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On weakly reversible rings and strongly reversible rings

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Abstract. We introduce the notion of weakly reversible rings which is a generalization of reversible rings, and investigate its properties. We first give some examples of weakly reversible rings. We next argue about the weak reversibility of some kinds of polynomial rings. A number of properties of this generalization are established. Furthermore, we introduce the concepts of α-strongly reversible rings and (α, δ)-strongly reversible rings to consider the strongly reversible property of an Ore extension $R[x; \alpha, \delta]$ of a ring R instead of the ring $R[x]$.

1. Introduction

In [1], Cohn introduced the notion of a reversible ring, a ring R is said to be reversible if $ab = 0$ implies $ba = 0$ for $a, b \in R$. Anderson–Camillo [2], observing the rings whose zero products commute, used the term ZC_2 for what is called reversible; while Krempa–Niewieczerzal [3] took the term C_0 for it. Recall that an endomorphism α of a ring R is called to be rigid if $a\alpha(a) = 0$ implies $a = 0$ for all $a \in R$. We call a ring R α-rigid if there exists a rigid endomorphism α of R. Note that every α-rigid ring is reduced (i.e., rings without no nonzero nilpotent elements in R). It is well known that every reduced ring is a reversible ring.

We consider a weak form of reversible rings in this paper. We call a ring R weakly reversible if $ab = 0$ implies that $r_1br_2ar_3$ is a nilpotent element of R for

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all $a, b \in R$ and $r_1, r_2, r_3 \in R$. As one would expect, reversible rings are weakly reversible. Examples are given to show that the converse is not true. We obtain the following implications of reversible rings :

reduced rings $\Longrightarrow$ strongly reversible rings $\Longrightarrow$ reversible rings $\Longrightarrow$ weakly reversible rings.

In general, each of these implications is irreversible. In [4, Lemma 1.4], it was claimed that all reversible rings are semicommutative, and we show that weakly reversible rings are not necessarily semicommutative in general. For a ring R, we prove that: (1) R is weakly reversible if and only if for any n, the n-by-n upper triangular matrix $T_n(R)$ is a weakly reversible ring. (2) If R is a semicommutative ring, then $R[x]$ is weakly reversible. (3) $R[x]$ is weakly reversible if and only if $R[x; x^{-1}]$ is weakly reversible.

Furthermore, we introduce the concept of α-strongly reversible rings for an endomorphism α of a ring R. We do this by considering the strongly reversible property on polynomials in the skew polynomial ring $R[x; \alpha]$ instead of the ring $R[x]$ (without skewing the scalar multiplication). This provides us with an opportunity to study strongly reversible rings in a general setting.

Throughout this paper, R denotes an associative ring with identity and α denotes a nonzero and non identity endomorphism, unless specified otherwise. For a ring R, we denote by $\mathrm{nil}(R)$ the set of all nilpotent elements of R.

2. Examples

Our focus in this section is to introduce the concept of a weakly reversible ring and give some examples of weakly reversible rings. We start with the following definition.

Definition 2.1. Let R be a ring, R is said to be a weakly reversible ring if $ab = 0$ then $r_1 b r_2 a r_3 \in \mathrm{nil}(R)$ for all $a, b \in R$ and $r_1, r_2, r_3 \in R$.

Clearly, every reversible ring is weakly reversible. In the following we will see that the converse is not true. Note that the class of weakly reversible rings is closed under subrings and finite direct products.

Example 2.1. Let R be a reduced ring. Then for any $n \geq 2$, the n-by-n upper triangular matrix ring $T_n(R)$ is not reversible. But $T_n(R)$ is weakly reversible.

PROOF. First we give some claims. $\square$

Claim 2.1. *A ring R is a weakly reversible ring if and only if, for any $n \geq 2$, the n-by-n upper triangular matrix ring $T_n(R)$ is weakly reversible .*

We note that any subring of weakly reversible rings is weakly reversible. Thus if $T_n(R)$ is a weakly reversible ring, then so is R. Conversely, let

$$A_{i_1} = \begin{pmatrix} a_{11}^{i_1} & a_{12}^{i_1} & \dots & a_{1n}^{i_1} \\ 0 & a_{22}^{i_1} & \dots & a_{2n}^{i_1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn}^{i_1} \end{pmatrix} \in T_n(R), \quad A_{i_2} = \begin{pmatrix} a_{11}^{i_2} & a_{12}^{i_2} & \dots & a_{1n}^{i_2} \\ 0 & a_{22}^{i_2} & \dots & a_{2n}^{i_2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn}^{i_2} \end{pmatrix} \in T_n(R)$$

with $A_{i_1}A_{i_2}$=0, and let

$$B_{j_k} = \begin{pmatrix} b_{11}^{j_k} & b_{12}^{j_k} & \dots & b_{1n}^{j_k} \\ 0 & b_{22}^{j_k} & \dots & b_{2n}^{j_k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_{nn}^{j_k} \end{pmatrix} \in T_n(R), k = 1, 2, 3.$$

Then we have $a_{ii}^{i_1}\, a_{ii}^{i_2} = 0$ for any $1 \le i \le n$. Since R is weakly reversible, there exists $m_i \in N$ such that $(b_{ii}^{j_1}\, a_{ii}^{i_2}\, b_{ii}^{j_2}\, a_{ii}^{i_1}\, b_{ii}^{j_3})^{m_i} = 0$ for any $i, i = 1, 2, \dots, n$. Let $m = max\{m_1, m_2, \dots, m_n\}$, then

$$\begin{aligned} &(B_{j_1}A_{i_2}B_{j_2}A_{i_1}B_{j_3})^m \\ &= \begin{pmatrix} b_{11}^{j_1}\, a_{11}^{i_2}\, b_{11}^{j_2}\, a_{11}^{i_1}\, b_{11}^{j_3} & * & \dots & * \\ 0 & b_{22}^{j_1}\, a_{22}^{i_2}\, b_{22}^{j_2}\, a_{22}^{i_1}\, b_{22}^{j_3} & \dots & * \\ \dots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & b_{nn}^{j_1} a_{nn}^{i_2}\, b_{nn}^{j_2}\, a_{nn}^{i_1}\, b_{nn}^{j_3} \end{pmatrix}^m \\ &= \begin{pmatrix} 0 & * & \dots & * \\ 0 & 0 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \end{aligned}$$

So we have $((B_{j_1}A_{i_2}B_{j_2}A_{i_1}B_{j_3})^m)^n = 0$, this implies that $T_n(R)$ is a weakly reversible ring.

The following is a corollary of Claim 2.1.

Claim 2.2. *Let R be a reduced ring, then, for any $n \ge 2$, $T_n(R)$ is weakly reversible.*

It is straightforward to verify that $T_n(R)$ is not reversible (see, e.g., [4, Example 1.5]). But $T_n(R)$ is weakly reversible by Claim 2.2.

It is well known that for a ring R and any positive integer n, if R is reduced then $R[x]/(x^n)$ is reversible, where (x^n) is the ideal of $R[x]$ generated by x^n. Based on it we may suspect that if R is reversible then $R[x]/(x^n)$ is reversible $(n \geq 2)$. But the following example eliminates the possibility.

Example 2.2. Let H be the Hamilton quaternions over the real number field and R be the trivial extension of H by H. Then R is reversible. But $S = T(R,R) \cong \left(\begin{smallmatrix} r & m \\ 0 & r \end{smallmatrix}\right)$ is not reversible by [4, Example 1.7]. Thus we have

$$R[x]/(x^n) \cong \left\{ \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} \\ 0 & a_0 & a_1 & \dots & a_{n-2} \\ 0 & 0 & a_0 & \dots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & a_0 \end{pmatrix} \mid a_i \in R,\ i = 0, 1, \dots, n-1 \right\}$$

is not reversible for any $n \geq 2$. However since R is weakly reversible by Claim 2.1, $R[x]/(x^n)$ is weakly reversible.

Another generalization of a reversible ring is a McCoy ring. Nielsen [5] called a ring R right McCoy if the equation $f(x)g(x) = 0$, where $f(x), g(x) \in R[x] \setminus \{0\}$, implies that there exists $s \in R \setminus \{0\}$ such that $f(x)s = 0$. Left McCoy rings are defined analogously. McCoy rings are both left and right McCoy rings. Every reversible ring is McCoy by [5, Theorem 2]. Based on this fact, one may suspect that if R is weakly reversible then R is a McCoy ring. But this is not true by the following example.

Example 2.3. Let R be a reduced ring and let

$$T_n(R) = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & a_{nn} \end{pmatrix} \mid a_{ij} \in R \right\}.$$

Then R is not McCoy by [6, Theorem 2.1], but R is weakly reversible by Claim 2.2.

According to [7], a ring R is called semicommutative if for all $a, b \in R$, $ab = 0$ implies $aRb = 0$. Since every reversible ring is semicommutative [4, Lemma 1.4], we may conjecture that weakly reversible rings may be semicommutative. But the following example shows that there exists a weakly reversible R such that R is not semicommutative.

Example 2.4. Let F be a division ring and we consider the 2-by-2 upper triangular matrix ring $R = \left(\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right)$. It is clear that R is not a semicommutative ring, but R is weakly reversible.

3. Polynomial extensions of weakly reversible rings

Now we will study some conditions under which polynomial rings may be weakly reversible. According to [8], a ring R is called Armendariz if whenever polynomials $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m, g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j. In [9], Liu introduced the notion of a weak Armendariz ring which is a generalization of Armendariz rings. A ring R is called weak Armendariz if whenever polynomials $f(x) = a_0+a_1x+a_2x^2+\cdots+a_mx^m, g(x) = b_0+b_1x+b_2x^2+\cdots+b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then a_ib_j is a nilpotent element of R for each i, j. Semicommutative rings are weak Armendariz rings [9, Corollary 3.4].

We conclude this section by investigating how the weak reversibility of a weakly reversible ring behaves with respect to polynomial rings. The following Lemma was proved in [9, Lemma 3.1].

Lemma 3.1. *Let R be a semicommutative ring. Then* $\mathrm{nil}(R)$ *is an ideal of R.*

The following is [9, Lemma 3.7].

Lemma 3.2. *Let R be a semicommutative ring, $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in R[x]$. If $a_0, a_1, \ldots, a_n \in \mathrm{nil}(R)$, then $f(x) \in \mathrm{nil}(R[x])$.*

It was proved in [7, Example 2] that if R is a semicommutative ring, then $R[x]$ need not be semicommutative (hence not be reversible). However, we have the following.

Proposition 3.1. *Let R be a semicommutative ring. Then $R[x]$ is weakly reversible.*

Proof. Let $f(x) = \sum_{i=0}^{m} a_ix^i$, $g(x) = \sum_{j=0}^{n} b_jx^j \in R[x]$ satisfy $f(x)g(x) = 0$. And let $h_1(x) = \sum_{k=0}^{p_1} c_kx^k$, $h_2(x) = \sum_{t=0}^{p_2} d_tx^t$, $h_3(x) = \sum_{s=0}^{p_3} e_sx^s$ be elements of $R[x]$. Since semicommutative rings are weak Armendariz, there exists $n_{ij} \in N$ such that $(a_ib_j)^{n_{ij}} = 0$ for any i and j, and hence $(c_kb_jd_ta_ie_s)^{n_{ij}} = 0$ by the semicommutativity of R. Note that

$$\begin{aligned}
&h_1(x)g(x)h_2(x)f(x)h_3(x)\\
&\quad=\Big(\sum_{k=0}^{p_1}c_kx^k\Big)\Big(\sum_{j=0}^{n}b_jx^j\Big)\Big(\sum_{t=0}^{p_2}d_tx^t\Big)\Big(\sum_{i=0}^{m}a_ix^i\Big)\Big(\sum_{s=0}^{p_3}e_sx^s\Big)\\
&\quad=\sum_{p=0}^{m+n+p_1+p_2+p_3}\Big(\sum_{i+j+k+t+s=p}c_kb_jd_ta_ie_s\Big)x^p.
\end{aligned}$$

We can see that $\sum_{i+j+k+t+s=p}c_kb_jd_ta_ie_s\in \mathrm{nil}(R)$ for any p by Lemma 3.1. Thus $h_1(x)g(x)h_2(x)f(x)h_3(x)\in \mathrm{nil}\,(R[x])$ by Lemma 3.2. This shows that $R[x]$ is weakly reversible. □

Let R be a ring and $\triangle$ be a multiplicative monoid in R consisting of central regular elements, and let $\triangle^{-1}R=\{u^{-1}a|u\in\triangle, a\in R\}$, then $\triangle^{-1}R$ is a ring. For it, we have the following result.

Proposition 3.2. *Let R be a ring, e is a central idempotent of R. Then the following statements are equivalent.*

(1) *R is weakly reversible.*

(2) *eR and $(1-e)R$ are weakly reversible.*

(3) *$\triangle^{-1}R$ is weakly reversible.*

Proof. (1) $\Longleftrightarrow$ (2) is straightforward since subrings and finite direct products of weakly reversible rings are weakly reversible.

(3) $\Longrightarrow$ (1) is obvious since R is a subring of $\triangle^{-1}R$.

(1) $\Longrightarrow$ (3) Let $\alpha\beta=0$ with $\alpha=u^{-1}a$, $\beta=v^{-1}b$, $u,v\in\triangle$ and $a,b\in R$, and let $r_i=w_ic_i$ be any element of $\triangle^{-1}R$, $i=1,2,3$, $w_i\in\triangle, c_i\in R$. Since $\triangle$ is contained in the center of R, we have $0=\alpha\beta=u^{-1}av^{-1}b=(u^{-1}v^{-1})ab$, and so $ab=0$. But R is weakly reversible, so there exists $n\in N$ such that $(c_1bc_2ac_3)^n=0$. Then

$$\begin{aligned}
(r_1\beta r_2\alpha r_3)^n&=(w_1^{-1}c_1v^{-1}bw_2^{-1}c_2u^{-1}aw_3^{-1}c_3)^n\\
&=((w_1^{-1}v^{-1}w_2^{-1}u^{-1}w_3^{-1})(c_1bc_2ac_3))^n=((w_3uw_2vw_1)^{-1}(c_1bc_2ac_3))^n\\
&=((w_3uw_2vw_1)^{-1})^n(c_1bc_2ac_3)^n=0.
\end{aligned}$$

Hence $\triangle^{-1}R$ is weakly reversible. □

The ring of Laurent polynomials in x, with coefficients in a ring R, consists of all formal sum $\sum_{i=k}^{n}m_ix^i$ with obvious addition and multiplication, where $m_i\in R$ and k,n are (possibly negative) integers. Denote it by $R[x;x^{-1}]$.

Corollary 3.1. *For a ring R, $R[x]$ is weakly reversible if and only if $R[x;x^{-1}]$ is weakly reversible.*

Proof. It suffices to establish necessity since $R[x]$ is a subring of $R[x;x^{-1}]$. Let $\triangle = \{1, x, x^2, \ldots\}$. Then clearly $\triangle$ is a multiplicative closed subset of $R[x]$. Since $R[x;x^{-1}] = \triangle^{-1}R[x]$, it follows that $R[x;x^{-1}]$ is weakly reversible by Proposition 3.2. $\square$

Remark. The following is another direct proof of Corollary 3.1.

Let $f(x), g(x) \in R[x;x^{-1}]$ with $f(x)g(x) = 0$, and let $h_i(x) \in R[x;x^{-1}]$ be any element with $i = 1,2,3$. Then there exists $s \in N$ such that $f_1(x) = f(x)x^s, g_1(x) = g(x)x^s$ and $h_i'(x) = h_i(x)x^s \in R[x]$, $i = 1,2,3$. Since $R[x]$ is weakly reversible and $f_1(x)g_1(x) = 0$ by the hypothesis, there exists $n \in N$ such that

$$(h_1'(x)g_1(x)h_2'(x)f_1(x)h_3'(x))^n = 0.$$

Then we have

$$\begin{aligned}(h_1(x)g(x)h_2(x)f(x)h_3(x))^n &= (x^{-5s}(h_1'(x)g_1(x)h_2'(x)f_1(x)h_3'(x)))^n \\ &= (x^{-5s})^n(h_1'(x)g_1(x)h_2'(x)f_1(x)h_3'(x))^n = 0.\end{aligned}$$

And so $R[x;x^{-1}]$ is weakly reversible.

It was proved in [4, Proposition 2.4] that if R is an Armendariz ring, then R is reversible if and only if $R[x]$ is reversible if and only if $R[x;x^{-1}]$ is reversible. Accordingly, we have the equivalence on weak reversibility in another situation.

Corollary 3.2. *Let R be a ring, then the following statements are equivalent:*

(1) *R is weakly reversible.*

(2) *$R[x]$ is weakly reversible.*

(3) *$R[x;x^{-1}]$ is weakly reversible.*

Now we consider D. A. Jordan's construction of the ring $A(R,\alpha)$ (see [10] for more details). Let $A(R,\alpha)$ or A be the subset $\{x^{-i}rx^i | r \in R, i \geq 0\}$ of the skew Laurent polynomial ring $R[x,x^{-1};\alpha]$, where $\alpha : R \to R$ is an injective ring endomorphism of a ring R. Elements of $R[x,x^{-1};\alpha]$ are finite sums of elements of the form $x^{-i}rx^i$ where $r \in R$ and i is a non-negative integer. Multiplication is subject to $xr = \alpha(r)x$ and $rx^{-1} = x^{-1}\alpha(r)$ for all $r \in R$. Note that for each $j \geq 0$, $x^{-i}rx^i = x^{-(i+j)}\alpha^j(r)x^{(i+j)}$. It follows that the set $A(R,\alpha)$ of all such elements forms a subring of $R[x,x^{-1};\alpha]$ with

$$x^{-i}rx^i + x^{-j}sx^j = x^{-(i+j)}(\alpha^j(r) + \alpha^i(s))x^{(i+j)}$$

$$(x^{-i}rx^i)(x^{-j}sx^j) = x^{-(i+j)}(\alpha^j(r)\alpha^i(s))x^{(i+j)}$$

for $r, s \in R$ and $i, j \geq 0$. Note that α is actually an automorphism of $A(R, \alpha)$.

Following [11], a ring R is α-compatible if for each $a, b \in R, a\alpha(b) = 0 \Leftrightarrow ab = 0$. A ring R is α-rigid if and only if R is α-compatible and reduced [11, Lemma 2.2]. Thus the α-compatible ring is a generalization of an α-rigid ring to the more general case where R is not assumed to be reduced.

Proposition 3.3. *If R is an α-rigid ring, then $A(R, \alpha)$ is a weakly reversible ring.*

Proof. Let $a = x^{-i}rx^i, b = x^{-j}sx^j \in A(R, \alpha)$. Suppose that $(x^{-i}rx^i)(x^{-j}sx^j) = 0$. Then $x^{-(i+j)}(\alpha^j(r)\alpha^i(s))x^{(i+j)} = 0$ and so $\alpha^j(r)\alpha^i(s) = 0$. Let $r_1 = x^{-m}px^m, r_2 = x^{-n}qx^n, r_3 = x^{-t}hx^t$ be elements of $A(R, \alpha)$, where $p, q, h \in R$ and m, n, t are non-negative integers. Since R is an α-rigid ring, we know that R is α-compatible and reduced. It is obvious that every reduced ring is weakly reversible, then there exists $k \in N$ such that

$$(\alpha^{n+i+j+t}(p)\alpha^i(s)\alpha^{m+i+j+t}(q)\alpha^j(r)\alpha^{m+n+i+j}(h))^k = 0.$$

Using [11, Lemma 2.1] and the fact that every reduced ring is reversible, we obtain

$$\begin{aligned}(\alpha^{k(m+n+i+j+t)-m}(p)\alpha^{k(m+n+i+j+t)-j}(s)\alpha^{k(m+n+i+j+t)-n}(q)\\ \alpha^{k(m+n+i+j+t)-i}(r)\alpha^{k(m+n+i+j+t)-t}(h))^k = 0.\end{aligned}$$

It follows that

$$\begin{aligned}&(r_1br_2ar_3)^k = ((x^{-m}px^m)(x^{-j}sx^j)(x^{-n}qx^n)(x^{-i}rx^i)(x^{-t}hx^t))^k\\ &= (x^{-(m+n+i+j+t)}(\alpha^{n+i+j+t}(p)\alpha^{m+n+i+t}(s)\alpha^{m+i+j+t}(q)\alpha^{m+n+j+t}(r)\\ &\quad \alpha^{m+n+i+j}(h))x^{(m+n+i+j+t)})^k\\ &= x^{-k(m+n+i+j+t)}(\alpha^{k(m+n+i+j+t)-m}(p)\alpha^{k(m+n+i+j+t)-j}(s)\alpha^{k(m+n+i+j+t)-n}(q)\\ &\quad \alpha^{k(m+n+i+j+t)-i}(r)\alpha^{k(m+n+i+j+t)-t}(h))^k x^{k(m+n+i+j+t)} = 0.\end{aligned}$$

This proves that $A(R, \alpha)$ is a weakly reversible ring. $\square$

4. α-strongly reversible rings and (α, δ)-strongly reversible rings

In this section, we consider the strongly reversible property on polynomials in the skew polynomial ring $R[x; \alpha]$ instead of the ring $R[x]$ (without skewing the scalar multiplication). We begin with the following.

Definition 4.1. Let α be an endomorphism of a ring R. R is called α-strongly reversible if for $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j$ in $R[x;\alpha]$, $f(x)g(x) = 0$ implies $g(x)f(x) = 0$.

According to [12], a ring R is called strongly reversible, if whenever polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) = 0$, then $g(x)f(x) = 0$. Since reduced rings are strongly reversible by [12], every α-rigid ring is α-strongly reversible [13, Proposition 6]. For the identity endomorphism I_R of a ring R, R is strongly reversible if and only if R is I_R-strongly reversible. Reduced rings are always I_R-strongly reversible.

In [14], the reversible property of a ring is extended to a ring endomorphism as follows: an endomorphism α of a ring R is called right (resp., left) reversible if whenever $ab = 0$ for $a, b \in R, b\alpha(a) = 0$ (resp., $\alpha(b)a = 0$). A ring R is called right (resp., left) α-reversible if there exists a right (resp., left) reversible endomorphism α of R. R is α-reversible if it is both right and left α-reversible.

The next example shows that weakly reversible rings need not be α-reversible.

Example 4.1. Let $\mathbb{Z}$ be the ring of integers. Consider the ring

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}.$$

Let $\alpha : R \to R$ be an endomorphism defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Then R is not left α-reversible by [14, Example 2.2]. But R is weakly reversible by Claim 2.1. It is obvious that R is not reversible, so R is not strongly reversible.

In [15], Hong called a ring an α-Armendariz ring if whenever any polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$,$g(x) = \sum_{j=0}^{n} b_j x^j$ in $R[x;\alpha]$, $f(x)g(x) = 0$ implies $a_i b_j = 0$ for all $0 \le i \le m$ and $0 \le j \le n$. The following example shows that there exists a strongly reversible ring which is not α-strongly reversible.

Example 4.2. Let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, where $\mathbb{Z}_2$ is the ring of integers modulo 2. Then R is a commutative reduced ring. Thus it is strongly reversible. Let $\alpha : R \to R$ be an endomorphism defined by $\alpha((a, b)) = (b, a)$. Then for $f(x) = (0, 1) + (1, 0)x$ and $g(x) = (1, 0)x$ in $R[x;\alpha]$, we have $f(x)g(x) = 0$, but $g(x)f(x) \neq 0$. So R is not α-strongly reversible. Note that $R[x;\alpha]$ is not reduced.

Moreover, R is not α-Armendariz. In fact, for $f(x) = (1, 0) + (1, 0)x$ and $g(x) = (0, 1) + (1, 0)x$ in $R[x;\alpha], f(x)g(x) = 0$ but $(1, 0)(1, 0) \neq (0, 0) \in R$.

A ring R can be extended to a ring

$$S = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\}$$

and an endomorphism α of a ring R is also extended to the endomorphism $\bar{\alpha}$: $S \to S$ defined by $\bar{\alpha}(a_{ij}) = (\alpha(a_{ij}))$.

Note that Example 4.2 also shows that strongly reversible rings need not be α-Armendariz. The next example shows that α-Armendariz rings need not be strongly reversible.

Example 4.3. Let α be an endomorphism of a ring R. If R is an α-rigid ring, then

$$S = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\}$$

is $\bar{\alpha}$-Armendariz [15, Proposition 2.1]. But S is not strongly reversible by [12, Example 3.7]. Note that S is not reduced, so S is not $\bar{\alpha}$-rigid.

We can obtain the following equivalence in the case of a reduced α-strongly reversible ring.

Proposition 4.1. *R is an α-rigid ring if and only if R is a reduced α-strongly reversible ring.*

Proof. It is enough to show that R is α-rigid when R is a reduced α-strongly reversible ring. Assume $a\alpha(a) = 0$ for $a \in R$. Then for $f(x) = ax$ and $g(x) = a$ in $R[x;\alpha]$, $f(x)g(x) = axa = a\alpha(a)x = 0$. Since R is an α-strongly reversible ring, $a^2 = 0$. Thus, $a = 0$ since R is reduced. Therefore, R is α-rigid. $\square$

Proposition 4.2. *Let R be an α-Armendariz ring. Then the following statements are equivalent:* (1) *R is reversible.* (2) *R is α-strongly reversible. As a corollary, let R be an Armendariz ring. Then the following statements are equivalent:* (1) *R is reversible.* (2) *R is strongly reversible.*

Proof. It follows from [15, Theorem 3.6] and [15, Corollary 3.7]. $\square$

Note that in Proposition 4.2, the statement R is α-Armendariz is not superfluous by Example 4.2.

In [16], Hong et al. defined a ring R with an endomorphism α to be α-skew Armendariz if whenever $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j$ in $R[x;\alpha]$, $f(x)g(x) = 0$ implies $a_i\alpha^i(b_j) = 0$ for all $0 \le i \le m$ and $0 \le j \le n$.

The next example shows that there exists an endomorphism α of a ring R such that (1) R is α-skew Armendariz, (2) R is not α-strongly reversible, (3) R is not α-Armendariz.

Example 4.4. Consider the ring of polynomials over $\mathbb{Z}_2, R = \mathbb{Z}_2[x]$. Let $\alpha : R \to R$ an endomorphism defined by $\alpha(f(x)) = f(0)$. Then

(1) R is α-skew Armendariz [16, Example 5].

(2) R is not α-strongly reversible: Let $p = ay, q = b \in R[y; \alpha]$ with $a = \bar{1} + x$ and $b = x$, then $pq = ayb = a\alpha(b)y = 0$. But $qp = bay = x(\bar{1} + x)y \neq 0$.

(3) R is not α-Armendariz [15, Example 1.9].

An Ore extension of a ring R is denoted by $R[x; \alpha, \delta]$, where α is an endomorphism of R and δ is a α-derivation, i.e., $\delta : R \to R$ is an additive map such that $\delta(ab) = \alpha(a)\delta(b) + \delta(a)b$ for all $a, b \in R$. Recall that elements of $R[x; \alpha, \delta]$ are the polynomials $\sum_{i=0}^{n} r_i x^i, r_i \in R$, where addition is defined as usual and multiplication by $xa = \alpha(a)x + \delta(a)$ for all $a \in R$.

For any $0 \leq i \leq j$ $(i, j \in \mathbb{N}), f_i^j \in End(R, +)$ will denote the map which is the sum of all positive words in α, δ built with i letters α and $j - i$ letters δ (e.g., $f_n^n = \alpha^n$ and $f_0^n = \delta^n, n \in \mathbb{N}$).

Lemma 4.1 ([17, Lemma 4.1])**.** *For any $n \in \mathbb{N}$ and $r \in R$ we have $x^n r = \Sigma_{i=0}^{n} f_i^n(r) x^i$ in the ring $R[x; \alpha, \delta]$.*

Following Hashemi and Moussavi [18], a ring R is called (α, δ)-skew Armendariz ring if whenever polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x; \alpha, \delta]$ satisfy $f(x)g(x) = 0$ then $a_i x^i b_j x^j = 0$ for each i, j.

The following Lemma extends [13, Proposition 6].

Lemma 4.2. *Let R be a ring, α an endomorphism of R and δ a α-derivation. Suppose that R is (α, δ)-skew Armendariz and (α, δ)-compatible. Let $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x; \sigma, \delta]$. Then $f(x)g(x) = 0$ if and only if $a_i b_j = 0$ for all $0 \leq i \leq m$ and $0 \leq j \leq n$.*

Proof. Assume that $f(x)g(x) = 0$. Since R is (α, δ)-skew Armendariz, $a_i x^i b_j x^j = 0$ for all i, j. Then we have the following equation by Lemma 4.1:

$$a_i x^i b_j x^j = a_i \sum_{t=0}^{i} f_t^i(b_j) x^{j+t} = a_i \alpha^i(b_j) x^{i+j} + h(x) = 0,$$

where $h(x)$ is a polynomial of degree strictly less than $i+j$. Therefore, $a_i \alpha^i(b_j)$=0. Since R is (α, δ)-compatible, $a_i b_j = 0$ for all $0 \leq i \leq m$ and $0 \leq j \leq n$.

Conversely, suppose that $a_ib_j = 0$ for all i, j. Then

$$a_ix^ib_jx^j = \sum_{t=0}^{i} a_if_t^i(b_j)x^{j+t} = 0$$

by (α, δ)-compatibility of R and Lemma 4.1. This implies that $a_if_t^i(b_j) = 0$ for all i, j, t. Therefore

$$f(x)g(x) = \left(\sum_{i=0}^{m} a_ix^i\right)\left(\sum_{j=0}^{n} b_jx^j\right) = \sum_{k=0}^{m+n}\left(\sum_{i+j=k} a_ix^ib_jx^j\right) = 0. \qquad \square$$

We call a ring R (α, δ)-strongly reversible if whenever polynomials $f(x) = \sum_{i=0}^{m} a_ix^i$, $g(x) = \sum_{j=0}^{n} b_jx^j \in R[x; \alpha, \delta]$ satisfy $f(x)g(x) = 0$ then $g(x)f(x) = 0$.

Proposition 4.3. *Let R be a ring, α an endomorphism of R and δ a α-derivation. Suppose that R is (α, δ)-skew Armendariz and (α, δ)-compatible. Then R is reversible if and only if R is (α, δ)-strongly reversible.*

Proof. It suffices to show that if R is reversible, then R is (α, δ)-strongly reversible. Let $f(x) = \sum_{i=0}^{m} a_ix^i$, $g(x) = \sum_{j=0}^{n} b_jx^j \in R[x; \alpha, \delta]$ such that $f(x)g(x) = 0$, by Lemma 4.2, we have $a_ib_j = 0$ for all $0 \le i \le m$ and $0 \le j \le n$. Since R is reversible, $b_ja_i = 0$ for all $0 \le i \le m$ and $0 \le j \le n$. Consequently $g(x)f(x) = 0$, this shows that R is (α, δ)-strongly reversible. $\square$

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