preserving connection is tantamount to the area-preserving connection.

In Section 3 we show how the curvature tensor $\rho_k{}^n{}_{ij}$ of the space F_2 can be explicated from the commutator of the covariant derivative arisen, yielding the astonishingly simple representation (3.14).

In Section 4 we outline Riemannian counterparts. It appears that the tensor $\rho_{knij} = g_{nm}\rho_k{}^m{}_{ij}$ is factorable, in accordance with (4.19).

In Conclusions several ideas are emphasized.

The possibility of global realization of the angle-preserving connection implies high regularity properties of the Finsler metric function and the angle function. Such a lucky possibility occurs in the Finsleroid-regular space $FR^{PD}_{g;c}$ (introduced and studied in [7]–[10]). The Finsler metric function $K = K(x, y)$ of the space $FR^{PD}_{g;c}$ involves a Riemannian metric tensor a_{mn} and the vector field $b^n = b^n(x)$ which represents the distribution of axis of indicatrix. We have two scalars, namely the characteristic scalar $g = g(x)$ and the norm $c = c(x) = \|b\| \equiv \sqrt{a^{mn}b_m b_n}$ of the 1-form $b = b_i(x)y^i$. The metric function K is not absolute homogeneous.

Attentive calculation presented in Appendix B in [11] has shown that the partial derivative $\partial K/\partial x^n$ obeys the total regularity with respect to the vector variable y (see (B.59) in [11]). The same regularity is shown by the partial derivative $\partial\theta/\partial x^n$ of the involved angle function $\theta = \theta(x, y)$ (see (B.83) in [11]). Therefore, all the ingredients in the coefficients N^k_n of the form proposed by (2.6) are of this high regularity. Thus we observe the remarkable phenomenon that the space $FR^{PD}_{g;c}$ possesses the angle-preserving connection of the C^∞-regular status globally regarding the y-dependence. Arbitrary (smooth) dependence on x in $g = g(x)$, $b_i = b_i(x)$, and $a_{ij} = a_{ij}(x)$ is permitted. Using an appropriate regular atlas of charts in the space $R_{\{x\}} := \{T_xM, g_{ij}(x,y)\}$, it proves possible to verify that over all the space $T_xM\backslash 0$ the function $\theta = \theta(x,y)$ is smooth of class C^∞ with respect to y. The entailed two-vector angle $\theta_2 - \theta_1$ is symmetric and additive. The θ is represented by means of integral and is not obtainable through composition of elementary functions.

Quite similar evaluation can be performed for the Randers metric function (Appendix C in [11]), yielding again the angle-preserving connection of the C^∞-regular status globally regarding the y-dependence.

More detail of calculation can be found in [11].

2. Proposal of connection coefficients

It is convenient to proceed with the skew-symmetric tensorial object $\epsilon_{ik} = \sqrt{\det(g_{mn})}\,\gamma_{ik}$, where $\gamma_{11} = \gamma_{22} = 0$ and $\gamma_{12} = -\gamma_{21} = 1$, to construct

$$m_i = -\epsilon_{ik} l^k. \tag{2.1}$$

The angular metric tensor $h_{ij} = g_{ij} - l_i l_j$ and the Cartan tensor C_{ijk} are factorized, and the Finsler metric tensor g_{ij} is expanded, according to

$$h_{ij} = m_i m_j, \qquad A_{ijk} = I m_i m_j m_k, \qquad g_{ij} = l_i l_j + m_i m_j. \tag{2.2}$$

It is also convenient to introduce the $\theta = \theta(x,y)$ by the help of the equation

$$F\frac{\partial \theta}{\partial y^n} = m_n, \tag{2.3}$$

assuming that the function θ is positively homogeneous of degree zero with respect to the variable y. These formulas are known from Section 6.6 of the book [1]. We denote the main scalar by I, instead of J used in the book. Our θ is the φ of Section 6.6 of [1]. The object ϵ_{ik} is a pseudo-tensor, whence m_i is a pseudo-vector and I, θ are pseudo-scalars. However, we don't consider the coordinate reflections and, therefore, we are entitled to refer to these objects as to "the vector m_i" and "the scalars I, θ".

We need the coefficients $N^k_n = N^k_n(x,y)$ to construct the operator

$$d_n = \frac{\partial}{\partial x^n} + N^k_n \frac{\partial}{\partial y^k} \tag{2.4}$$

which generates a covariant vector $d_n W$ when is applied to an arbitrary differentiable scalar $W = W(x,y)$. We shall use also the derivative coefficients

$$N^k{}_{nm} = \frac{\partial N^k_n}{\partial y^m}, \qquad N^k{}_{nmi} = \frac{\partial N^k{}_{nm}}{\partial y^i}. \tag{2.5}$$

Proposal. *Take the coefficients N^k_n according to the expansion*

$$N^k_n = -l^k \frac{\partial F}{\partial x^n} - F m^k \breve{P}_n \tag{2.6}$$

with

$$\breve{P}_n = \frac{\partial \theta}{\partial x^n} - k_n, \tag{2.7}$$

where $k_n = k_n(x)$ is a covariant vector field, such that the equalities

$$d_n F = 0, \qquad d_n \theta = k_n, \qquad y_k N^k_{nmi} = 0 \tag{2.8}$$

be realized.

The vanishing $d_n F = 0$ and the equality $d_n \theta = k_n$ just follow from the choice (2.6). Considering two values $\theta_1 = \theta(x, y_1)$ and $\theta_2 = \theta(x, y_2)$, we have

$$d_n \theta_1 = \frac{\partial \theta_1}{\partial x^n} + N^k_n(x, y_1) \frac{\partial \theta_1}{\partial y^k_1}, \qquad d_n \theta_2 = \frac{\partial \theta_2}{\partial x^n} + N^k_n(x, y_2) \frac{\partial \theta_2}{\partial y^k_2}, \tag{2.9}$$

and from $d_n \theta = k_n$ we may conclude that *the preservation*

$$d_n(\theta_2 - \theta_1) = 0 \tag{2.10}$$

holds because the vector field k_n is independent of tangent vectors y.

From (2.6) it follows directly that

$$N^k{}_{nm} = -l^k \frac{\partial l_m}{\partial x^n} - l_m m^k \breve{P}_n - F \frac{\partial m^k}{\partial y^m} \breve{P}_n - m^k \frac{\partial m_m}{\partial x^n}. \tag{2.11}$$

It is convenient to use the identities

$$F \frac{\partial m_k}{\partial y^m} = -l_k m_m + I m_m m_k, \qquad F \frac{\partial m^k}{\partial y^m} = -I m^k m_m - l^k m_m \tag{2.12}$$

(they are tantamount to the identities written in formula (6.22) of chapter 6 in [1]), together with their immediate implication $F \partial(m^k m_n)/\partial y^m = -(l_n m^k m_m + l^k m_n m_m)$. Short evaluations show that

$$N^k{}_{nmi} = \frac{1}{F} m^k m_m \left(F \frac{\partial I}{\partial y^i} \breve{P}_n - m_i \frac{\partial I}{\partial x^n} \right). \tag{2.13}$$

Indeed, from (2.11) it follows straightforwardly that

$$N^k{}_{nmi} = -\frac{1}{F} h^k_i \partial_n l_m - l^k \partial_n \left(\frac{1}{F} h_{mi} \right) - \frac{1}{F} h_{mi} m^k \breve{P}_n$$

$$
\begin{aligned}
&+l_m\Bigl(I\,m^k m_i + l^k m_i\Bigr)\breve{P}_n - l_m m^k \partial_n\left(\frac{1}{F}m_i\right)\\
&+\frac{\partial I}{\partial y^i}\,m^k m_m \breve{P}_n + I\,m^k m_m \partial_n\left(\frac{1}{F}m_i\right) - I l^k m_i m_m \breve{P}_n - I m^k l_m m_i \breve{P}_n\\
&+\frac{1}{F}h_i^k m_m \breve{P}_n - l^k(l_m - I m_m)m_i \breve{P}_n + l^k m_m \partial_n\left(\frac{1}{F}m_i\right)\\
&+(I m^k + l^k)m_i \frac{1}{F}\partial_n m_m + m^k \partial_n\Bigl(\frac{1}{F}l_m m_i - \frac{1}{F}I\,m_i m_m\Bigr),
\end{aligned}
$$

where ∂_n means $\partial/\partial x^n$. The identity $h_{mi} = m_m m_i$ has been taken into account. Canceling similar terms leads to (2.13). Owing to the identity $l_k m^k = 0$, the vanishing $l_k N^k{}_{nmi} = 0$ holds fine.

Because of $y^i \partial_{y^i} I = 0$, the equality (2.13) can be written in the concise form

$$N^k{}_{nmi} = -\frac{1}{F}m^k m_m m_i d_n I \equiv -\frac{1}{F}A^k{}_{mi} d_n \ln|I|. \tag{2.14}$$

The sought *Finsler connection*

$$FC = \{N_n^k, D^k{}_{nm}\} \tag{2.15}$$

involves also the coefficients $D^k{}_{nm} = D^k{}_{nm}(x,y)$ which are required to construct the operator of the covariant derivative D_n which action is exemplified in the conventional way:

$$D_n w^k{}_m := d_n w^k{}_m + D^k{}_{nh} w^h{}_m - D^h{}_{nm} w^k{}_h, \tag{2.16}$$

where $w^k{}_m = w^k{}_m(x,y)$ is an arbitrary differentiable (1,1)-type tensor.

If we differentiate the vanishing $d_i F = 0$ with respect to the variable y^j and multiply the result by F, we obtain the vanishing

$$D_i y_j := \frac{\partial y_j}{\partial x^i} + N_i^k g_{kj} - D^h{}_{ij} y_h = 0 \tag{2.17}$$

when the choice

$$D^k{}_{in} = -N^k{}_{in} \tag{2.18}$$

is made. Differentiating (2.17) with respect to y^njust manifests that the choice is also of success to fulfill the metricity condition

$$D_i g_{jn} := d_i g_{jn} - D^h{}_{ij} g_{hn} - D^h{}_{in} g_{jh} = 0, \tag{2.19}$$

because of $y_k N^k_{nmi} = 0$.

If we contract (2.18) by y^n and take into account the definition of the coefficients $N^k{}_{in}$ indicated in (2.5), we obtain the equality

$$N_i^k = -D^k{}_{in} y^n. \tag{2.20}$$

The contravariant version of the vanishing (2.17) is obtained through the chain

$$D_i y^j := d_i y^j + D^j{}_{ih} y^h = N_i^j + D^j{}_{ih} y^h = 0. \tag{2.21}$$

Because of $D_i h^{nk} = 0$, applying the derivative D_i to the equality $h^{nk} = m^n m^k$ (see (2.2)) and contracting the result by m_n, we conclude that

$$D_i m^k = 0, \quad \text{which means} \quad d_i m^k = N^k{}_{ih} m^h. \tag{2.22}$$

Because of the homogeneity, the unit tangent vector components $l^n = l^n(x, y)$ can obviously be regarded as functions $l^n = L^n(x, \theta)$ of the pair (x, θ). Let us denote $l^n_\theta = \partial L^n/\partial\theta$. Since $\partial F/\partial\theta = 0$ and $l_n l^n_\theta = 0$, we may conclude from (2.3) that $l^n_\theta = m^n$. Measuring the length of the indicatrix (which is defined by $F = 1$) by means of a parameter s, so that $ds = \sqrt{g_{ij} dl^i dl^j}$, we obtain $ds = \sqrt{g_{ij} l^i_\theta l^j_\theta}\, d\theta = d\theta$, assuming $ds > 0$ and $d\theta > 0$. Thus

$$ds = d\theta \quad \text{along the indicatrix}, \tag{2.23}$$

which explains us that the θ *measures the length of indicatrix.*

If at a fixed x we introduce in the tangent space $T_x M$ the coordinates $z^A = \{z^1 = F, z^2 = \theta\}$ and consider the respective transforms

$$G_{AB} = g_{ij} \frac{\partial y^i}{\partial z^A} \frac{\partial y^i}{\partial z^B}, \quad A = 1, 2, \ B = 1, 2, \tag{2.24}$$

of the Finsler metric tensor components g_{ij}, we obtain simply

$$G_{11} = 1, \quad G_{12} = 0, \quad G_{22} = F^2. \tag{2.25}$$

With these components, it is easy to calculate the area of domain of the tangent space $T_x M$ by using the integral measure

$$\int \sqrt{\det(G_{AB})}\, dz^1 dz^2 = \int F dF d\theta. \tag{2.26}$$

In particular, for the sector $\sigma_{\{y_1, y_2\}} \subset T_x M$ bounded by the vectors y_1, y_2 and the indicatrix arc we obtain by integration the area $\sum_{\{y_1, y_2\}}$ which is given by

$$\sum_{\{y_1, y_2\}} = \frac{1}{2}(\theta_2 - \theta_1), \tag{2.27}$$

so that in the two-dimensional case the angle in the Finsler geometry can be defined by the area just in the same way as in the Riemannian geometry. The difference

$$\theta_2 - \theta_1 = \theta(x, y_2) - \theta(x, y_1) \tag{2.28}$$

can naturally be regarded as the value of angle between two vectors $y_1, y_2 \in T_xM$, the *two-vector angle* for short. The formula (2.27) is equivalent to Definition (2) of [6] which was proposed to define angle by area; we use the right orientation of angle.

As well as the area is attributed to the tangent space by means of the integral measure (2.26) and the conditions $d_nF = 0$ and $d_n(\theta_2 - \theta_1) = 0$ are fulfilled, *the angle-preserving connection keeps the area under parallel transports* along curves joining point to point in the background manifold. Thus we are entitled to set forth the thesis: *the angle-preserving connection is the area-preserving connection.*

3. Curvature tensor

With arbitrary coefficients $\{N^k_n, D^k{}_{nm}\}$, commuting the covariant derivative (2.16) yields the equality

$$(D_iD_j - D_jD_i)\, w^n{}_k = M^h{}_{ij}\frac{\partial w^n{}_k}{\partial y^h} - E_k{}^h{}_{ij}w^n{}_h + E_h{}^n{}_{ij}w^h{}_k \tag{3.1}$$

with the tensors

$$M^n{}_{ij} := d_iN^n_j - d_jN^n_i \tag{3.2}$$

and

$$E_k{}^n{}_{ij} := d_iD^n{}_{jk} - d_jD^n{}_{ik} + D^m{}_{jk}D^n{}_{im} - D^m{}_{ik}D^n{}_{jm}. \tag{3.3}$$

If the choice $D^k{}_{in} = -N^k{}_{in}$ is made (see (2.18)), the tensor (3.2) can be written in the form

$$M^n{}_{ij} = \frac{\partial N^n_j}{\partial x^i} - \frac{\partial N^n_i}{\partial x^j} - N^h_i D^n{}_{jh} + N^h_j D^n{}_{ih}, \tag{3.4}$$

which entails the equality

$$E_k{}^n{}_{ij} = -\frac{\partial M^n{}_{ij}}{\partial y^k}. \tag{3.5}$$

By applying the commutation rule (3.1) to the vanishing set $\{D_iF = d_iF = 0,$ $D_iy^n = 0,\ D_iy_k = 0,\ D_ig_{nk} = 0\}$, we respectively obtain the identities

$$y_nM^n{}_{ij} = 0, \quad y^kE_k{}^n{}_{ij} = -M^n{}_{ij}, \quad y_nE_k{}^n{}_{ij} = g_{kn}M^n{}_{ij},$$
$$E_{mnij} + E_{nmij} = 2C_{mnh}M^h{}_{ij}. \tag{3.6}$$

In case of the coefficients $\{N^k_n, D^k{}_{nm}\}$ proposed by (2.6) and (2.18) the direct calculation of the right-hand parts in (3.2) and (3.3) results in

Theorem 3.1. *The tensors $M^n{}_{ij}$ and $E_k{}^n{}_{ij}$ are represented by the following simple and explicit formulas:*

$$M^n{}_{ij} = Fm^n M_{ij} \tag{3.7}$$

and

$$E_k{}^n{}_{ij} = \bigl(-l_k m^n + l^n m_k + I m_k m^n\bigr) M_{ij}, \tag{3.8}$$

where

$$M_{ij} = \frac{\partial k_j}{\partial x^i} - \frac{\partial k_i}{\partial x^j}. \tag{3.9}$$

To verify this theorem it is convenient to use (2.22)) and (2.11) and obtain

$$d_i m^k = -m^h l^k \frac{\partial l_h}{\partial x^i} + (Im^k + l^k)\check{P}_i - m^h m^k \frac{\partial m_h}{\partial x^i}.$$

Full evaluations can be found in Appendix A of [11].

It proves pertinent to replace in the commutator (3.1) the partial derivative $\partial w^n{}_k/\partial y^h$ by the definition

$$S_h w^n{}_k = \frac{\partial w^n{}_k}{\partial y^h} + C^n{}_{hk} w^h{}_k - C^m{}_{hk} w^n{}_m \tag{3.10}$$

which has the meaning of the covariant derivative in the tangent Riemannian space $R_{\{x\}}$. With the *curvature tensor*

$$\rho_k{}^n{}_{ij} = E_k{}^n{}_{ij} - M^h{}_{ij} C^n{}_{hk}, \tag{3.11}$$

the commutator takes on the form

$$(D_i D_j - D_j D_i)\, w^n{}_k = M^h{}_{ij} S_h w^n{}_k - \rho_k{}^h{}_{ij} w^n{}_h + \rho_h{}^n{}_{ij} w^h{}_k. \tag{3.12}$$

The skew-symmetry

$$\rho_{mnij} = -\rho_{nmij} \tag{3.13}$$

holds (cf. the last item in (3.6)).

If we take into account the form of the tensor C_{ijk} indicated in (2.2), from (3.8) and (3.11) we may conclude that the curvature tensor is of the following astonishingly simple form:

$$\rho_k{}^n{}_{ij} = (l^n m_k - l_k m^n) M_{ij} \equiv \epsilon^n{}_k M_{ij}. \tag{3.14}$$

The tensor $\rho_{knij} = g_{nm}\rho_k{}^m{}_{ij}$ can be represented in the form

$$\rho_{knij} = \epsilon_{nk} M_{ij}. \tag{3.15}$$

We have $l^k m_n \rho_k{}^n{}_{ij} = -M_{ij}$.

4. Riemannian counterparts

If the Finsler space F_2 is a Riemannian space with a Riemannian metric function $S = \sqrt{a_{ij}y^iy^j}$, where $a_{ij} = a_{ij}(x)$ is a Riemannian metric tensor, we can consider the Riemannian precursor coefficients

$$L_n^k = N_n^k\Big|_{\text{Riemannian limit}}. \tag{4.1}$$

From (2.6) it follows that

$$L_n^k = -y^k \frac{1}{S}\frac{\partial S}{\partial x^n} - Sm^k\left(\frac{\partial \theta}{\partial x^n} - k_n\right). \tag{4.2}$$

On the other hand, denoting by $a^k{}_{nh}$ the Riemannian Christoffel symbols constructed from the Riemannian metric tensor a_{mn}, we can obtain the equality

$$a^k{}_{nh}y^h = \frac{1}{S}\frac{\partial S}{\partial x^n}y^k + \left(\frac{\partial \theta}{\partial x^n} + n^h\nabla_n\widetilde{b}_h\right)Sm^k \tag{4.3}$$

(see [11]), where $\widetilde{b}_h = \widetilde{b}_h(x)$ is a vector field chosen to fulfill

$$\theta(x, \widetilde{b}(x)) = 0 \tag{4.4}$$

and the pair $n_i, \widetilde{b}_i$ is orthonormal with respect to the tensor a_{ij}. The reciprocal pair is $\{\widetilde{b}^i, n^i\}$ with $\widetilde{b}^i = a^{ij}\widetilde{b}_j$ and $n^i = a^{ij}n_j$, where a^{ij} is the inverse of a_{ij}. The ∇_n stands for the Riemannian covariant derivative taken with $a^k{}_{nh}$. We get

$$L_n^k = Sm^kT_n - a^k{}_{nh}y^h \equiv L^k{}_{nh}y^h \tag{4.5}$$

with

$$L^k{}_{nh} = -a^{kj}\epsilon_{jh}^{\text{Riem}}T_n - a^k{}_{nh} \tag{4.6}$$

and

$$T_n = n^h\nabla_n\widetilde{b}_h + k_n. \tag{4.7}$$

$\epsilon_{jh}^{\text{Riem}} = \sqrt{\det(a_{mn})}\,\gamma_{jh}$, where $\gamma_{11} = \gamma_{22} = 0$ and $\gamma_{12} = -\gamma_{21} = 1$. The metricity property

$$\frac{\partial a_{mn}}{\partial x^i} + L^s{}_{im}a_{sn} + L^s{}_{in}a_{ms} = 0 \tag{4.8}$$

holds independently of presence of the vector T_n. In contrast to the Christoffel symbols $a^k{}_{nh}$, the coefficients $L^k{}_{nh}$ obtained are not symmetric with respect to the subscripts.

Let us take the coefficients $\bar{L}^n{}_{ik} = -L^n{}_{ik}$ from (4.6) and construct the tensor

$$\bar{L}_k{}^n{}_{ij} = \frac{\partial \bar{L}^n{}_{jk}}{\partial x^i} - \frac{\partial \bar{L}^n{}_{ik}}{\partial x^j} + \bar{L}^m{}_{jk}\bar{L}^n{}_{im} - \bar{L}^m{}_{ik}\bar{L}^n{}_{jm} \equiv \bar{L}_k{}^n{}_{ij}(x). \tag{4.9}$$

We obtain

$$\bar{L}_k{}^n{}_{ij} = (\nabla_i T_j - \nabla_j T_i)\, a^{nt}\epsilon^{\text{Riem}}_{tk} + a_k{}^n{}_{ij}, \tag{4.10}$$

where

$$a_k{}^n{}_{ij} = \frac{\partial a^n{}_{jk}}{\partial x^i} - \frac{\partial a^n{}_{ik}}{\partial x^j} + a^m{}_{jk}a^n{}_{im} - a^m{}_{ik}a^n{}_{jm} \tag{4.11}$$

is the Riemannian curvature tensor constructed from the Riemannian metric tensor a_{mn}. We have taken into account the vanishing $\nabla_i \epsilon^{\text{Riem}}_{tk} = 0$.

From the equalities

$$\nabla_i \widetilde{b}^k = -n^k \widetilde{b}_m \nabla_i n^m, \qquad \nabla_i n^k = -\widetilde{b}^k n_m \nabla_i \widetilde{b}^m \tag{4.12}$$

it follows that

$$\nabla_i(n^t \nabla_j \widetilde{b}_t) - \nabla_j(n^t \nabla_i \widetilde{b}_t) = n^t(\nabla_i\nabla_j \widetilde{b}_t - \nabla_j \nabla_i \widetilde{b}_t) = -n^t \widetilde{b}_l a_t{}^l{}_{ij}. \tag{4.13}$$

Therefore, taking the T_i from (4.7), we find that

$$\nabla_i T_j - \nabla_j T_i = M_{ij} - n^t \widetilde{b}_l a_t{}^l{}_{ij}.$$

Noting the equality

$$n^t \widetilde{b}_l a_t{}^l{}_{ij} a^{nt} \epsilon^{\text{Riem}}_{tk} = a_k{}^n{}_{ij} \tag{4.14}$$

(see [11]), we conclude that the tensor (4.10) can be read merely

$$\bar{L}_k{}^n{}_{ij} = a^{nt}\epsilon^{\text{Riem}}_{tk} M_{ij}. \tag{4.15}$$

If we want to have $\bar{L}^s{}_{ij} = a^s{}_{ij}$, we must make the choice

$$k_n = -n^h \nabla_n \widetilde{b}_h, \tag{4.16}$$

which entails $T_i = 0$, in which case the tensor $\bar{L}_k{}^n{}_{ij}$ given by (4.9) is the ordinary Riemannian curvature tensor $a_k{}^n{}_{ij}$.

If the Finsler space F_2 is not a Riemannian space, it is possible to introduce the *θ-associated Riemannian space* $R_{\{\theta\}}$ as follows.

The angle function $\theta = \theta(x, y)$ is defined (from equation (2.3)) up to an arbitrary integration constant which may depend on x, which is the reason why $d_n\theta$ should not be put to be zero in (2.8) (in distinction from the vanishing

$d_n F = 0$). There exists the freedom to make the redefinition $\theta \to \theta + C(x)$. To specify the value of θ unambiguously in a fixed tangent space T_xM, we need in this T_xM an axis from which the value is to be measured. Let the distribution of these axes over the base manifold be assigned by means of a contravariant vector field $b^i = b^i(x)$. Then we obtain precisely the equality $\theta(x, b(x)) = 0$ which does not permit the redefinitions anymore.

It is appropriate to construct the *osculating* Riemannian metric tensor $a_{mn}(x) = g_{mn}(x, b(x))$ and introduce the normalized vector $\widetilde{b}^i = b^i/\sqrt{a_{mn}b^m b^n}$. Because of the homogeneity, $g_{mn}(x, b(x)) = g_{mn}\big(x, \widetilde{b}(x)\big)$. The vector $n_i(x)$ can be taken to equal the value of the derivative $\partial\theta/\partial y^i$ at the argument pair $\big(x, \widetilde{b}(x)\big)$. Then, because of $\theta(x, \widetilde{b}(x)) = 0$ and $g_{ij} = l_i l_j + m_i m_j$ (see (2.2) and (2.3)), the pair $\{\widetilde{b}_i, n_i\}$ thus introduced is orthonormal with respect to the tensor a_{ij} produced by osculation. This tensor a_{ij} introduces the Riemannian space $R_{\{\theta\}}$ on the base manifold M. We obtain the equalities

$$a_{mn}\widetilde{b}^m\widetilde{b}^n = 1, \quad a_{mn}n^m n^n = 1, \quad a_{mn}\widetilde{b}^m n^n = 0, \tag{4.17}$$

and $F\big(x, \widetilde{b}(x)\big) = 1$ together with

$$\begin{gathered} a_{mn}(x) = g_{mn}\big(x, \widetilde{b}(x)\big), \quad \theta(x, \widetilde{b}(x)) = 0, \\ \frac{\partial\theta}{\partial y^i}\big(x, \widetilde{b}(x)\big) = n_i(x), \quad \frac{\partial\theta}{\partial y^i}\big(x, n(x)\big) = -\widetilde{b}_i(x). \end{gathered} \tag{4.18}$$

The arisen expansion $y^m = \widetilde{b}\widetilde{b}^m + nn^m$ is convenient to use in many fragments of evaluations. The last equality in the list (4.18) is explicated from (2.1).

Now we can download in the space $R_{\{\theta\}}$ all the relations (4.2)–(4.16) formulated above in the Riemannian precursor space. On doing so, we can conclude after comparing (4.15) with (3.15) that the tensor $\rho_{knij} = g_{nm}\rho_k{}^m{}_{ij}$ is factorable, namely

$$\rho_{knij} = f_1 \bar{L}_{knij} \quad \text{with} \quad \bar{L}_{knij} = a_{nh}\bar{L}_k{}^h{}_{ij} \equiv \bar{L}_{knij}(x), \tag{4.19}$$

where

$$f_1 = \sqrt{\frac{\det(g_{hl})}{\det(a_{mn})}}. \tag{4.20}$$

We have arrived at the following theorem.

Theorem 4.1. *The curvature tensor $\rho_k{}^n{}_{ij}$ of the Finsler space F_2 equipped with the angle-preserving connection is such that the tensor $\rho_{knij} = g_{nm}\rho_k{}^m{}_{ij} = \rho_{knij}(x, y)$ is proportional to the tensor $\bar{L}_{knij} = a_{nh}\bar{L}_k{}^h{}_{ij} = \bar{L}_{knij}(x)$ which does not involve any dependence on tangent vectors. The factor of proportionality f_1 is expressed through the determinants of metric tensors, according to* (4.20).

5. Conclusions

In the Riemannian geometry the contraction $a^k{}_{nh}y^h$ of the Christoffel symbols $a^k{}_{nh}$ with the tangent vector y admits the angle representation (4.3)–(4.4). Why don't lift the representation to the Finsler level to take the coefficients $N_n^k = N_n^k(x,y)$ in the operator $d_n = \partial_{x^n} + N_n^k(x,y)\partial_{y^k}$ to be of the similar form? Our proposal in (2.6) was of this kind. At any k_n in (2.6), the vanishing $d_nF = d_n\theta - k_n = d_n(\theta_2 - \theta_1) = 0$ immediately ensues from this proposal. It is a big (and good) surprise that the vanishing $y_kN^k{}_{nmi} = 0$ ensues also, which enables us to obtain the covariant derivative D_n possessing the metric property $D_ng_{ij} = 0$, where $D_ng_{ij} = d_ng_{ij} - D^h{}_{ni}g_{hj} - D^h{}_{nj}g_{ih}$ with the connection coefficients $D^h{}_{ni} = -N^h{}_{ni}$. If we want to obtain torsionless coefficients in the Riemannian limit of these $D^h{}_{ni}$, we should take the vector field $k_n(x)$ to be the field $t_n(x) = -n^h\nabla_n\widetilde{b}_h$ in accordance with (4.16).

The induced parallel transports of the objects $\{F,\, \theta_2 - \theta_1,\, g_{ij}\}$ along the horizontal curves (running on the base manifold M) are represented infinitesimally by the elements $\{dx^nd_nF,\, dx^nd_n(\theta_2 - \theta_1),\, dx^nD_ng_{ij}\}$ which all are the naught because of $d_nF = d_n(\theta_2 - \theta_1) = D_ng_{ij} = 0$. Therefore the transports realize isometries of the tangent Riemannian spaces $R_{\{x\}}$ supported by points $x \in M$, taking indicatrices into indicatrices. The coefficients N_n^k given by (2.6) are in general non-linear with respect to the variable y.

In the Riemannian case, the right-hand part of the Riemannian coefficients (4.3) can be expressed through the Christoffel symbols and, therefore, can be constructed from the first derivatives of the metric tensor. This is the privilege of the Riemannian geometry which lives in the ground floor of the Finsler building, – the right-hand part of the Finsler coefficients (2.6) is not a composition of partial derivatives of the Finsler metric tensor. In distinction to the Riemannian geometry which provides us with the simple and explicit angle $\theta = \arctan(n/b)$, the Finsler angle function $\theta = \theta(x,y)$ is defined by the partial differentiable equation (2.3) which cannot be integrated explicitly, except for rare particular cases of the Finsler metric function.

The second big surprise is that the angle-preserving connection obtained in this way admits the C^∞-regular realization globally regarding the dependence on tangent vectors. Such a realization takes place for the Finsleroid-regular metric function, the Randers metric function, and probably for many other Finsler metric functions.

The Finsler connection obtained does not need any facility which could be provided by the geodesic spray coefficients. Due attention to the angle wisdom is sufficient: the Finsler space is connected by its angle structure, similarly to the well known property of Riemannian geometry.

Our consideration was restricted by the dimension 2. Development of due extensions to higher dimensions is the problem of urgent kind.

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