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Connected topologies on finite sets and polynomial irreducibility

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Abstract. It is known (cf. [10]) that any topology on a finite set with a prime number of open sets is connected. In this note, a generalization of this result is done by using a new criterion of irreducibility for polynomials with non-negative integer coefficients (cf. [5], [6], [7]), which we apply to polynomials naturally associated to topologies on finite sets.

1. Preliminaries

Throughout this note topologies on finite sets are considered, which will be called finite topologies. We will also talk about connected topologies instead of connected topological spaces. If $\langle A, \tau\rangle$, $\langle B, \sigma\rangle$ are two topological spaces with $A \cap B = \emptyset$, we call sum space the space $\langle A \cup B, \rho\rangle$, where $\rho = \{U \cup V : U \in \tau,\ V \in \sigma\}$ (cf. [3], [8]). We will say that ρ is the sum topology of τ and σ, and we will denote $\rho = \tau \oplus \sigma$. For instance, if $A = \{a_1, a_2, \dots , a_n\}$ and $\tau_d(n)$ is the discrete topology on A, we have $\tau_d(n) = \bigoplus_{i=1}^{n} \tau_d(1)$. Moreover, if $B \subseteq A$ we symbolize with $\tau\,|_B$ the topology induced by τ on B, i.e. $\tau\,|_B = \{U \cap B : U \in \tau\}$.

The following notation will be useful: if $A = \{a_1, a_2, \dots , a_n\}$ and $U \subseteq A$, say $U = \{a_{i_1}, a_{i_2}, \dots , a_{i_k}\}$, the symbol X^U indicates the monomial $X_{i_1} X_{i_2} \dots X_{i_k}$. Particularly, we agree that $X^{\emptyset} = 1$.

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Given the space $\langle A, \tau \rangle$, with $A = \{a_1, a_2, \ldots, a_n\}$, the polynomial $F_\tau \in \mathbb{Z}[X_1, X_2, \ldots, X_n]$ is defined as

$$F_\tau(X_1, X_2, \ldots, X_n) = \sum_{U \in \tau} X^U. \tag{1}$$

For example, if $A = \{a_1, a_2, a_3\}$ and τ has the Hasse diagram

then

$$F_\tau(X_1, X_2, X_3) = 1 + X_1 + X_2 + X_1 X_2 + X_1 X_2 X_3.$$

If $B \subseteq A$, say $B = \{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\}$, the result of specializing in 0 the indeterminates $X_{i_1}, X_{i_2}, \ldots, X_{i_k}$ in $F_\tau(X_1, X_2, \ldots, X_n)$ is the polynomial associated to $\tau \mid_{\tilde{B}}$, where $\tilde{B}$ designs the complement to B in A.

Given $F \in \mathbb{Z}[X_1, X_2, \ldots, X_n]$ and $G \in \mathbb{Z}[Y_1, Y_2, \ldots, Y_m]$, we define the tensor product $F \otimes G \in \mathbb{Z}[X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_m]$ as usual:

$$\begin{aligned}(F \otimes G)(X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_m) = \\ = F(X_1, X_2, \ldots, X_n)\ G(Y_1, Y_2, \ldots, Y_m).\end{aligned}$$

The following result, whose proof is straightforward, states the fundamental property of the assignation $\tau \mapsto F_\tau$.

Proposition 1.1. *If τ is a topology on $A = \{a_1, a_2, \ldots, a_n\}$ and σ is another topology on $Y = \{b_1, b_2, \ldots, b_m\}$, then*

$$F_{\tau \oplus \sigma} = F_\tau \otimes F_\sigma.$$

Further we shall consider polynomials with integral coefficients. Let be $F \in \mathbb{Z}[X]$, $F \neq 0$. We will call F non-negative if its coefficients are non-negative. In what follows we assume F non-negative. Now, we shall treat the divisibility and irreducibility relations between non-negative polynomials through the notions of (+)-divisibility and (+)-irreducibility, which can be defined in the following terms (cf. [5], [6], [7]):

Definition 1.2. If G is non-negative, we will say that G is a $(+)$-divisor of F $(G\,|_{(+)}\,F)$, if G divides F $(G \mid F)$ and $G^* = F/G$ is non-negative. It will be said that F is $(+)$-irreducible if $F \neq 1$ and

$$G\,|_{(+)}\,F \implies G = 1 \text{ or } G = F.$$

The following result has an immediate proof.

Proposition 1.3. *If $F(p)$ is prime for some $p \in \mathbb{N}$, then F is $(+)$-irreducible.*

In order to lay down another useful criterion of $(+)$-irreducibility, we introduce certain restrictions of the divisibility relation to the positive integers. To this purpose, let a, c and p be any positive integers with $p \geq 2$.

By the p-expansion of c we mean the representation of c in base p, let us say

$$c = (c_m c_{m-1} \dots c_0)_p = c_m p^m + c_{m-1} p^{m-1} + \cdots + c_0,$$

where $0 \leq c_k < p$; $k = 0, 1, \dots, m$, and $c_m \neq 0$. The non-negative polynomial

$$c_{\langle p \rangle} = c_m X^m + c_{m-1} X^{m-1} + \cdots + c_0$$

is called p-polynomial of c.

With this terminology at hands we can now state the following definition:

Definition 1.4. We will say that c is a p-divisor of a $(c\,|_p\,a)$ if c divides a $(c \mid a)$ and, for each $k = 0, 1, \dots$, the integers a, c *y* $c^* = a/c$ verify

$$a_k = c_k c_0^* + c_{k-1} c_1^* + \cdots + c_0 c_k^*,$$

where a_k, c_k *y* c_k^*; $k = 0, 1, \dots$, denote the digits of the p-expansions of a, c and c^* respectively. It will be said that a is p-irreducible if $a \neq 1$ and

$$c\,|_p\,a \implies c = 1 \text{ or } c = a.$$

Analogously to Proposition 1.3, we deduce the following result:

Proposition 1.5. *If a is prime, then a is p-irreducible.*

It should be observed that a divisor c of a is a p-divisor of a if and only if the product of the p-expansions of c and c^* can be made, by operating on the base p, without carries.

Examples 1.6. The 4-divisors of 30 are 1, 5, 6, 30, because in base 4, we have $5 = (11)_4$, $6 = (21)_4$, $30 = (132)_4$ and $(132)_4 = (11)_4(21)_4$. Note that in the products between 2 and $15((2)_4(33)_4)$, 3 and $10((3)_4(22)_4)$ there are carries. Besides, 6 is 4-irreducible since in the product $(2)_4(3)_4$ there is a carry.

A criterion to distinguish among the divisors of a those which are p-divisors, can be given by considering the sum of the digits of the p-expansions. The integer

$$|c|_p = c_m + c_{m-1} + \cdots + c_0$$

is called p-height of $c = (c_m c_{m-1} \cdots c_0)_p$. A theorem of C. de Polignac (cf. [4], pg. 269) establishes that the product (operating in base p) of the p-expansions of a and c is made without carries if and only if $|ac|_p = |a|_p |c|_p$. This result can be restated as follows:

Theorem 1.7. *Suppose that $c \mid a$ and let $c^* = a/c$. Then*

$$c \mid_p a \iff |a|_p = |c|_p \, |c^*|_p.$$

The maximum of the coefficients of F is called radius of F and is denoted by $r(F)$. Symbolically, we have $r(F) = \max\limits_{0 \le i \le n} a_i$ whenever $F = \sum\limits_{i=0}^{n} a_i X^i$.

The (+)-irreducibility criterion above mentioned can now be established in the following terms (cf. [5], [6], [7]):

Theorem 1.8. *Let F be with integral coefficients, and let $p \in \mathbb{Z}$, $p > r(F)$. Then*

$$F \text{ is } (+)\text{-irreducible} \iff F(p) \text{ is } p\text{-irreducible.}$$

2. Results

In the sequel, τ denotes an arbitrary topology on $A = \{a_1, a_2, \dots, a_n\}$. Let $F_\tau \in \mathbb{Z}[X_1, X_2, \dots, X_n]$ be the polynomial (1) associated to τ. Then, the following theorem holds:

Theorem 2.1. *τ is connected if and only if F_τ is irreducible in $\mathbb{Z}[X_1, X_2, \dots, X_n]$.*

Proof. $\Longleftarrow$) If τ is not connected and $A_1, A_2, \dots, A_r$ $(r \ge 2)$ are the connected components of $\langle A, \tau \rangle$, then we have

$$\tau = \tau\mid_{A_1} \oplus \tau\mid_{A_2} \oplus \cdots \oplus \tau\mid_{A_r},$$

and according to Proposition 1.1

$$F_\tau = F_{\tau|_{A_1}} \otimes F_{\tau|_{A_2}} \otimes \cdots \otimes F_{\tau|_{A_r}},$$

from where we obtain that F_τ is reducible.

$\Longrightarrow$) Suppose that $F_\tau = GH$, with G and H polynomials of positive degree. Because F is linear in each of its indeterminates, each of these appears in G or in H and then $F_\tau = G \otimes H$. By specializing in 0 the indeterminates of H, which up to a permutation can be supposed $X_{k+1}, X_{k+2}, \dots, X_n$, we obtain

$$F_\tau(X_1, X_2, \dots, X_k, 0, 0, \dots, 0) = G(X_1, X_2, \dots, X_k). \tag{2}$$

Since the left hand side of (2) is the polynomial associated to $\tau\,|_{\{a_1,a_2,\dots,a_k\}}$, it follows that

$$G = F_{\tau|_{\{a_1,a_2,\dots,a_k\}}}. \tag{3}$$

A similar reasoning, this time for H, leads to

$$H = F_{\tau|_{\{a_{k+1},a_{k+2},\cdots,a_n\}}}. \tag{4}$$

From (3) and (4) we obtain

$$\tau = \tau\,|_{\{a_1,a_2,\cdots,a_k\}} \oplus \tau\,|_{\{a_{k+1},a_{k+2},\cdots,a_n\}},$$

that is, τ is disconnected. $\square$

Note 2.2. Theorem 2.1 remains valid if F_τ is considered as a polynomial with coefficients on $\mathbb{Z}_2$ and the irreducibility is understood in $\mathbb{Z}_2[X_1, X_2, \dots, X_n]$.

Let $\hat{F}_\tau(X) = F_\tau(X, X, \dots, X)$ be the non-negative polynomial obtained by replacing by X the indeterminates of (1). We can now, by using the notion of (+)-irreducibility, prove a sufficient condition of connectedness for a finite topology τ.

Theorem 2.3. *If $\hat{F}_\tau$ is (+)-irreducible, then τ is connected.*

Proof. If τ is disconnected, say $\tau = \tau_1 \oplus \tau_2$, then Proposition 1.1 implies

$$F_\tau = F_{\tau_1} \otimes F_{\tau_2},$$

from where

$$\hat{F}_\tau = \hat{F}_{\tau_1} \hat{F}_{\tau_2},$$

with $\hat{F}_{\tau_1}$ and $\hat{F}_{\tau_2}$ of positive degree, which completes the proof. $\square$

The converse of Theorem 2.3 is false. In fact, the topology τ on $A = \{a_1, a_2, a_3\}$, which has the following Hasse diagram is obviously connected and $\hat{F}_\tau(X) = 1 + X + X^2 + X^3 = (1 + X)(1 + X^2)$ is reducible.

Remark 2.4. In order to establish a converse of Theorem 2.3, it would be useful to characterize the polynomials $\hat{F}_\tau = \sum_{i=0}^{n} a_i X^i$ with τ a topology on a set of cardinal $n \in \mathbb{N}$. Excepting immediate properties (for instance, $0 \le a_i \le \binom{n}{i}$; $i = 0, 1, \dots, n$), we have been unable generally to answer this question, which remains an interesting open problem.

Next, we state two important consequences of Theorem 2.3.

Corollary 2.5. *If $\hat{F}_\tau(p)$ is prime for some $p \in \mathbb{N}$, then τ is connected.*

Proof. According to what has been stated in the previous section, $\hat{F}_\tau$ is (+)-irreducible when $\hat{F}_\tau(p)$ is prime for some $p \in \mathbb{N}$. $\square$

Corollary 2.6. *Let be $r_k = |\{U \in \tau : |U| = k\}|$; $k = 0, 1, \dots, n$ and $r = \max_{0 \le k \le n} r_k$. If $p > r$, then*

$$\hat{F}_\tau(p) \text{ } p\text{-irreducible} \quad \Longrightarrow \quad \tau \text{ connected.}$$

Proof. By virtue of the definition of r_k, we can write $\tilde{F}_\tau(X) = \sum_{k=0}^{n} r_k X^k$. Now, the conclusion follows from Theorem 1.8. $\square$

From Corollary 2.5 or from Corollary 2.6 and Theorem 1.7 we deduce the following known result:

Corollary 2.7. *If $|\tau|$ is prime, then τ is a connected topology.*

In some cases we can establish sufficient conditions for the (+)-reducibility of $\hat{F}_\tau$ based on criteria of disconnectedness of the associated topology. In the sequel an example of this situation is given.

Proposition 2.8. *Let τ be a topology on $\{a_1, a_2, \dots, a_n\}$ such that $|\tau| > 2^{n-1} + 1$. Then, $\hat{F}_\tau$ is (+)-reducible.*

Proof. Any connected topology τ on $\{a_1, a_2, \dots, a_n\}$ verifies $|\tau| \le 2^{n-1} + 1$, (cf. [10]). Because of Theorem 2.3, $\hat{F}_\tau$ cannot be (+)-irreducible. $\square$

Next, we will show some simple applications.

Examples 2.9. 1) The (+)-reducibility of the polynomial $F(X) = 1 + X + X^2 + X^3$ can be established through Theorem 1.8 by observing that $F = \hat{F}_\tau$ where τ is the disconnected topology with diagram

2) Any topology τ on $\{a_1, a_2, a_3, a_4\}$ such that $\hat{F}_\tau(X) = 1 + X + 3X^2 + 3X^3 + X^4$ is connected because $\hat{F}_\tau(9) = 9001$ is prime. We arrive to the same conclusion by observing that $\hat{F}_\tau(4) = 501$ is 4-irreducible. In fact, $501 = 3 \cdot 167$ or, in base 4, $(13311)_4 = (3)_4(2213)_4$; thus, $|501|_4 = 9 \neq 3 \cdot 8 = |3|_4|167|_4$, which implies (according to Theorem 1.7) that 501 is 4-irreducible.

3) There exists a topology τ on $\{a_1, a_2, a_3, a_4, a_5\}$ such that $\hat{F}_\tau(X) = 1 + 3X + 5X^2 + 5X^3 + 3X^4 + X^5$. Since $|\tau| = 18 > 17 = 2^{5-1} + 1$, it results from Corollary 2.8 that $\hat{F}_\tau$ is (+)-reducible. In fact, we find that $\hat{F}_\tau(X) = (1 + X)(1 + X + X^2)^2$.

Note 2.10. A result analogous to the previous theorem can be obtained (cf. [7]) by considering polynomials in $\mathbb{Z}_p[X]$ (i.e. the polynomials whose coefficients are integers modulo p, with p not necessarily prime). More precisely, let F, G be arbitrary non-zero polynomials in $\mathbb{Z}_p[X]$. We say that G is a divisor of F ($G \mid F$), if there exists $H \in \mathbb{Z}_p[X]$ such that $F = GH$; in this case H is called a complement of G in F (H is not necessarily unique when p is not prime). We denote by $\hat{F}$ the polynomial in $\mathbb{Z}[X]$ whose coefficient of order k is the minimal non-negative integer in the class corresponding to the coefficient of order k of F. The following definitions (p-divisibility, p-irreducibility and p-height) are convenient.

We say that G is a p-divisor of F ($G \mid_p F$) , if $G \mid F$ and there exists a complement H of G in F such that $\hat{F} = \hat{G}\hat{H}$. We say that F is p-irreducible if $F \neq 1$ and

$$G \mid_p F \implies G = 1 \text{ or } G = F.$$

The p-height of F is the integer defined by $|F|_p = \hat{F}(1)$.

In the first place, we have the following analogous of Theorem 1.7:

Theorem 1.7*. *Assume that $G \mid F$. Then $G \mid_p F \iff |F|_p = |G|_p\, |H|_p$ for some complement H of G in F.*

Theorems 1.7 and 1.7* imply the following analogous of Theorem 1.8:

Theorem 1.8*. *$\hat{F}$ is $(+)$-irreducible $\iff$ F is p-irreducible.*

Now, if we consider F_τ as a polynomial in $\mathbb{Z}_p[X_1, X_2, \dots, X_n]$, we have $\hat{F}_\tau \in \mathbb{Z}_p[X]$. Thus, we obtain the following analogous of Theorem 2.3:

Theorem 2.3*. *If $\hat{F}_\tau$ is p-irreducible, then τ is connected.*

Remark 2.11. Here we will briefly indicate how the results obtained up to now can be transferred to a more general context. If $\mathcal{A}$ is a family of sets such that $\bigcup_{A\in\mathcal{A}} A$ is finite, we can, similarly to what was made above for a finite topology, associate to $\mathcal{A}$ a polynomial $F_{\mathcal{A}}(X_1, X_2, \dots, X_n)$, where $n = \left| \bigcup_{A\in\mathcal{A}} A \right|$. Now, if $\mathcal{A}$ and $\mathcal{B}$ are two families where $\bigcup_{A\in\mathcal{A}} A$ and $\bigcup_{B\in\mathcal{B}} B$ are finite and disjoint (disjoint families, to be brief), there can be defined a "product" $\mathcal{A} \otimes \mathcal{B}$ as follows:

$$\mathcal{A} \otimes \mathcal{B} = \{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}\}.$$

With respect to this product, the associated polynomials have the following property (corresponding to Proposition 1.1):

$$F_{\mathcal{A}\otimes\mathcal{B}} = F_{\mathcal{A}} \otimes F_{\mathcal{B}}.$$

This allows, after making an appropriate change in the terminology, an immediate generalization of the results previously obtained. For instance, if we say that a family $\mathcal{A} \neq \{\emptyset\}$ is irreducible when there do not exist disjoint families $\mathcal{B} \neq \{\emptyset\}$ and $\mathcal{C} \neq \{\emptyset\}$ such that $\mathcal{A} = \mathcal{B} \otimes \mathcal{C}$, one can prove the following analogous to Theorem 2.3:

$$\hat{F}_{\mathcal{A}} \ (+)\text{-irreducible} \implies \mathcal{A} \text{ irreducible},$$

where $\hat{F}_{\mathcal{A}}(X) = F_{\mathcal{A}}(X, X, \dots, X)$.

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