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On exponentiation in concrete categories

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Generally speaking our terminology will follow the book of Herrlich and Strecker [3]. By a concrete category $\mathcal{K}$ we understand such a category whose objects are structured sets, i.e. pairs consisting of underlying sets and $\mathcal{K}$-structures on them, and whose morphisms are structure-compatible maps. For any two objects A, B of a concrete category $\mathcal{K}$ the set of all morphisms of A into B in $\mathcal{K}$ will be denoted by $\mathrm{Mor}_{\mathcal{K}}(A,B)$. A binary operation on objects of a concrete category $\mathcal{K}$ assigning to any two objects $A, B \in \mathcal{K}$ an object $A^B \in \mathcal{K}$ whose underlying set is $\mathrm{Mor}_{\mathcal{K}}(B,A)$ is called an exponentiation in $\mathcal{K}$ and the object A^B is called a power of A and B. By a concrete category with exponentiation we shall mean a concrete category in which an exponentiation is defined.

An exponentiation in a concrete category can be defined in many ways. If a concrete category $\mathcal{K}$ is finitely productive (i.e. has finite cartesian products), then there arises the natural problem how to define an exponentiation in $\mathcal{K}$ for the relation $A^{B\times C} \simeq (A^B)^C$ (where $\simeq$ denotes the isomorphism in $\mathcal{K}$) to hold for any objects $A, B, C \in \mathcal{K}$. A solution of this problem is well known for cartesian closed topological categories — see [2]. In the present note we solve a more special problem: for three given objects A, B, C of a finitely productive concrete category with exponentiation we find sufficient conditions under which the relation $A^{B\times C} \simeq (A^B)^C$ is valid. In the second part of the paper we show an application of the obtained result to categories of relational systems.

Let $\mathcal{K}$ be a finitely productive concrete category with exponentiation and $A, B \in \mathcal{K}$ its objects. By the evaluation map for the power A^B we understand the map $e : B \times A^B \to A$ defined by $e(y,f) = f(y)$ (see [3]).

Definition. Let $\mathcal{K}$ be a finitely productive concrete category with exponentiation and $A, B \in \mathcal{K}$ its objects. The power A^B is called regular if the following two conditions are fulfilled:

(i) The evaluation map for A^B is a morphism of $B \times A^B$ into A in $\mathcal{K}$.

(ii) If $C \in \mathcal{K}$ is an object for which each constant map of B into C is a morphism in $\mathcal{K}$ and if $f \in \mathrm{Mor}_{\mathcal{K}}(B\times C, A)$ is a morphism, then the map $f^* : C \to A^B$ defined by $f^*(z)(y) = f(y,z)$ is a morphism in $\mathcal{K}$, too.

From the definition it immediately follows that a topolgical category $\mathcal{K}$ is cartesian closed iff there exists an exponentiation in $\mathcal{K}$ with all powers regular.

Theorem. *Let $\mathcal{K}$ be a finitely productive concrete category with exponentiation and $A, B, C \in \mathcal{K}$ its objects. Let all constant maps of B into C, of C into $A^{B\times C}$ and of $B \times C$ into $(A^B)^C$ be morphisms in $\mathcal{K}$. Let all the three powers A^B, $A^{B\times C}$, $(A^B)^C$ be regular. Then $A^{B\times C} \simeq (A^B)^C$ is valid.*

Proof. First of all, let Y and Z denote the underlying sets of B and C respectively. For each morphism $f \in \mathrm{Mor}_{\mathcal{K}}(B \times C, A)$ put $\varphi(f) = f^*$ where $f^* : C \to A^B$ is the map defined by $f^*(z)(y) = f(y,z)$. As any constant map of B into C is a morphism in $\mathcal{K}$ and as A^B is regular, there holds $\varphi(f) \in \mathrm{Mor}_{\mathcal{K}}(C, A^B)$ for every $f \in \mathrm{Mor}_{\mathcal{K}}(B \times C, A)$. Hence φ maps $A^{B\times C}$ into $(A^B)^C$ and it is evident that φ is an injection. Let $g \in \mathrm{Mor}_{\mathcal{K}}(C, A^B)$ be an arbitrary morphism and put $f(y,z) = g(z)(y)$ for each $y \in Y$ and $z \in Z$. Next, put $\hat{g}(y,z) = (y, g(z))$ for all $y \in Y$ and $z \in Z$. Then clearly $\hat{g} \in \mathrm{Mor}_{\mathcal{K}}(B \times C, B \times A^B)$. Let e_1 be the evaluation map for A^B. Since A^B is regular, we have $e_1 \in \mathrm{Mor}_{\mathcal{K}}(B \times A^B, A)$. Hence $e_1 \circ \hat{g} \in \mathrm{Mor}_{\mathcal{K}}(B\times C, A)$. There holds $e_1(\hat{g}(y,z)) = e_1(y, g(z)) = g(z)(y) = f(y,z)$ for each $y \in Y$ and $z \in Z$. Thus $e_1 \circ \hat{g} = f$, i.e. $f \in \mathrm{Mor}_{\mathcal{K}}(B\times C, A)$. As $g = f^* = \varphi(f)$, φ is a surjection.

We have proved that φ is a bijection. Let e_2 be the evaluation map for $A^{B\times C}$. Then $e_2 \in \mathrm{Mor}_{\mathcal{K}}(B\times C\times A^{B\times C}, A)$ because $A^{B\times C}$ is regular. Put $e_2^*(z,f)(y) = e_2(y,z,f)$ for every $y \in Y,\ z \in Z$ and $f \in \mathrm{Mor}_{\mathcal{K}}(B \times C, A)$. As all constant maps of B into C and of C into $A^{B\times C}$ are morphisms in $\mathcal{K}$, any constant map of B into $C \times A^{B\times C}$ is a morphism in $\mathcal{K}$, too. Now, the regularity of A^B yields $e_2^* \in \mathrm{Mor}_{\mathcal{K}}(C \times A^{B\times C}, A^B)$. Next, put $e_2^{**}(f)(z) = e_2^*(z,f)$ for any $z \in Z$ and $f \in \mathrm{Mor}_{\mathcal{K}}(B \times C, A)$. Since any constant map of C into $A^{B\times C}$ is a morphism in $\mathcal{K}$ and since $(A^B)^C$ is regular, we have $e_2^{**} \in \mathrm{Mor}_{\mathcal{K}}(A^{B\times C}, (A^B)^C)$. But $e_2^{**}(f)(z)(y) = e_2^*(z,f)(y) = e_2(y,z,f) = f(y,z) = f^*(z)(y) = \varphi(f)(z)(y)$ for all $y \in Y$, $z \in Z$ and $f \in \mathrm{Mor}_{\mathcal{K}}(B \times C, A)$. Hence $e_2^{**} = \varphi$ and therefore $\varphi \in \mathrm{Mor}_{\mathcal{K}}(A^{B\times C}, (A^B)^C)$.

Finally, we are to show that $\varphi^{-1} \in \mathrm{Mor}_{\mathcal{K}}((A^B)^C, A^{B\times C})$. On that account, let $h : B \times C \times (A^B)^C \to A$ be the map defined by $h(y,z,g) = g(z)(y)$. Let e_3 be the evaluation map for $(A^B)^C$ and put $\hat{e}_3(y,z,g) = (y, e_3(z,g))$ for each $y \in Y$, $z \in Z$ and $g \in \mathrm{Mor}_{\mathcal{K}}(C, A^B)$. As $(A^B)^C$ is regular, there holds $e_3 \in \mathrm{Mor}_{\mathcal{K}}(C \times (A^B)^C, A^B)$ and consequently $\hat{e}_3 \in \mathrm{Mor}_{\mathcal{K}}(B \times C \times (A^B)^C, B \times A^B)$. Thus $e_1 \circ \hat{e}_3 \in \mathrm{Mor}_{\mathcal{K}}(B \times C \times (A^B)^C, A)$. We have $e_1(\hat{e}_3(y,z,g)) = e_1(y, e_3(z,g)) = e_1(y, g(z)) = g(z)(y) = h(y,z,g)$ for any $y \in Y$, $z \in Z$ and $g \in \mathrm{Mor}_{\mathcal{K}}(C, A^B)$. This yields $h = e_1 \circ \hat{e}_3$ and $h \in \mathrm{Mor}_{\mathcal{K}}(B \times C \times (A^B)^C, A)$. Put $h^*(g)(y,z) = h(y,z,g)$ whenever $y \in Y$, $z \in Z$ and $g \in \mathrm{Mor}_{\mathcal{K}}(C, A^B)$. Since any constant map of $B \times C$ into $(A^B)^C$ is a morphism in $\mathcal{K}$, from the regularity of $A^{B\times C}$ we get $h^* \in \mathrm{Mor}_{\mathcal{K}}((A^B)^C, A^{B\times C})$. There holds $h^*(g)(y,z) = h(y,z,g) = g(z)(y) = \varphi^{-1}(g)(y,z)$ for all $y \in Y$, $z \in Z$ and $g \in \mathrm{Mor}_{\mathcal{K}}(C, A^B)$. Therefore $h^* = \varphi^{-1}$. Consequently $\varphi^{-1} \in \mathrm{Mor}_{\mathcal{K}}((A^B)^C, A^{B\times C})$ which completes the proof.

Now we shall present an application of the Theorem.

Let X and I be non-empty sets. Any set of maps $\varrho \subseteq X^I$ is called a relation on X. The set I is called the domain (or index set) of ϱ. By a relational system we understand a pair $F = (X, \varrho)$ where X is a non-empty set and ϱ is a relation on X. By the domain of a relational system $F = (X, \varrho)$ we mean the domain of the relation ϱ. If $F = (X, \varrho)$ and $G = (Y, \sigma)$ are two relational systems with the same domain I and $h : X \to Y$ a map, then h is called a morphism of F into G if for each map $f \in \varrho$ there holds $h \circ f \in \sigma$. The class of all relational systems with the same domain I together with morphisms defined above forms a concrete category which we denote by $\mathcal{R}_I$. For objects $F, G \in \mathcal{R}_I$ we write $\mathrm{Mor}_I(F,G)$ instead of $\mathrm{Mor}_{\mathcal{R}_I}(F,G)$. Of course, $\mathcal{R}_I$ has finite cartesian products: if $F = (X, \varrho)$ and $G = (Y, \sigma)$ are two objects of $\mathcal{R}_I$, then $F \times G = (X \times Y, \tau)$ where $\tau \subseteq (X \times Y)^I$ is the relation defined by $h \in (X \times Y)^I$, $h \in \tau \iff$ there exist $f \in \varrho$ and $g \in \sigma$ such that $h(i) = (f(i), g(i))$ for each $i \in I$. In [5] an exponentiation in $\mathcal{R}_I$ is defined as follows: for $F = (X, \varrho)$ and $G = (Y, \sigma)$ of $\mathcal{R}_I$ we put $F^G = (\mathrm{Mor}_I(G,F), \tau)$ where $\tau \subseteq (\mathrm{Mor}_I(G,F))^I$ is defined by $h \in (\mathrm{Mor}_I(G,F))^I$, $h \in \tau \iff {}^y h \in \varrho$ for each $y \in Y$; here ${}^y h : I \to X$ is the map defined by ${}^y h(i) = h(i)(y)$. Obviously, this exponentiation is an extension of Birkhoff's cardinal exponentiation for ordered sets [1] onto relational systems. In [5] there are also defined the following two properties of relational systems: A relational system $F = (X, \varrho)$ with domain I is said to be

(a) reflexive if for any constant map $c : I \to X$ there holds $c \in \varrho$,

(b) diagonal if the following condition is fulfilled: if $\{f_i \mid i \in I\}$ is a family of elements of ϱ and if the family $\{g_j \mid j \in I\}$ of elements of X^I defined by $g_j(i) = f_i(j)$ whenever $i, j \in I$ fulfils $g_j \in \varrho$ for each $j \in I$, then putting $h(i) = f_i(i)$ for every $i \in I$ we get $h \in \varrho$.

For n-ary relations (i.e. relations with finite domains) the diagonality coincides with the diagonal property defined in [4]. In particular, if ϱ is a binary relation on a non-empty set X, then (X, ϱ) is diagonal iff ϱ is transitive.

For relational systems the following statement is valid:

Lemma. *Let $F, G \in \mathcal{R}_I$ be relational systems. If F is diagonal and G is reflexive, then the power F^G is regular.*

Proof. (i) Denote $F = (X, \varrho)$, $G = (Y, \sigma)$ and $G \times F^G = (Z, \tau)$. Let $p \in \tau$. Then there exist $g \in \sigma$ and $h \in (\mathrm{Mor}_I(G, F))^I$ with ${}^y h \in \varrho$ for each $y \in Y$ such that $p(i) = (g(i), h(i))$ for every $i \in I$. For each $i \in I$ put $f_i = {}^{g(i)}h$. Then $f_i \in \varrho$ for all $i \in I$. For any $i, j \in I$ put $g_j(i) = f_i(j)$. We have $g_j(i) = f_i(j) = {}^{g(i)}h(j) = h(j)(g(i))$ whenever $i, j \in I$. Hence $g_j = h(j) \circ g$ for each $j \in I$. Consequently, $g_j \in \varrho$ for all $j \in I$. Let e be the evaluation map for F^G. For any $i \in I$ we have $e(p(i)) = e(g(i), h(i)) = h(i)(g(i)) = {}^{g(i)}h(i) = f_i(i)$. Thus, putting $h(i) = f_i(i)$ for each $i \in I$ we get $h = e \circ p$. As F is diagonal there holds $h \in \varrho$. Therefore $e \in \mathrm{Mor}_I(G \times F^G, F)$.

(ii) Let $H = (U, \xi)$ be a relational system for which each constant map of G into H is a morphism in $\mathcal{R}_I$. Let $f \in \mathrm{Mor}_I(G \times H, F)$. Let $h \in \xi$ and denote $(V, \eta) = G \times H$. For any $y \in Y$ and any $i \in I$ put $s_y(i) = (y, h(i))$. Then the reflexivity of G implies $s_y \in \eta$ for each $y \in Y$. Thus, for each $y \in Y$ there holds $f \circ s_y \in \varrho$. Let $f^* : H \to F^G$ be the map defined by $f^*(z)(y) = f(y, z)$. For any elements $h \in \xi$, $y \in Y$ and $i \in I$ we have ${}^y(f^* \circ h)(i) = (f^* \circ h)(i)(y) = f^*(h(i))(y) = f(y, h(i)) = f(s_y(i))$. Hence ${}^y(f^* \circ h) = f \circ s_y \in \varrho$ for all $y \in Y$. Consequently, denoting $(W, \nu) = F^G$ we have $f^* \circ h \in \nu$. Therefore $f^* \in \mathrm{Mor}_I(H, F^G)$. The proof is complete.

Finally, as an application of Theorem we obtain the following assertion (which is proved in [5], Theorem 9, but in another, more laborious way):

Proposition. *Let $F, G, H \in \mathcal{R}_I$ be relational systems. A sufficient condition for $F^{G\times H} \simeq (F^G)^H$ to be valid is that F is diagonal and both G and H are reflexive.*

PROOF. Clearly, if a relational system $L \in \mathcal{R}_I$ is reflexive, then any constant map of any relational system of $\mathcal{R}_I$ into L is a morphism in $\mathcal{R}_I$. As G and H are reflexive, $F^{G\times H}$ and $(F^G)^H$ are reflexive, too. Hence all constant maps of G into H, of H into $F^{G\times H}$ and of $G \times H$ into $(F^G)^H$ are morphisms in $\mathcal{R}_I$. Further, since $G \times H$ is reflexive and since the diagonality of F evidently implies the diagonality of F^G, all three powers F^G, $F^{G\times H}$, $(F^G)^H$ are regular according to the Lemma. Now the assertion follows from the Theorem.

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