On the practical stability of dependent parameter perturbed systems

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Abstract. We investigate in this paper the global uniform practical asymptotic stability for a class of dependent parameter perturbed systems where the associated linear system is globally exponentially stable and the perturbation term is subject to two conditions. An example is given to illustrate the result of this paper formulated in the form of Linear matrix inequalities.

1. Introduction

The analysis of stability of time-varying systems is an important topic in systems theory ([10], [12], [15]), specially in the case of the uncertain system. One of the most effective approaches for studying stability of uncertain systems ([2], [4], [8], [9], [20]) is the Lyapunov approach ([5], [7], [13], [16], [17], [18], [19]) where the problem of stability of dependent parameter systems has been subject to considerable research efforts. Continuous-time linear systems whose dynamic matrices are affected by bounded uncertain time-varying parameters have been investigated through sufficient Linear matrix inequalities (LMIs) stability conditions [3] as for instance in ([5], [18]) where the system is written with affine dependence on the uncertain parameter.

In this paper, we discuss the problem of stability for a class of perturbed systems which depend on a parameter. This discussion is done through Lyapunov

functions, $LMIs$ conditions and by imposing some restrictions on the perturbation term. We shall develop an algorithm, when the nonlinearities are bounded, to ensure the global uniform practical exponential stability of a perturbed system whose linear system satisfies some $LMIs$. Moreover, we are motivated to give, under some works of ([1], [6], [11] and [14]), a numerical example.

2. Problem formulation

Consider the system

$$\dot{x}(t) = A(\alpha(t))x(t) + f(t, \alpha(t), x). \tag{1}$$

The matrix $A(\alpha(t)) \in \mathbb{R}^{n \times n}$ is defined as

$$A(\alpha(t)) = \alpha_1(t)A_1 + \alpha_2(t)A_2$$

where $\alpha_i(t)$, $i = 1, 2$ are continuous functions such that $\alpha_i(t) \geq 0$, $\alpha_1(t) + \alpha_2(t) = 1$, $|\dot{\alpha_1}(t)| \leq \rho$ with $\rho \in \mathbb{R}_+$ and A_1, $A_2 \in \mathbb{R}^{n \times n}$ are constant matrices.

Note that, since $\alpha_2(t) = 1 - \alpha_1(t)$, then the linear system can be written as

$$A(\alpha(t)) = \alpha_1(t)A_1 + (1 - \alpha_1(t))A_2. \tag{2}$$

Moreover, the continuous function $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is the perturbation of system (2) and it result in general from modelling errors, aging of parameters or uncertainties and disturbances.

The main objective of this paper is to study the stability of the system (1) by imposing some conditions on the term of perturbation f. In fact, we begin by investigating the global uniform practical exponential stability of perturbed system and after that, an illustrative numerical example is given to illustrate the applicability of these results where the stability conditions are given in terms of $LMIs$ which can be easily checked by LMI Control Toolbox in Matlab.

Notations: The following notations will be used throughout this paper. For a real square matrix X, the notation $X > 0$ (respectively $X < 0$) means that X is positive definite (respectively negative definite). $\lambda_{\min}(X)$ and $\lambda_{\max}(X)$ denote the minimum and the maximum eigenvalues of X respectively.

3. General definitions

Consider the non-autonomous system

$$\dot{x} = g(t, x) \tag{3}$$

where $g : [0, \infty) \times D \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x on $[0, \infty) \times D$, and $D \subset \mathbb{R}^n$ is a domain that contains the origin $x = 0$. The origin is an equilibrium point for (3), if

$$g(t, 0) = 0, \quad \forall\, t \geq 0.$$

We define $x(t_0)$ as the solution of system (3) at $t = t_0$ and $x(t) = x(t, t_0, x_0)$ for any solution of system (3).

We give now the definition of uniform stability and uniform attractivity of system (3) towards a ball $B_r = \{x \in \mathbb{R}^n / \|x\| \leq r\}$.

Definition 1 (Uniform stability of B_r). B_r is uniformly stable if for all $\varepsilon > r$, there exists $\delta = \delta(\varepsilon) > 0$, such that

$$\|x(t_0)\| < \delta \implies \|x(t)\| < \varepsilon, \quad \forall\, t \geq t_0. \tag{4}$$

Definition 2 (Uniform attractivity of B_r). B_r is uniformly attractive, if for $\varepsilon > r$, $t_0 > 0$ and $x(t_0) \in D$, there exists $T(\varepsilon, x(t_0)) > 0$, such that

$$\|x(t)\| < \varepsilon, \quad \forall\, t \geq t_0 + T(\varepsilon, x(t_0)). \tag{5}$$

B_r is globally uniformly attractive if (5) is satisfied for $x(t_0) \in \mathbb{R}^n$.

Definition 3 (Practical stability). System (3) is said uniformly practically asymptotically stable, if there exists $B_r \subset \mathbb{R}^n$, such that B_r is uniformly stable and uniformly attractive. It is globally uniformly practically asymptotically stable if $x(t_0) \in \mathbb{R}^n$.

Definition 4. System (3) is said uniformly exponentially convergent to B_r, if there exist $\gamma > 0$ and $k \geq 0$, such that

$$\|x(t)\| \leq k\|x(t_0)\| \exp(-\gamma(t - t_0)) + r, \quad \forall\, t \geq t_0, \ \forall\, x(t_0) \in D. \tag{6}$$

If $x(t_0) \in \mathbb{R}^n$, the system is globally uniformly exponentially convergent to B_r.

We say that the system is globally uniformly practically exponentially stable if for $r > 0$, it is globally uniformly exponentially convergent to B_r.

Definition 5. System (3) is said uniformly exponentially convergent to zero, if there exist $\gamma > 0$ and $k \geq 0$, such that

$$\|x(t)\| \leq k\|x(t_0)\| \exp(-\gamma(t-t_0)) + r(t), \quad \forall\, t \geq t_0, \ \forall\, x(t_0) \in D \qquad (7)$$

with $\lim_{t \to +\infty} r(t) = 0$.

Definition 6 (Linear matrix inequality). A linear matrix inequality (LMI) is any constraint of the form

$$A(x) := A_0 + x_1 A_1 + \cdots + x_N A_N < 0 \qquad (8)$$

where

- $x = (x_1, \ldots, x_N)$ is a vector of unknown scalars (the decision or optimization variables)
- $A_0, \ldots, A_N$ are given symmetric matrices
- < 0 stands for "negative definite", i.e., the largest eigenvalue of $A(x)$ is negative.

Note that the constraints $A(x) > 0$ and $A(x) < B(x)$ are special cases of (8) since they can be rewritten as $-A(x) < 0$ and $A(x) - B(x) < 0$, respectively.

Moreover, in most control applications, LMIs do not naturally arise in the canonical form (8), but rather in the form

$$L(X_1, \ldots, X_n) < R(X_1, \ldots, X_n)$$

where $L(.)$ and $R(.)$ are affine functions of some structured matrix variables $X_1, \ldots, X_n$.

4. Global uniform practical exponential stability via linear matrix inequalities

In this section, our object is to study the global uniform practical exponential stability of system (1). This study is done through three steps. Firstly, we begin by the use of two $LMIs$. Secondly, we treat the case of three $LMIs$ and finally we study the case of four $LMIs$. In fact, we give an algorithm that shows the robustness of this method to obtain other classes of strongly practically stable perturbed systems by considering a large number of $LMIs$ where the radius of the ball decreases when the number of $LMIs$ increases.

Consider the system (1), assume that, there exists $M > 0$ such that for all $t \geq 0$, $x \in \mathbb{R}^n$,

$$(\mathcal{H}_1) \qquad\qquad\qquad \|f(t, \alpha(t), x)\| \leq M.$$

Here, the perturbation term is bounded which implies that the origin could not be an equilibrum point. Let

$$V(x, \alpha) = x^T P(\alpha)x, \quad P(\alpha) = P^T(\alpha) > 0,$$

a candidate Lyapunov function. The time derivative of V along the trajectories of perturbed system is given by

$$\dot{V}(x, \alpha) = x^T[P(\alpha)A(\alpha) + A^T(\alpha)P(\alpha) + \dot{P}(\alpha)]x + 2x^T P(\alpha)f(t, \alpha(t), x).$$

We will use Linear matrix inequalities and by imposing some conditions on the linear part of system (1), we give some classes of perturbed systems which are globally uniformly practically exponentially stable.

Lemma 1. *Let*
$$V(x, \alpha) = x^T P(\alpha)x,$$

a quadratic positive definite function with $P(\alpha)$ is a positive definite matrix that satisfy
$$P(\alpha) > \lambda_{\min}(P_i)I$$

where P_i is a constant matrix.
If one has
$$\dot{v}(t) \leq -\eta_1 v(t) + Mr \tag{9}$$

where $v(t) = \sqrt{V(x, \alpha)}$, $\eta_1 > 0$, $M > 0$, $r > 0$, then the solutions of system (1) satisfiy
$$\|x(t)\| \leq \eta_2 \|x(t_0)\| e^{-\eta_1(t-t_0)} + M \frac{r}{\eta_1 \lambda_{\min}(P_i)},$$

$\forall t \geq t_0 \geq 0$, $\eta_1 > 0$, $\eta_2 > 0$, $M > 0$ and $r > 0$.

We will start our study by the simplest case: the case of constant matrices.

Proposition 1. *Suppose that, for given reals $l_1 \in \mathbb{R}_+^*$ and $l_2 \in \mathbb{R}_+^*$, there exists a symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$, such that*

$$\left(A_1^T + \frac{l_1}{2}I\right)P + P\left(A_1 + \frac{l_1}{2}I\right) < 0 \tag{10}$$

$$\left(A_2^T + \frac{l_2}{2}I\right)P + P\left(A_2 + \frac{l_2}{2}I\right) < 0 \tag{11}$$

and suppose that the assumption $(\mathcal{H}_1)$ holds, then any solution $x(t)$ of system (1) converges exponentially towards the ball $B(0, r_1)$ where

$$r_1 = \frac{2M\lambda_{\max}^2(P)}{l\lambda_{\min}^2(P)}$$

with $l = \inf(l_1, l_2)$.

PROOF. Let $P(\alpha) = P$. The time-derivative of V along the trajectories of perturbed system (1) is given by

$$\dot{V}(x, \alpha) = x^T[\alpha_1(t)(A_1^T P + PA_1) + \alpha_2(t)(A_2^T P + PA_2)]x + 2x^T Pf(t, \alpha(t), x)$$
$$\leq x^T[\alpha_1(t)(-l_1 P) + \alpha_2(t)(-l_2 P)]x + 2M\lambda_{\max}(P)\|x\|$$
$$\leq -l\lambda_{\min}(P)\|x\|^2 + 2M\lambda_{\max}(P)\|x\|$$
$$\leq \frac{-l\lambda_{\min}(P)}{\lambda_{\max}(P)}V(x, \alpha) + 2M\frac{\lambda_{\max}(P)}{\sqrt{\lambda_{\min}(P)}}\sqrt{V(x, \alpha)}.$$

Let

$$v(t) = \sqrt{V(x, \alpha)},$$

hence,

$$\dot{v}(t) \leq \frac{-l\lambda_{\min}(P)}{2\lambda_{\max}(P)}v(t) + M\frac{\lambda_{\max}(P)}{\sqrt{\lambda_{\min}(P)}}.$$

A simple computation shows that,

$$v(t) \leq v(t_0)\exp\left(\frac{-l}{2}\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}(t - t_0)\right) + \frac{2M\lambda_{\max}^2(P)}{l\sqrt{\lambda_{\min}(P)}\lambda_{\min}(P)}$$

which implies that

$$\|x(t)\| \leq \frac{\sqrt{\lambda_{\max}(P)}}{\sqrt{\lambda_{\min}(P)}}\|x(t_0)\|\exp\left(\frac{-l}{2}\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}(t - t_0)\right) + \frac{2M\lambda_{\max}^2(P)}{l\lambda_{\min}^2(P)}.$$

Therefore, from Definition 4, any solution $x(t)$ of system (1) converges exponentially towards the ball $B(0, r_1)$ whose radius is given by

$$r_1 = \frac{2M\lambda_{\max}^2(P)}{l\lambda_{\min}^2(P)}. \qquad \square$$

Less conservative conditions to evaluate the practical stability of time-varying systems subject to bounded rates of variation are given by the next proposition.

Proposition 2. *Assume that, for strict positive reals l_1, l_2, l_3 and a given parameter $\rho \in \mathbb{R}^+$, there exist symmetric positive definite matrices $P_1 \in \mathbb{R}^{n \times n}$, $P_2 \in \mathbb{R}^{n \times n}$ such that $(P_1 - P_2)$ is a symmetric positive definite matrix where $\lambda_{\min}(P_1) > \lambda_{\min}(P_2)$, and satisfying*

$$\left(A_1^T + \frac{l_1}{2}I\right)P_1 + P_1\left(A_1 + \frac{l_1}{2}I\right) - l_1 P_2 + \rho(P_1 - P_2) < 0 \qquad (12)$$

$$\left(A_2^T - \frac{l_2}{2}I\right)P_2 + P_2\left(A_2 - \frac{l_2}{2}I\right) + l_2 P_1 + \rho(P_1 - P_2) < 0 \qquad (13)$$

and

$$(A_1^T - l_3 I)P_2 + P_2(A_1 - l_3 I) + (A_2^T + l_3 I)P_1 + P_1(A_2 + l_3 I) + 2\rho(P_1 - P_2) < 0. \qquad (14)$$

Suppose that $(\mathcal{H}_1)$ holds, then, any solution $x(t)$ of system (1) converges exponentially towards the ball $B(0, r_2)$ where

$$r_2 = \frac{2M(\lambda_{\max}(P_1) + \lambda_{\max}(P_2))^2}{l\lambda_{\min}(P_2)\lambda_{\min}(P_1 - P_2)}$$

with $l = \inf(l_1, l_2, l_3)$.

PROOF. Let $P(\alpha) = \alpha_1(t)P_1 + \alpha_2(t)P_2$. On the one hand, we have

$$P(\alpha) \geq \alpha_1(t)\lambda_{\min}(P_1)I + \alpha_2(t)\lambda_{\min}(P_2)I \geq \lambda_{\min}(P_2)I$$

and

$$V(x, \alpha) \leq (\lambda_{\max}(P_1) + \lambda_{\max}(P_2))\|x\|^2.$$

On the other hand, multiplying the time derivative of $P(\alpha)$ by $(\alpha_1(t) + \alpha_2(t))^2$ which equal to 1, the time derivative of V along the trajectories of system (1) is given by

$$
\begin{aligned}
\dot{V}(x, \alpha) = x^T \big(& \alpha_1^2(t)(A_1^T P_1 + P_1 A_1 + \dot{\alpha}_1(t)(P_1 - P_2)) \\
& + \alpha_2^2(t)(A_2^T P_2 + P_2 A_2 + \dot{\alpha}_1(t)(P_1 - P_2)) \\
& + \alpha_1(t)\alpha_2(t)(A_1^T P_2 + P_2 A_1 + A_2^T P_1 + P_1 A_2 + 2\dot{\alpha}_1(t)(P_1 - P_2)))x \\
& + 2x^T(\alpha_1(t)P_1 + \alpha_2(t)P_2)f(t, \alpha(t), x).
\end{aligned}
$$

Since $|\dot{\alpha}_1(t)| \le \rho$, we get

$$
\begin{aligned}
\dot{V}(x,\alpha) &\le x^T(-l(P_1 - P_2)(\alpha_1^2(t) + \alpha_2^2(t) + 2\alpha_1(t)\alpha_2(t)))x + 2M(\lambda_{\max}(P_1) \\
&\quad + \lambda_{\max}(P_2))\|x\| \le -l\lambda_{\min}(P_1 - P_2)\|x\|^2 + 2M(\lambda_{\max}(P_1) + \lambda_{\max}(P_2))\,\|x\| \\
&\le \frac{-l\lambda_{\min}(P_1 - P_2)}{\lambda_{\max}(P_1) + \lambda_{\max}(P_2)}V(x,\alpha) + \frac{2M(\lambda_{\max}(P_1) + \lambda_{\max}(P_2))}{\sqrt{\lambda_{\min}(P_2)}}\sqrt{V(x,\alpha)}.
\end{aligned}
$$

Let

$$ v(t) = \sqrt{V(x,\alpha)} $$

hence,

$$ \dot{v}(t) \le \frac{-l\lambda_{\min}(P_1 - P_2)}{2(\lambda_{\max}(P_1) + \lambda_{\max}(P_2))}v(t) + \frac{M(\lambda_{\max}(P_1) + \lambda_{\max}(P_2))}{\sqrt{\lambda_{\min}(P_2)}}. $$

An integration between t_0 and t shows that

$$
\begin{aligned}
v(t) &\le v(t_0)\exp\left(\frac{-l}{2}\frac{\lambda_{\min}(P_1 - P_2)}{\lambda_{\max}(P_1) + \lambda_{\max}(P_2)}(t - t_0)\right) \\
&\quad + \frac{2M(\lambda_{\max}(P_1) + \lambda_{\max}(P_2))^2}{l\sqrt{\lambda_{\min}(P_2)}\lambda_{\min}(P_1 - P_2)}
\end{aligned}
$$

and consequently,

$$
\begin{aligned}
\|x(t)\| &\le \frac{\sqrt{\lambda_{\max}(P_1) + \lambda_{\max}(P_2)}}{\sqrt{\lambda_{\min}(P_2)}}\,\|x(t_0)\| \cdot \exp\left(\frac{-l\lambda_{\min}(P_1 - P_2)(t - t_0)}{2(\lambda_{\max}(P_1) + \lambda_{\max}(P_2))}\right) \\
&\quad + \frac{2M(\lambda_{\max}(P_1) + \lambda_{\max}(P_2))^2}{l\lambda_{\min}(P_2)\lambda_{\min}(P_1 - P_2)}
\end{aligned}
$$

Therefore, any solution $x(t)$ of system (1) converges exponentially towards the ball $B(0, r_2)$ whose radius is given by

$$ r_2 = \frac{2M(\lambda_{\max}(P_1) + \lambda_{\max}(P_2))^2}{l\lambda_{\min}(P_2)\lambda_{\min}(P_1 - P_2)}. \qquad \square $$

Now, in order to obtain the less conservative result and following the ideas used in the previous propositions, a new sufficient condition based on quadratically parameter dependent Lyapunov functions is studied in the next theorem.

Theorem 1. *Suppose that, for strict positive reals l_1, l_2, l_3, l_4 and a given parameter $\rho \in \mathbb{R}^+$, there exist symmetric positive definite matrices $P_1 \in \mathbb{R}^{n \times n}$, $P_2 \in \mathbb{R}^{n \times n}$, $P_3 \in \mathbb{R}^{n \times n}$ such that $(P_1 - P_2)$, $(2P_1 - P_3)$, $(P_3 - 2P_2)$ and $(P_1 + P_2 - P_3)$ are also symmetric positive definite matrices where $\lambda_{\min}(P_1) > \lambda_{\min}(P_2)$, $\lambda_{\min}(P_3) > 2\lambda_{\min}(P_2)$, and satisfying*

$$\left(A_1^T + \frac{l_1}{2}I\right)P_1 + P_1\left(A_1 + \frac{l_1}{2}I\right) + l_1(P_2 - P_3) + \rho(2P_1 - P_3) < 0 \tag{15}$$

$$\left(A_1^T - \frac{3}{2}l_2 I\right)P_3 + P_3\left(A_1 - \frac{3}{2}l_2 I\right) + \left(A_2^T + \frac{3}{2}l_2 I\right)P_1 + P_1\left(A_2 + \frac{3}{2}l_2 I\right)$$
$$+\, 3l_2 P_2 + \rho(4P_1 - P_3 - 2P_2) < 0 \tag{16}$$

$$\left(A_1^T + \frac{3}{2}l_3 I\right)P_2 + P_2\left(A_1 + \frac{3}{2}l_3 I\right) + \left(A_2^T - \frac{3}{2}l_3 I\right)P_3 + P_3\left(A_2 - \frac{3}{2}l_3 I\right)$$
$$+\, 3l_3 P_1 + \rho(2P_1 + P_3 - 4P_2) < 0 \tag{17}$$

and
$$\left(A_2^T + \frac{l_4}{2}I\right)P_2 + P_2(A_2 + \frac{l_4}{2}I) + l_4(P_1 - P_3) + \rho(P_3 - 2P_2) < 0 \tag{18}$$

Suppose that $(\mathcal{H}_1)$ holds, then any solution $x(t)$ of system (1) converges exponentially towards the ball $B(0, r_3)$ where

$$r_3 = \frac{2M(\lambda_{\max}(P_1) + \lambda_{\max}(P_2) + \lambda_{\max}(P_3))^2}{l\lambda_{\min}(P_2)\lambda_{\min}(P_1 + P_2 - P_3)}$$

with $l = \inf(l_1, l_2, l_3, l_4)$.

PROOF. Let $P(\alpha) = \alpha_1^2(t)P_1 + \alpha_2^2(t)P_2 + \alpha_1(t)\alpha_2(t)P_3$. It satisfies

$$P(\alpha) \geq \alpha_1^2(t)\lambda_{\min}(P_1)I + \alpha_2^2(t)\lambda_{\min}(P_2)I + \alpha_1(t)\alpha_2(t)\lambda_{\min}(P_3)I$$
$$\geq \alpha_1^2(t)\lambda_{\min}(P_1)I + (1 - 2\alpha_1(t)\alpha_2(t) - \alpha_1^2(t))\lambda_{\min}(P_2)I$$
$$+ \alpha_1(t)\alpha_2(t)\lambda_{\min}(P_3)I\alpha_1^2(t)(\lambda_{\min}(P_1) - \lambda_{\min}(P_2))I$$
$$+ \alpha_1(t)\alpha_2(t)(\lambda_{\min}(P_3) - 2\lambda_{\min}(P_2))I + \lambda_{\min}(P_2)I \geq \lambda_{\min}(P_2)I$$

and

$$\dot{P}(\alpha) = 2\alpha_1(t)\dot{\alpha}_1(t)P_1 + \dot{\alpha}_1(t)\alpha_2(t)P_3 - \alpha_1(t)\dot{\alpha}_1(t)P_3 - 2\alpha_2(t)\dot{\alpha}_1(t)P_2.$$

Multiplying last equality by $(\alpha_1(t) + \alpha_2(t))^2$ which equal to 1, one has

$$\dot{P}(\alpha) = \alpha_1^3(t)(\dot{\alpha}_1(t)(2P_1 - P_3)) + \alpha_1^2(t)\alpha_2(t)(\dot{\alpha}_1(t)(4P_1 - P_3 - 2P_2))$$
$$+ \alpha_1(t)\alpha_2^2(t)(\dot{\alpha}_1(t)(2P_1 + P_3 - 4P_2)) + \alpha_2^3(t)(\dot{\alpha}_1(t)(P_3 - 2P_2)).$$

The time-derivative of $V(x, \alpha)$ along the trajectories of system (1) is given by

$$
\begin{aligned}
\dot{V}(x, \alpha) = x^T \big(&\alpha_1^3(t)(A_1^T P_1 + P_1 A_1 + \dot{\alpha}_1(t)(2P_1 - P_3)) + \alpha_1^2(t)\alpha_2(t)(A_1^T P_3 \\
&+ P_3 A_1 + A_2^T P_1 + P_1 A_2 + \dot{\alpha}_1(t)(4P_1 - P_3 - 2P_2)) \\
&+ \alpha_2^2(t)\alpha_1(t)(A_1^T P_2 + P_2 A_1 + A_2^T P_3 + P_3 A_2 + \dot{\alpha}_1(t)(2P_1 + P_3 - 4P_2)) \\
&+ \alpha_2^3(t)(A_2^T P_2 + P_2 A_2 + \dot{\alpha}_1(t)(P_3 - 2P_2))\big) x \\
&+ 2x^T(\alpha_1^2(t)P_1 + \alpha_2^2(t)P_2 + \alpha_1(t)\alpha_2(t)P_3)f(t, \alpha(t), x).
\end{aligned}
$$

Since $|\dot{\alpha}_1(t)| \leq \rho$, we have

$$
\begin{aligned}
\dot{V}(x, \alpha) \leq x^T \big(&- l_1\alpha_1^3(t)(P_1 + P_2 - P_3) - 3l_2\alpha_1^2(t)\alpha_2(t)(P_1 + P_2 - P_3) \\
&- 3l_3\alpha_1(t)\alpha_2^2(t)(P_1 + P_2 - P_3) - l_4\alpha_2^3(t)(P_1 + P_2 - P_3)\big) x \\
&+ 2M\left(\lambda_{\max}(P_1) + \lambda_{\max}(P_2) + \lambda_{\max}(P_3)\right)\|x\| \leq -l\lambda_{\min}(P_1 + P_2 - P_3)\|x\|^2 \\
&+ 2M\left(\lambda_{\max}(P_1) + \lambda_{\max}(P_2) + \lambda_{\max}(P_3)\right)\|x\| \\
\leq &\frac{-l\lambda_{\min}(P_1 + P_2 - P_3)}{\lambda_{\max}(P_1) + \lambda_{\max}(P_2) + \lambda_{\max}(P_3)}V(x, \alpha) \\
&+ \frac{2M\left(\lambda_{\max}(P_1) + \lambda_{\max}(P_2) + \lambda_{\max}(P_3)\right)}{\sqrt{\lambda_{\min}(P_2)}}\sqrt{V(x, \alpha)}.
\end{aligned}
$$

Let

$$
v(t) = \sqrt{V(x, \alpha)}
$$

hence,

$$
\begin{aligned}
\dot{v}(t) \leq &\frac{-l\lambda_{\min}(P_1 + P_2 - P_3)}{2(\lambda_{\max}(P_1) + \lambda_{\max}(P_2) + \lambda_{\max}(P_3))}v(t) \\
&+ \frac{M\left(\lambda_{\max}(P_1) + \lambda_{\max}(P_2) + \lambda_{\max}(P_3)\right)}{\sqrt{\lambda_{\min}(P_2)}}.
\end{aligned}
$$

Integrating between t_0 and t, one obtains

$$
v(t) \leq v(t_0)\exp\left(\frac{-l}{2}\frac{\lambda_{\min}(P_1 + P_2 - P_3)}{\lambda_{\max}(P_1) + \lambda_{\max}(P_2) + \lambda_{\max}(P_3)}(t - t_0)\right) + \mu_1
$$

where

$$
\mu_1 = \frac{2M(\lambda_{\max}(P_1) + \lambda_{\max}(P_2) + \lambda_{\max}(P_3))^2}{l\sqrt{\lambda_{\min}(P_2)}\lambda_{\min}(P_1 + P_2 - P_3)},
$$

which implies that,

$$
\|x(t)\| \leq \mu_2\|x(t_0)\|\exp\left(\frac{-l}{2}\frac{\lambda_{\min}(P_1 + P_2 - P_3)}{\lambda_{\max}(P_1) + \lambda_{\max}(P_2) + \lambda_{\max}(P_3)}(t - t_0)\right)
$$

$$+\frac{2M(\lambda_{\max}(P_1) + \lambda_{\max}(P_2) + \lambda_{\max}(P_3))^2}{l\lambda_{\min}(P_2)\lambda_{\min}(P_1 + P_2 - P_3)}$$

with

$$\mu_2 = \sqrt{\frac{\lambda_{\max}(P_1) + \lambda_{\max}(P_2) + \lambda_{\max}(P_3)}{\lambda_{\min}(P_2)}}.$$

Thus, any solution $x(t)$ converges exponentially towards the ball $B(0, r_3)$ whose radius is given by

$$r_3 = \frac{2M(\lambda_{\max}(P_1) + \lambda_{\max}(P_2) + \lambda_{\max}(P_3))^2}{l\lambda_{\min}(P_2)\lambda_{\min}(P_1 + P_2 - P_3)}. \qquad \square$$

Now, as a generalization to the previous work, we give a general algorithm, with the aim of getting the radius of the ball as small as possible, by using N linear matrix inequalities. In fact, we begin by taking a Lyapunov function candidate in the form

$$V(x, \alpha) = x^T P(\alpha)x$$

where $P(\alpha) = \sum_{h=1}^{\eta} \beta_h(t)P_h$ is a symmetric positive definite matrix such that $\beta_h(t)$ are positive continuous functions that verify $\beta_h(t) \leq 1$ and P_h are constant symmetric positive definite matrices.

In the case of constant matrices, i.e., $P(\alpha) = P$, we use two $LMIs$ to prove the practical stability of system (1). These 2 $LMIs$ are

$$A_i^T P + PA_i + l_i P < 0, \quad i = 1, 2$$

where l_i are strict positive reals.

Otherwise, for the general case, i.e., $P(\alpha) = \sum_{h=1}^{\eta} \beta_h(t)P_h$, we have two steps to follow:

Step 1: For available values of the strict positive reals l_k, $k = 1, \ldots, N$, suppose that there exist symmetric positive definite matrices P_h, $h = 1, \ldots, \eta$, such that some combinations of these matrices are also symmetric positive definite, and that satisfy this system of $N - LMIs$:

$$A_i^T P_i + P_i A_i + \rho \sum_{h=1}^{\eta} c_{hi}P_h + l_k \sum_{h=1}^{\eta} d_{hi}P_h < 0$$

for $i = 1, 2$, c_{hi}, d_{hi} are constant coefficients in $\mathbb{R}$ and $\sum_{h=1}^{\eta} c_{hi}P_h$ and $\sum_{h=1}^{\eta} d_{hi}P_h$ are combinations of η matrices.

$$A_1^T P_j + P_j A_1 + A_2^T P_r + P_r A_2 + \rho \sum_{h=1}^{\eta} c_{hj}P_h + l_k \sum_{h=1}^{\eta} d_{hj}P_h < 0$$

for $j, r \in \{1, \ldots, \eta\}$, $j \neq 1$, $r \neq 2$ and c_{hj}, d_{hj} are constant coefficients in $\mathbb{R}$.

Step 2: Take the following value of the radius of the attractive ball

$$r = \frac{2M}{l} \frac{\left(\sum_{h=1}^{\eta} \lambda_{\max}(P_h)\right)^2}{\lambda_{\min}(P_s) \, \lambda_{\min}\left(\sum_{h=1}^{\eta} d_{hj} P_h\right)}$$

where $l = \inf_{k=1,\ldots,N} l_k$ and P_s, $s \in \{1, \ldots, \eta\}$ is the symmetric positive definite matrix that satisfies $P(\alpha) \geq \lambda_{\min}(P_s)I$.

Remark 1. Note that, if we replace M by $M(t)$ with $\lim_{t \to +\infty} M(t) = 0$, then the system (1) is uniformly exponentially convergent to zero.

Example 1. Let the system

$$\dot{x}(t) = (\alpha_1(t)A_1 + \alpha_2(t)A_2)x(t) + f(t, \alpha(t), x)$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, the matrices A_1, A_2 are given by

$$A_1 = \begin{pmatrix} 1 & -9 \\ 11 & -3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -6.6 & 8.5 \\ 3.4 & -6 \end{pmatrix}$$

and $f(t, \alpha(t), x) = frac\sqrt{2}.10^{-5}1 + t^2(\alpha_1(t)\sin^2 x_1, \alpha_2(t)\cos^2 x_2)$.

It is clear that f satisfies

$$\|f(t, \alpha(t), x)\|_2 \leq 2.10^{-5}.$$

When we take the conditions of Proposition 1 for $l_1 = 0.3$ and $l_2 = 0.001$, there exists a symmetric positive definite matrix

$$P = \begin{pmatrix} 0.0454 & -0.0081 \\ -0.0081 & 0.0399 \end{pmatrix}$$

which satisfies the $LMIs$ given in (10)-(11).

We get also $\lambda_{\min}(P) = 0.0341$, $\lambda_{\max}(P) = 0.0512$, which implies that the solutions $x(t)$ converge exponentially towards the ball $B(0, r_1)$ where

$$r_1 = 9.02 \times 10^{-2}$$

If we take into account the conditions of Proposition 2, it results that, for $\rho = 10$, $l_1 = 3.8$, $l_2 = 0.5$ and $l_3 = 3.7$, there exist symmetric positive definite matrices

P_1, P_2 and $(P_1 - P_2)$ which satisfy the $LMIs$ (12)–(13)–(14). These solutions are given by

$$P_1 = \begin{pmatrix} 3.5403 & -0.6367 \\ -0.6367 & 3.1370 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 3.2327 & -0.3958 \\ -0.3958 & 2.8180 \end{pmatrix}$$

and

$$P_1 - P_2 = \begin{pmatrix} 0.3076 & -0.2409 \\ -0.2409 & 0.3190 \end{pmatrix}.$$

We get also $\lambda_{\min}(P_1) = 2.6708 > \lambda_{\min}(P_2) = 2.5786$, $\lambda_{\min}(P_1 - P_2) = 0.0723$, $\lambda_{\max}(P_1) = 4.0066$ and $\lambda_{\max}(P_2) = 3.4722$ which implies that any solution $x(t)$ of system (1) converges exponentially towards the ball $B(0, r_2)$ where

$$r_2 = 2.4 \times 10^{-2}$$

Finally, when we use conditions of Theorem 1 for $\rho = 21$, $l_1 = 165$, $l_2 = 880$, $l_3 = 255$ and $l_4 = 200$, it results that, there exist symmetric positive definite matrices P_1, P_2, P_3 such that $(P_1 - P_2)$, $(2P_1 - P_3)$, $(P_3 - 2P_2)$ and $(P_1 + P_2 - P_3)$ are also symmetric positive definite matrices that satisfy the $LMIs$ (15)–(16)–(17)–(18). These matrices are given by

$$P_1 = \begin{pmatrix} 4.6708 & -0.8744 \\ -0.8744 & 4.3851 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 4.4169 & -0.6539 \\ -0.6539 & 4.1170 \end{pmatrix}$$

and

$$P_3 = \begin{pmatrix} 9.0703 & -1.5117 \\ -1.5117 & 8.4836 \end{pmatrix}$$

which allows us to conclude that the solutions $x(t)$ of system (1) converge exponentially towards the ball $B(0, r_3)$ where

$$r_3 = 1.92 \times 10^{-2}$$

Therefore, we have $r_1 > r_2 > r_3$.

Now, in the same context of studying the global uniform practical exponential stability and by using the same $LMIs$, we will impose the following condition on the perturbation term, instead of $(\mathcal{H}_1)$.

$(\mathcal{H}_2)$ There exists a continuous function $\rho(t, \alpha(t))$, such that

$$\|f(t, \alpha(t), x)\| \leq \rho(t, \alpha(t))$$

with

$$\int_0^{+\infty} \rho(s, \alpha(s))ds \leq M' < \infty, \quad M' > 0.$$

Proposition 3. *Assume that, for $l_1 \in \mathbb{R}_+^*$, $l_2 \in \mathbb{R}_+^*$, there exists a symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$ which satisfies the LMIs given in (10)–(11) and suppose that the assumption $(\mathcal{H}_2)$ holds, then any solution $x(t)$ of system (1) converges exponentially towards the ball $B(0, r_4)$ where*

$$r_4 = \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} M'.$$

PROOF. Let $P(\alpha) = P$. We have

$$\dot{V}(x, \alpha) \leq -l\lambda_{\min}(P)\|x\|^2 + 2\lambda_{\max}(P)\rho(t, \alpha(t))\, \|x\|$$

$$\leq \frac{-l\lambda_{\min}(P)}{\lambda_{\max}(P)} V(x, \alpha) + \frac{2\lambda_{\max}(P)\rho(t, \alpha(t))}{\sqrt{\lambda_{\min}(P)}} \sqrt{V(x, \alpha)}.$$

Let

$$\upsilon(t) = \sqrt{V(x, \alpha)}$$

So,

$$\dot{\upsilon}(t) \leq \frac{-l\lambda_{\min}(P)}{2\lambda_{\max}(P)} \upsilon(t) + \frac{\lambda_{\max}(P)}{\sqrt{\lambda_{\min}(P)}} \rho(t, \alpha(t))$$

and consequently, we get

$$\upsilon(t) \leq \upsilon(t_0) \exp\left(-\frac{l\lambda_{\min}(P)}{2\lambda_{\max}(P)}(t - t_0)\right) + \frac{\lambda_{\max}(P)}{\sqrt{\lambda_{\min}(P)}} \int_{t_0}^{t} \rho(s, \alpha(s))$$

$$\times \exp\left(-\frac{l\lambda_{\min}(P)}{2\lambda_{\max}(P)}(t - s)\right) ds$$

$$\leq \upsilon(t_0) \exp\left(-\frac{l\lambda_{\min}(P)}{2\lambda_{\max}(P)}(t - t_0)\right) + \frac{\lambda_{\max}(P)}{\sqrt{\lambda_{\min}(P)}} \int_{t_0}^{t} \rho(s, \alpha(s)) ds$$

$$\leq \upsilon(t_0) \exp\left(-\frac{l\lambda_{\min}(P)}{2\lambda_{\max}(P)}(t - t_0)\right) + \frac{\lambda_{\max}(P)}{\sqrt{\lambda_{\min}(P)}} M'.$$

Thus,

$$\|x(t)\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \|x(t_0)\| \exp\left(-\frac{l}{2} \frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}(t - t_0)\right) + \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} M',$$

which implies that the solutions $x(t)$ of system (1) converge exponentially towards the ball $B(0, r_4)$ whose radius is given by

$$r_4 = \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} M'. \qquad \square$$

Remark 2. In this case, we can use the same process described with assumption $(\mathcal{H}_1)$ in order to obtain less conservative results. In fact, we take the same Step 1 described before in the case where the nonlinearity is bounded, and for Step 2, we take the following value of the radius of the attractif ball

$$r = \frac{\sum_{h=1}^{\eta} \lambda_{\max}(P_h)}{\lambda_{\min}(P_s)} \; M'.$$

where P_s, $s \in \{1, \ldots, \eta\}$ verifies $P(\alpha) \geq \lambda_{\min}(P_s)I$.

5. Conclusion

Based on Lyapunov stability and LMI technique, new criteria are derived to ensure the practical stability of perturbed systems with parameter dependence. The effectiveness of the proposed criterion is verified in a numerical example.

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