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On a characterization theorem on Abelian groups

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Abstract. Let $\xi_1, \xi_2, \dots, \xi_n,\ n \ge 2$ be independent identically distributed random variables. It is well known that if $\bar{\xi} = \frac{1}{n}\sum_{j=1}^{n} \xi_j$ and $\mathbf{v} = (\xi_1 - \bar{\xi}, \xi_2 - \bar{\xi}, \dots, \xi_n - \bar{\xi})$ are independent, then all ξ_j are Gaussian. We give a complete description of second countable locally compact Abelian groups for which a group analogue of this characterization theorem holds true.

1. Introduction

It is well known that if $\xi_1, \xi_2, \dots, \xi_n,\ n \ge 2$ are independent identically distributed Gaussian random variables, then $\bar{\xi} = \frac{1}{n}\sum_{j=1}^{n} \xi_j$ and $\mathbf{v} = (\xi_1 - \bar{\xi}, \xi_2 - \bar{\xi}, \dots, \xi_n - \bar{\xi})$ are independent. Assume now that $\xi_1, \xi_2, \dots, \xi_n,\ n \ge 2$ are independent identically distributed random variables such that $\bar{\xi}$ and $\mathbf{v}$ are independent. Then $\bar{\xi}$ and $s^2 = \frac{1}{n}\sum_{j=1}^{n}(\xi_j - \bar{\xi})^2$ are also independent. This implies by Geary's theorem ([6], [11], [10], [13]) that the random variables ξ_j are Gaussian. Thus, a Gaussian measure on the real line is characterized by the independence of $\bar{\xi}$ and $\mathbf{v}$.

The article deals with a generalization of this characterization theorem to the case when independent random variables take values in a locally compact Abelian group. Since an arbitrary Abelian group generally is not a group with unique division by n, instead of $\bar{\xi}$ and $\mathbf{v}$ we consider $S = \sum_{j=1}^{n} \xi_j$ and $\mathbf{V} = (n\xi_1 - S, \dots, n\xi_n - S)$.

We will use in the article the standard results on structure of locally compact Abelian groups and the duality theory ([7]). Agree on notation. For an arbitrary

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locally compact Abelian group X denote by $Y = X^*$ its character group, and by (x, y) the value of a character $y \in Y$ at an element $x \in X$. If H is a closed subgroup of the group Y, then denote by $A(X, H) = \{x \in X : (x, y) = 1 \text{ for all } y \in H\}$ its annihilator. Denote by c_X the connected component of zero of the group X, and by b_X the set of all compact elements of X. Let n be a natural number. Put $X_{(n)} = \{x \in X : nx = 0\}$ and $X^{(n)} = \{x \in X : x = n\widetilde{x},\ \widetilde{x} \in X\}$. Denote by $\mathbb{Z}$ the group of integers, by $\mathbb{R}$ the group of real numbers and by $\mathbb{T}$ the circle group (the one-dimensional torus). Let Y be an arbitrary Abelian group, $f(y)$ be a function on Y, h be an element of Y. Denote by Δ_h the finite difference operator

$$\Delta_h f(y) = f(y + h) - f(y).$$

A function $f(y)$ on Y is called a polynomial if for a nonnegative integer m $f(y)$ satisfies the equation

$$\Delta_h^{m+1} f(y) = 0, \quad y, h \in Y.$$

The minimal m for which this equality holds is called the degree of the polynomial $f(y)$.

We will assume in the article that X is a second countable locally compact Abelian group. Denote by $\mathrm{M}^1(X)$ the convolution semigroup of probability distributions on X. For $\mu \in \mathrm{M}^1(X)$ denote by

$$\widehat{\mu}(y) = \int_X (x, y) d\mu(x)$$

its characteristic function. Note that if ξ is a random variable with values in X and distribution μ, then the characteristic function of the distribution μ is the mathematical expectation

$$\widehat{\mu}(y) = \mathbf{E}[(\xi, y)].$$

Denote by E_x the degenerate distribution concentrated at the point $x \in X$. The set of all degenerate distributions on the group X denote by $D(X)$. For $\mu \in \mathrm{M}^1(X)$ define the distribution $\bar{\mu} \in \mathrm{M}^1(X)$ by the formula $\bar{\mu}(B) = \mu(-B)$ for any Borel set B. Denote by $\sigma(\mu)$ the support of a distribution μ.

A probability measure γ on the group X is called Gaussian (in the sense of Parthasarathy) ([12, Ch. 4.6]), if its characteristic function can be represented in the form

$$\hat{\gamma}(y) = (x, y) \exp\{-\varphi(y)\}, \tag{1}$$

where $x \in X$, and $\varphi(y)$ is a continuous nonnegative function on Y satisfying the equation

$$\varphi(u + v) + \varphi(u - v) = 2[\varphi(u) + \varphi(v)], \quad u, v \in Y. \tag{2}$$

Taking into account that in the article we will deal only with Gaussian measures in the sense of Parthasarathy we will name them Gaussian. Denote by $\Gamma(X)$ the set of Gaussian measures on the group X. Denote by m_K the normalized Haar measure of a compact subgroup K of the group X, and by $I(X)$ the set of shifts of such measures.

We will say that a distribution $\mu \in \Gamma_n(X)$ if there exist independent identically distributed random variables ξ_j, $j = 1, 2, \dots, n$, $n \geq 2$ with values in the group X and a distribution μ such that S and $\mathbf{V}$ are independent.

It is not difficult to verify that for a group X the inclusion $\Gamma(X) \subset \Gamma_n(X)$ holds. In §2 we completely describe groups X which have the following property: if $\mu \in \Gamma_n(X)$ and the characteristic function $\widehat{\mu}(y)$ does not vanish, then $\mu \in \Gamma(X)$. We apply the results of §2 in §3 to give the complete description of locally compact Abelian groups X for which any distribution $\mu \in \Gamma_n(X)$ is invariant with respect to a compact subgroup K of the group X and under the natural homomorphism $X \mapsto X/K$ induces on the factor-group X/K a Gaussian measure. We can consider the obtained class of groups as the widest subclass of locally compact Abelian groups on which an analogue of the theorem about characterization of a Gaussian measure on the real line by the independence of $\bar{\xi} = \frac{1}{n}\sum_{j=1}^{n}\xi_j$ and $\mathbf{v} = (\xi_1 - \bar{\xi}, \xi_2 - \bar{\xi}, \dots, \xi_n - \bar{\xi})$ holds.

The above mentioned problems are reduced to solving of a functional equation in the class of continuous positive definite functions on the group $Y = X^*$.

2. The characteristic function $\widehat{\mu}(y)$ does not vanish

Lemma 1. *A distribution $\mu \in \Gamma_n(X)$ if and only if its characteristic function $\widehat{\mu}(y)$ satisfies the equation*

$$\prod_{j=1}^{n} \widehat{\mu}(u + nv_j - (v_1 + \cdots + v_n)) = \widehat{\mu}^n(u) \prod_{j=1}^{n} \widehat{\mu}(nv_j - (v_1 + \cdots + v_n)), \quad u, v_1, \dots, v_n \in Y. \quad (3)$$

Proof. Note that S and $\mathbf{V}$ are independent if and only if the equality

$$\mathbf{E}[(S, u)(\mathbf{V}, (v_1, \dots, v_n))] = \mathbf{E}[(S, u)]\mathbf{E}[(\mathbf{V}, (v_1, \dots, v_n))] \quad (4)$$

holds for all $u, v_1, \dots, v_n \in Y$. Taking into account that the random variables

$\xi_1, \dots, \xi_n$ are independent, we transform the left-hand side of (4):

$$\mathbf{E}[(S,u)(\mathbf{V},(v_1,\dots,v_n))] = \mathbf{E}[(\xi_1+\cdots+\xi_n,u)((n\xi_1-(\xi_1+\cdots+\xi_n),$$
$$\dots, n\xi_n-(\xi_1+\cdots+\xi_n)),(v_1,\dots,v_n))]$$
$$= \mathbf{E}\left[\prod_{j=1}^n (\xi_j, u+nv_j-(v_1+\cdots+v_n))\right] = \prod_{i=1}^n \widehat{\mu}(u+nv_i-(v_1+\cdots+v_n)).$$

Analogously we transform the right-hand side of (4):

$$\mathbf{E}[(S,u)]\mathbf{E}[(\mathbf{V},(v_1,\dots,v_n))] = \mathbf{E}[(\xi_1+\cdots+\xi_n,u)]\mathbf{E}[((n\xi_1-(\xi_1+\cdots+\xi_n),$$
$$\dots, n\xi_n-(\xi_1+\cdots+\xi_n)),(v_1,\dots,v_n))]$$
$$= \prod_{j=1}^n \mathbf{E}[(\xi_j,u)]\mathbf{E}\left[\prod_{j=1}^n (\xi_j, nv_j-(v_1+\cdots+v_n))\right]$$
$$= \prod_{j=1}^n \mathbf{E}[(\xi_j,u)]\prod_{j=1}^n \mathbf{E}[(\xi_j, nv_j-(v_1+\cdots+v_n))]$$
$$= \widehat{\mu}^n(u)\prod_{i=1}^n \widehat{\mu}(nv_i-(v_1+\cdots+v_n)). \qquad \square$$

Suppose that $\gamma \in \Gamma(X)$ and the characteristic function $\widehat{\gamma}(y)$ has representation (1). By the function $\varphi(y)$ we can construct a symmetric 2-additive function by the formula

$$\psi(u,v) = \frac{1}{2}[\varphi(u+v)-\varphi(u)-\varphi(v)].$$

Then $\varphi(y) = \psi(y,y)$. Using this representation for the function $\varphi(y)$ one can check directly that the characteristic function $\widehat{\gamma}(y)$ satisfies equation (3). Hence, by Lemma 1 the inclusion

$$\Gamma(X) \subset \Gamma_n(X) \tag{5}$$

holds. The main result of this section is the complete description of groups X which have the property: if $\mu \in \Gamma_n(X)$ and the characteristic function $\widehat{\mu}(y)$ does not vanish, then $\mu \in \Gamma(X)$. The following proposition is valid.

Proposition 1. *Assume that $\mu \in \Gamma_2(X)$ and the characteristic function $\widehat{\mu}(y)$ does not vanish. This implies that $\mu \in \Gamma(X)$ if and only if the group X contains no subgroup topologically isomorphic to the two-dimensional torus $\mathbb{T}^2$. Assume that $\mu \in \Gamma_n(X)$, where $n \geq 3$. This implies that $\mu \in \Gamma(X)$ if and only if the group X contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$.*

We need some lemmas to prove Proposition 1.

Lemma 2. *Let $X = \mathbb{T}$ and $n \geq 3$. Then there exists a distribution $\mu \in \Gamma_n(X)$ such that the characteristic function $\widehat{\mu}(y)$ does not vanish and $\mu \notin \Gamma(X)$.*

Proof. Let $n = 3$. Consider on the group $\mathbb{Z}$ the function

$$l(k) = \begin{cases} 1 & \text{if } k \in \mathbb{Z}^{(3)} \\ \exp\left\{\dfrac{2\pi i}{9}\right\} & \text{if } k \in 1 + \mathbb{Z}^{(3)} \\ \exp\left\{-\dfrac{2\pi i}{9}\right\} & \text{if } k \in 2 + \mathbb{Z}^{(3)}. \end{cases} \tag{6}$$

Obviously, (6) implies that

$$l^3(k) = \exp\left\{\frac{2\pi k i}{3}\right\}, \quad k \in \mathbb{Z}. \tag{7}$$

Taking into account (7) and the fact that $l(k+3p) = l(k)$, $k, p \in \mathbb{Z}$, we can verify directly that the function $l(k)$ satisfies equation (3) for $n = 3$.

Take $\sigma > 0$ in such a way that the inequality

$$\sum_{k \in \mathbb{Z},\ k \neq 0} \exp\{-\sigma k^2\} < 1 \tag{8}$$

holds. Put

$$\rho(t) = 1 + \sum_{k \in \mathbb{Z},\ k \neq 0} l(k) \exp\{-\sigma k^2 - ikt\}, \quad t \in \mathbb{R}.$$

Since $l(-k) = \overline{l(k)}$, $|l(k)| = 1$, $k \in \mathbb{Z}$, in view of (8) the inequality

$$\rho(t) > 0, \quad t \in \mathbb{R}$$

is valid. We also have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \rho(t)dt = 1.$$

Let μ be the distribution on the group $\mathbb{T}$ with density $r(e^{it}) = \rho(t)$ with respect to $m_{\mathbb{T}}$. By the construction the characteristic function of the distribution μ is of the form

$$\widehat{\mu}(k) = l(k) \exp\{-\sigma k^2\}, \quad k \in \mathbb{Z}.$$

Since the function $l(k)$ satisfies equation (3) for $n = 3$, the characteristic function $\widehat{\mu}(k)$ also satisfies equation (3) for $n = 3$. By Lemma 1 $\mu \in \Gamma_3(X)$. On the other hand, since the function $l(k)$ is not a character of the group $\mathbb{Z}$, we have $\mu \notin \Gamma(X)$.

Let $n > 3$. If $n = 2p - 1$ we put

$$l(k) = \begin{cases} 1 & \text{if } k \in q + \mathbb{Z}^{(n)},\ q = 0, 1, \dots, p-2, p+1, \dots, n-1, \\ \exp\left\{\dfrac{2\pi i}{n}\right\} & \text{if } k \in p - 1 + \mathbb{Z}^{(n)} \\ \exp\left\{-\dfrac{2\pi i}{n}\right\} & \text{if } k \in p + \mathbb{Z}^{(n)}. \end{cases}$$

If $n = 2p$ we put

$$l(k) = \begin{cases} 1 & \text{if } k \notin p + \mathbb{Z}^{(n)} \\ -1 & \text{if } k \in p + \mathbb{Z}^{(n)}. \end{cases}$$

Next we argue as in the case when $n = 3$. □

Remark 1. Let $n \geq 3$. It is easily seen that the function $l(k)$ constructed in the proof of Lemma 2 is the characteristic function of a signed measure on the group $\mathbb{T}$ concentrated in the subgroup $\mathbb{Z}(n)$ (the multiplicative group of nth roots of unity).

Lemma 3. *Assume that $\gamma \in \Gamma_n(X)$, and the characteristic function $\widehat{\gamma}(y) > 0$ for $y \in Y$. Then $\gamma \in \Gamma(X)$, and the function $\widehat{\gamma}(y)$ can be represented in the form* (1)*, where $x = 0$.*

Proof. Put $\psi(y) = -\log \widehat{\gamma}(y)$. By Lemma 1 the characteristic function $\widehat{\gamma}(y)$ satisfies equation (3). Taking the logarithm of both sides of (3), we get

$$\sum_{j=1}^{n} \psi(u + nv_j - (v_1 + \cdots + v_n)) = n\psi(y) + \sum_{j=1}^{n} \psi(nv_j - (v_1 + \cdots + v_n)), \quad u, v_j \in Y. \quad (9)$$

We use the finite difference method to solve equation (9). Let h_1 be an arbitrary element of the group Y. Substitute $u + h_1$ for u and $v_j + h_1$ for v_j, $j = 1, 2, \dots, n$ in equation (9). Subtracting equation (9) from the resulting equation we obtain

$$\sum_{j=1}^{n} \Delta_{h_1} \psi(u + nv_j - (v_1 + \cdots + v_n)) = n\Delta_{h_1}\psi(u), \quad u, h_1, v_j \in Y. \quad (10)$$

Let h_2 be an arbitrary element of the group Y. Substitute $u + h_2$ for u and $v_1 + h_2$ for v_1 in equation (9). Subtracting equation (10) from the resulting equation we get

$$\Delta_{nh_2}\Delta_{h_1}\psi(u + nv_1 - (v_1 + \cdots + v_n)) = n\Delta_{h_2}\Delta_{h_1}\psi(u), \quad u, h_1, h_2, v_j \in Y. \quad (11)$$

Let h_3 be an arbitrary element of the group Y. Substitute $u+h_3$ for u and v_2+h_3 for v_2 in equation (9). Subtracting equation (11) from the resulting equation we find

$$n\Delta_{h_3}\Delta_{h_2}\Delta_{h_1}\psi(u)=0. \quad u,h_1,h_2,h_3\in Y. \tag{12}$$

Put in (12) $h_1=h_2=h_3=h$. We get

$$\Delta_h^3\psi(u)=0, \quad u,h\in Y, \tag{13}$$

i.e. $\psi(y)$ is a continuous polynomial of degree ≤ 2. It is easy to see that each polynomial of degree ≤ 2, in particular $\psi(y)$, can be represented in the form

$$\psi(y)=\varphi(y)+l(y)+c, \quad y\in Y, \tag{14}$$

where the function $\varphi(y)$ satisfies equation (2), the function $l(y)$ satisfies equation

$$l(u+v)=l(u)+l(v), \quad u,v\in Y,$$

and $c=const$. Since $\widehat{\gamma}(0)=1$, we can assume that $c=0$. Since the function $\widehat{\gamma}(y)$ is real-valued, we have $\widehat{\gamma}(-y)=\overline{\widehat{\gamma}(y)}=\widehat{\gamma}(y)$. Hence, $\psi(-y)=\psi(y)$. This implies that in (14) $l(y)=0$, $y\in Y$. So, $\psi(y)=\varphi(y)$, $y\in Y$. This proves the lemma. □

A distribution $\mu\in \mathrm{M}^1(X)$ is called a Gaussian measure in the sense of Bernstein ([9, 5.3]) if μ has the following property: if ξ_1 and ξ_2 are independent identically distributed random variables with values in X and distribution μ, then their sum and difference are independent. We denote by $\Gamma_B(X)$ the set of Gaussian measures in the sense of Bernstein on the group X.

Lemma 4 ([9, 5.3])**.** *A distribution $\mu\in \mathrm{M}^1(X)$ belongs to the class $\Gamma_B(X)$ if and only if the characteristic function $\widehat{\mu}(y)$ satisfies the equation*

$$\widehat{\mu}(u+v)\widehat{\mu}(u-v)=\widehat{\mu}^2(u)|\widehat{\mu}(v)|^2, \quad u,v\in Y. \tag{15}$$

Lemma 5. $\Gamma_2(X)=\Gamma_B(X)$.

Proof. Let $\mu\in\Gamma_2(X)$. By Lemma 1 the characteristic function $\widehat{\mu}(y)$ satisfies equation (3) which takes the form

$$\widehat{\mu}(u+(v_1-v_2))\widehat{\mu}(u-(v_1-v_2))=\widehat{\mu}^2(u)|\widehat{\mu}(v_1-v_2)|^2, \quad u,v_1,v_2\in Y. \tag{16}$$

Substituting $v_1=v$, $v_2=0$ into (16), we obtain that the characteristic function $\widehat{\mu}(y)$ satisfies equation (15). Hence, by Lemma 4 $\mu\in\Gamma_B(X)$. Lemmas 1 and 4 also imply that if $\mu\in\Gamma_B(X)$, then $\mu\in\Gamma_2(X)$. □

Proof of Proposition 1. Let $n=2$. Applying Lemma 5 we reduce the proof of Proposition 1 to the proof of the corresponding statement for distributions from the class $\Gamma_B(X)$, but for such distributions this statement was proved in [2] (see also [5], Lemmas 9.6 and 9.7).

Let $n \geq 3$ and $\mu \in \Gamma_n(X)$. Put $\nu = \mu * \bar{\mu}$. It follows from Lemma 1 that $\nu \in \Gamma_n(X)$. Since $\widehat{\nu}(y) = |\widehat{\mu}(y)|^2 > 0$, by Lemma 3 $\nu \in \Gamma(X)$. If a group X contains no subgroup topologically isomorphic to $\mathbb{T}$, then by Cramer's theorem for locally compact Abelian groups ([1], see also [5, Theorem 4.6]), we get that $\mu \in \Gamma(X)$. Thus, the sufficiency when $n \geq 3$ is also proved. The necessity follows from Lemma 2. □

3. The main theorem

Put $I_n(X) = I(X) \cap \Gamma_n(X)$. It follows from (5) and Lemma 1 that the inclusion

$$\Gamma(X) * I_n(X) \subset \Gamma_n(X)$$

holds. The main problem solved in this section is the following: to describe all groups X for which

$$\Gamma(X) * I_n(X) = \Gamma_n(X). \tag{17}$$

Observe that if $\mu \in \Gamma(X) * I(X)$, then μ is invariant with respect to a compact subgroup K of the group X and under the natural homomorphism $X \mapsto X/K$ induces on the factor-group X/K a Gaussian measure. We formulate now the main theorem.

Theorem 1. *Equality* (17) *holds for a group X, when $n=2$, if and only if the connected component of zero c_X of X contains no more than one element of order* 2. *Equality* (17) *holds for a group X, when $n \geq 3$, if and only if c_X has the property*

$$\{x \in c_X : nx = 0\} = \{0\}. \tag{18}$$

To prove Theorem 1 we need some lemmas. First we will formulate as a lemma the following simple statement, but omit its proof.

Lemma 6. *Let n be a natural number, K be a compact subgroup of a group X. Then the following statements are equivalent:*

(i) $K^{(n)} = K$;

(ii) *if $ny \in A(Y,K)$, then $y \in A(Y,K)$.*

We note that the characteristic function of the Haar measure m_K is of the form

$$\widehat{m}_K(y) = \begin{cases} 1 & \text{if } y \in A(Y,K) \\ 0 & \text{if } y \notin A(Y,K). \end{cases} \tag{19}$$

Lemma 7. *Let K be a compact subgroup of a group X. Then the following statements are equivalent:*

(i) $m_K \in I_n(X)$;

(ii) $K^{(n)} = K$.

PROOF. (i) ⇒ (ii). By Lemma 1 the characteristic function $\widehat{m}_K(y)$ satisfies equation (3). Substituting $v_1 = u$, $v_2 = \cdots = v_n = 0$ into (3), we get

$$\widehat{m}_K(nu) = \widehat{m}_K(u)\widehat{m}_K((n-1)u), \quad u \in Y. \tag{20}$$

It follows from (19) and (20) that if $ny \in A(Y,K)$, then $y \in A(Y,K)$. Hence, by Lemma 6 $K^{(n)} = K$.

(ii) ⇒ (i). By Lemma 1 it suffices to check that the characteristic function $\widehat{m}_K(y)$ satisfies equation (3), which takes the form

$$\prod_{j=1}^{n} \widehat{m}_K(u + nv_j - (v_1 + \cdots + v_n)) = \widehat{m}_K(u) \prod_{j=1}^{n} \widehat{m}_K(nv_j - (v_1 + \cdots + v_n)), \quad u, v_1, \ldots, v_n \in Y. \tag{21}$$

Both sides of equation (21) take the values either 0 or 1. If the right-hand side of (21) is equal to 1, then $u \in A(Y,K)$, $nv_j - (v_1 + \cdots + v_n) \in A(Y,K)$, $j = 1, 2, \ldots, n$. Hence, $u + nv_j - (v_1 + \cdots + v_n) \in A(Y,K)$, $j = 1, 2, \ldots, n$. Therefore the left-hand side of (21) is also equal to 1.

Let the left-hand side of (21) be equal to 1. This implies that

$$u + nv_j - (v_1 + \cdots + v_n) \in A(Y,K), \quad j = 1, 2, \ldots, n. \tag{22}$$

It follows from (22) that

$$\sum_{j=1}^{n} (u + nv_j - (v_1 + \cdots + v_n)) = nu \in A(Y,K). \tag{23}$$

By Lemma 6 we get from (23) that $u \in A(Y,K)$. Then (22) implies that $nv_j - (v_1 + \cdots + v_n) \in A(Y,K)$, $j = 1, 2, \ldots, n$. Hence, the right-hand side of (21) is also equal to 1. □

Lemma 8 (compare with [4, Lemma 10.4])**.** *Let X be a discrete torsion free group. Then $\Gamma_n(X) = D(X)$.*

Proof. If X is a discrete torsion free group, then Y is a connected compact group ([7, (24.25)]). Assume first that $Y \not\cong \mathbb{T}$. Then there exists a continuous monomorphism $p : \mathbb{R} \mapsto Y$ such that its image $p(\mathbb{R})$ is everywhere dense in Y ([7, (25.18)]). Let $\mu \in \Gamma_n(X)$. Consider the restriction of the characteristic function $\widehat{\mu}(y)$ to $p(\mathbb{R})$. Put $f(t) = \widehat{\mu}(p(t))$, $t \in \mathbb{R}$. By Lemma 1 the characteristic function $\widehat{\mu}(y)$ satisfies equation (3). Therefore $f(t)$ is a characteristic function on $\mathbb{R}$ satisfying (3) too. Taking into account that $\Gamma_n(\mathbb{R})$=$\Gamma(\mathbb{R})$, we have

$$f(t) = \exp\{-\sigma t^2 + i\beta t\}, \quad \sigma \geq 0, \ \beta \in \mathbb{R}.$$

Since p is a monomorphism and $\overline{p(\mathbb{R})} = Y$, for any neighborhood of zero V of the group Y we can choose a sequence of real numbers $t_n \to \infty$ such that $p(t_n) \in V$ for all n. If $\sigma > 0$, then

$$|f(t_n)| = |\widehat{\mu}(p(t_n))| = \exp\{-\sigma t_n^2\} \to 0$$

when $t_n \to \infty$. Taking into account that V is an arbitrary neighborhood of zero of the group Y we come to contradiction with the continuity at zero of the function $\widehat{\mu}(y)$. Hence, $\sigma = 0$, and this implies that $|\widehat{\mu}(p(t))| = 1$ for $t \in \mathbb{R}$. Since $\overline{p(\mathbb{R})} = Y$, we have $|\widehat{\mu}(y)| = 1$ for $y \in Y$. It follows from this that $\mu \in D(X)$.

If $Y \cong \mathbb{T}$, then $X \cong \mathbb{Z}$. Since $\mathbb{Z}$ is a subgroup of $\mathbb{R}$, we have $\Gamma_n(\mathbb{Z}) \subset \Gamma_n(\mathbb{R}) = \Gamma(\mathbb{R})$. Taking into account that any Gaussian measure on $\mathbb{R}$ supported in $\mathbb{Z}$ is degenerated we completely prove the lemma. □

Corollary 1. *Let Y be a connected compact group, $f(y)$ be a characteristic function on Y satisfying equation* (3)*. Then $|f(y)| = 1$, $y \in Y$.*

Lemma 9. *Let $\mu \in \Gamma_n(X)$. Then μ is supported in a coset of a subgroup $G \cong R^m \times K$, where $m \geq 0$, and K is a compact subgroup of X such that $K^{(n)} = K$.*

Proof. Note first that if $\lambda \in \Gamma_n(X)$, then for any $x \in X$ we have $\lambda * E_x \in \Gamma_n(X)$. Therefore if it is necessary we can substitute a distribution $\lambda \in \Gamma_n(X)$ by its shift, and the shift also belongs to the class $\Gamma_n(X)$.

Let $\lambda \in \mathrm{M}^1(X)$ be an arbitrary distribution. Consider the set

$$E = \{y \in Y : |\widehat{\lambda}(y)| = 1\}.$$

Then E is a subgroup of Y, and there exists an element $x \in X$ such that $\widehat{\lambda}(y) = (x, y)$ for $y \in E$. It follows from what has been said that we can substitute the distribution μ by its shift and assume from the beginning that

$$E = \{y \in Y : |\widehat{\mu}(y)| = 1\} = \{y \in Y : \widehat{\mu}(y) = 1\}. \tag{24}$$

It follows from (24) that $\sigma(\mu) \subset G$, where $G = A(X, E)$. Put $H = G^*$. The distribution μ considering as a distribution on G has the property

$$\{y \in H : |\widehat{\mu}(y)| = 1\} = \{0\}. \tag{25}$$

We will check that G is the desired subgroup. It follows from the structure theorem for locally compact Abelian group that $H \cong R^m \times L$, where $m \geq 0$, and L contains a compact open subgroup [7, (24.30)]). Consider the restriction of equation (3) to the connected component of zero c_L of the group L. Since the group c_L is compact, by Corollary 1 $|\widehat{\mu}(y)| = 1$ for $y \in c_L$. Taking into account (25) this implies that $c_L = \{0\}$, i.e. the group L is totally disconnected. We will prove that the group L is discrete.

By Lemma 1 the characteristic function $\widehat{\mu}(y)$ satisfies equation (3). Substitute $v_1 = u$, $v_2 = \cdots = v_n = 0$ into (3). From the resulting equation we obtain

$$|\widehat{\mu}(nu)| = |\widehat{\mu}(u)|^{2n-1} |\widehat{\mu}((n-1)u)|, \quad u \in H.$$

This implies the inequality

$$|\widehat{\mu}(nu)| \leq |\widehat{\mu}(u)|^{2n-1}, \quad u \in H. \tag{26}$$

It follows from (26) that for any natural p the inequality

$$|\widehat{\mu}(n^p u)| \leq |\widehat{\mu}(u)|^{(2n-1)^p}, \quad u \in H \tag{27}$$

holds. Since $\widehat{\mu}(0) = 1$ and the function $\widehat{\mu}(y)$ is continuous, there exists a neighborhood of zero V of the group L such that $|\widehat{\mu}(y)| > 0$ for $y \in V$. Inasmuch as the group L is totally disconnected, for any neighborhood of zero of L, in particular for V, there exists a compact subgroup W such that $W \subset V$ ([7, (7.7)]). Thus, we have

$$|\widehat{\mu}(y)| > 0, \quad y \in W. \tag{28}$$

Assume that at a point $y_0 \in W$ the inequality

$$|\widehat{\mu}(y_0)| < 1 \tag{29}$$

holds. Since the group W is compact, the sequence $n^p y_0$, $p = 1, 2, \dots$ contains a converging subsequence $n^{p_j} y_0 \to \widetilde{y}$, $\widetilde{y} \in W$. It follows from (27) and (29) that $\widehat{\mu}(\widetilde{y}) = 0$, that contradicts (28). Hence, $|\widehat{\mu}(y)| = 1$ for $y \in W$. Taking into account (25), we get that $W = \{0\}$, i.e. the group L is discrete. Put $K = L^*$. Since L is discrete, this implies that K is compact.

Take $y_0 \in L_{(n)}$, i.e. $ny_0 = 0$. It follows from (26) that $|\widehat{\mu}(y_0)| = 1$. Taking into account (25), we obtain that $y_0 = 0$. Hence, $L_{(n)} = \{0\}$. This implies that $K^{(n)} = K$. □

Lemma 10. *Let $X = R^m \times K$, where $m \geq 0$ and K is a compact group such that $K^{(n)} = K$. Then $X_{(n)} \subset c_X$.*

Proof. We have $A(Y, Y^{(n)}) = X_{(n)}$, $A(Y, b_Y) = c_X$ ([7, (24.17)]). The lemma will be proved if we check that $Y^{(n)} \supset b_Y$. Put $L = K^*$. Obviously, b_Y is a discrete torsion group. Since $K^{(n)} = K$, we have $L_{(n)} = \{0\}$. Take $y_0 \in b_Y$. Let M be a finite cyclic subgroup generated by y_0. It follows from $L_{(n)} = \{0\}$ that the restriction of the mapping $y \mapsto ny$, $y \in Y$, to M is a monomorphism. Taking into account that the group M is finite, this mapping is an isomorphism. Hence, $y_0 \in Y^{(n)}$. □

Lemma 11. *Let X be a locally compact Abelian group. Then the following statements are equivalent:*

(i) *for any compact subgroup K of the group X, satisfying the condition $K^{(n)}=K$, the equality $(K^*)^n = K^*$ holds;*

(ii) *the connected component of zero c_X of the group X has the property* (18)*;*

(iii) *for any compact subgroup K of the group X, satisfying the condition $K^{(n)}=K$, the factor group X/K contains no subgroup topologically isomorphic to $\mathbb{T}$.*

Proof. The equivalence of (i) and (ii) was proved in [3] (see also [4, Lemma 11.15]). To prove the equivalence of (i) and (iii) note first that $K^{(n)} = K$ if and only if $K^{(p)} = K$ for any prime divisor p of the number n. Hence it suffices to prove the equivalence of (i) and (iii) assuming that n is a prime number.

Let us prove (iii) $\Rightarrow$ (i). Let n be a prime number and K be a compact subgroup of X such that $K^{(n)} = K$. Put $L = K^*$. Then L is a discrete group satisfying the condition $L_{(n)} = \{0\}$. It follows from this that if y_0 is an element of finite order in L, then $y_0 \in L^{(n)}$. We will check that any element y_0 of infinite order in L also belongs to $L^{(n)}$. Thus, the implication (iii) $\Rightarrow$ (i) will be proved.

Let y_0 be an element of infinite order in L such that $y_0 \notin L^{(n)}$. Consider the factor-group $L/L^{(n)}$. Since n is a prime number, all nonzero elements of the factor-group $L/L^{(n)}$ have order n. It is obvious that $[y_0] \neq 0$. Therefore $[y_0]$ has order n and hence, $ky_0 \notin L^{(n)}$, $k = 1, 2, \dots, n-1$. Denote by M the subgroup of L generated by y_0, i.e. $M = \{y \in L : y = ly_0,\ l \in \mathbb{Z}\}$. Let $h \in L$ and $nh \in M$. Then $nh = ly_0$, $l \in \mathbb{Z}$. We have $l = qn+k$, where $q \in \mathbb{Z}$, $k \in \{0, 1, \dots, n-1\}$. This implies that $ky_0 = n(h - qy_0) \in L^{(n)}$. Therefore $k = 0$, and hence $nh = qny_0$. Since $L_{(n)} = \{0\}$, we get $h = qy_0 \in M$. Thus, the subgroup M has the property: if $nh \in M$, then $h \in M$. By Lemma 6 it follows from this that for the annihilator $G = A(K, M)$ the equality $G^{(n)} = G$ holds. Note now that $(K/G)^* \cong M \cong \mathbb{Z}$. This implies that $K/G \cong \mathbb{T}$. Since K/G is a subgroup of X/G, this contradicts to (iii).

The implication (i) $\Rightarrow$ (iii) for $n = 2$ was proved in [5, Lemma 7.7]. The proof in the general case is the same as in the case when $n = 2$. □

Proof of Theorem 1. Let $n = 2$. By Lemma 5 $\Gamma_2(X) = \Gamma_B(X)$ and the statement of the theorem in this case was proved in [2] (see also [4, Theorem 9.10]).

Assume that $n \geq 3$ and let us prove the sufficiency. Let $\mu \in \Gamma_n(X)$. By Lemma 1 the characteristic function $\widehat{\mu}(y)$ satisfies equation (3). Substituting $v_1 = v,\ v_2 = -v, v_3 = \dots = v_n = 0$ in (3), we get

$$\widehat{\mu}(u + nv)\widehat{\mu}(u - nv)\widehat{\mu}^{n-2}(u) = \widehat{\mu}^n(u)|\widehat{\mu}(nv)|^2, \quad u, v \in Y. \tag{30}$$

By Lemma 9 we can assume that the group X is of the form $X = R^m \times K$, where $m \geq 0$, and K is a compact group such that $K^{(n)} = K$. Applying Lemma 10 we obtain $X_{(n)} \subset c_X$. Then (18) implies that $X_{(n)} = \{0\}$, and hence $A(Y, X_{(n)}) = \overline{Y^{(n)}} = Y^{(n)} = Y$. Taking this into account, (30) implies that the characteristic function $\widehat{\mu}(y)$ satisfies equation

$$\widehat{\mu}(u + v)\widehat{\mu}(u - v)\widehat{\mu}^{n-2}(u) = \widehat{\mu}^n(u)|\widehat{\mu}(v)|^2, \quad u, v \in Y. \tag{31}$$

It follows from (31) that the set

$$B = \{y \in Y : \widehat{\mu}(y) \neq 0\}$$

is an open subgroup of Y. Put $K = A(X, B)$. Since B is an open subgroup, the group K is compact. Inasmuch as the characteristic function $\widehat{\mu}(y)$ satisfies equation (3), the function $\widehat{\mu}(y)$ satisfies inequality (26), and it follows from (26) that the group B has the property: if $ny \in B$, then $y \in B$. Applying Lemma 6 we get $K^{(n)} = K$. By Lemma 11 the factor-group X/K contains no subgroup topologically isomorphic to $\mathbb{T}$. Note now that $(X/K)^* \cong B$ and consider the restriction of the characteristic function $\widehat{\mu}(y)$ to B. By Lemma 1 this restriction is the characteristic function of a distribution from $\Gamma_n(X/K)$. Since the factor-group X/K contains no subgroup topologically isomorphic to $\mathbb{T}$, we can apply Proposition 1 to the factor-group X/K. As a result we obtain the following representation for the characteristic function $\widehat{\mu}(y)$

$$\widehat{\mu}(y) = \begin{cases} (x, y)\exp\{-\varphi(y)\} & \text{if } y \in B \\ 0 & \text{if } y \notin B. \end{cases} \tag{32}$$

where $x \in X$, and $\varphi(y)$ is a continuous function on B, satisfying equation (2). It is well known that the function $\varphi(y)$ can be extended from the subgroup B to Y retaining its properties ([9, Lemma 5.2.5]). Denote by $\widetilde{\varphi}(y)$ the extended function. Let γ be a Gaussian measure on X with the characteristic function

$$\widehat{\gamma}(y) = (x, y)\exp\{-\widetilde{\varphi}(y)\}, \quad y \in Y. \tag{33}$$

Obviously, that $\widehat{\mu}(y) = \widehat{\gamma}(y)\widehat{m}_K(y)$. Hence, $\mu = \gamma * m_K$.

Let us prove the necessity. Assume that (18) is not fulfilled. By Lemma 11 there exists a compact subgroup K of the group X such that $K^{(n)} = K$ and the factor-group X/K contains a subgroup topologically isomorphic to $\mathbb{T}$. Therefore the distribution μ on the group $\mathbb{T}$, constructed in Lemma 2, can be considered as a distribution on the factor-group X/K. We retain the notation μ for this distribution. Since $(X/K)^* \cong A(Y, K)$, we may assume that the characteristic function $\widehat{\mu}(y)$ is defined on $A(Y, K)$. Consider on the group Y the function

$$h(y) = \begin{cases} \widehat{\mu}(y) & \text{if } y \in A(Y, K) \\ 0 & \text{if } y \notin A(Y, K). \end{cases}$$

Since the set $A(Y, K)$ is a subgroup and $\widehat{\mu}(y)$ is a positive definite function, $h(y)$ is also a positive definite function ([8, (32.43)]). Since K is a compact group, its annihilator $A(Y, K)$ is an open subgroup, and hence the function $h(y)$ is continuous. By Bochner's theorem there exists a distribution $\lambda \in \mathrm{M}^1(X)$ such that $\widehat{\lambda}(y) = h(y)$. We will check that $\lambda \in \Gamma_n(X)$. By Lemma 1 it suffices to verify that the function $h(y)$ satisfies equation (3).

Take $u \in A(Y, K)$. If $nv_{j_0} - (v_1 + \cdots + v_n) \notin A(Y, K)$ holds at least for one $j = j_0$, then $u + nv_{j_0} - (v_1 + \cdots + v_n) \notin A(Y, K)$, and both sides of equation (3) are equal to zero. Assume that $nv_j - (v_1 + \cdots + v_n) \in A(Y, K)$ for all $j = 1, 2, \ldots, n$. This implies that $n(v_j - v_1) \in A(Y, K)$, and hence by Lemma 6 $v_j - v_1 = h_j \in A(Y, K)$, $j = 1, 2, \ldots, n$. Substituting $v_j = v_1 + h_j$, $j = 1, 2, \ldots, n$ in equation (3) and taking into account that the function $\widehat{\mu}(y)$ satisfies equation (3) on $A(Y, K)$, we get the equality.

Take $u \notin A(Y,K)$. Then the right-hand side of equation (3) is equal to zero. If in this case the left-hand side of equation (3) does not vanish, then the inclusions $u + nv_j - (v_1 + \cdots + v_n) \in A(Y,K)$, $j = 1,2,\dots,n$ are fulfilled. This implies that $\sum_{j=1}^{n}(u + nv_j - (v_1 + \cdots + v_n)) = nu \in A(Y,K)$, and hence by Lemma 6 $u \in A(Y,K)$, contrary to the choice of u. So we proved that the function $h(y)$ satisfies equation (3). Since $\mu \notin \Gamma(X/K)$, obviously that $\lambda \notin \Gamma(X) * I(X)$. The theorem is completely proved. □

Remark 2. Let $\mu \in \Gamma_n(\mathbb{R})$, $n \geq 2$. Put $\gamma = \mu * \bar{\mu} \in \Gamma_n(\mathbb{R})$. Then $\widehat{\gamma}(y) = |\widehat{\mu}(y)|^2 \geq 0$. By Lemma 1 the characteristic function $\widehat{\gamma}(y)$ satisfies equation (3), and hence satisfies equation (30) too. Taking into account that $\mathbb{R}^{(n)} = \mathbb{R}$, (30) implies (31), and hence the set $B = \{y \in \mathbb{R} : \widehat{\gamma}(y) \neq 0\}$ is an open subgroup of $\mathbb{R}$. So, $B = \mathbb{R}$. Since $\widehat{\gamma}(y) > 0$, $y \in \mathbb{R}$, by Lemma 3 $\gamma \in \Gamma(\mathbb{R})$. This implies by Cramer's theorem that $\mu \in \Gamma(\mathbb{R})$. Thus, we proved the equality $\Gamma(\mathbb{R}) = \Gamma_n(\mathbb{R})$, $n \geq 2$, which we used in the proof of Lemma 8, and this proof is independent from Geary's theorem.

Remark 3. Comparing Proposition 1 and Theorem 1 we see that in both statements we have a particular case $n = 2$. If $n \geq 3$, then the description of the corresponding class of groups in Proposition 1 does not depend on n, and does not depend on n in Theorem 1.

Remark 4. Observe that if μ is an infinitely divisible distribution and $\mu \in \Gamma_n(X)$, then $\mu \in \Gamma(X) * I_n(X)$. Indeed, since μ is an infinitely divisible distribution, the set $B = \{y \in Y : \widehat{\mu}(y) \neq 0\}$ is an open subgroup of Y ([12, p. 106]). Put $\nu = \mu * \bar{\mu}$ and consider the restriction of the characteristic function $\widehat{\nu}(y)$ to B. Since $\widehat{\nu}(y) = |\widehat{\mu}(y)|^2 > 0$ for $y \in B$, by Lemma 3 $\widehat{\nu}(y) = \exp\{-\varphi(y)\}$, $y \in B$. Note now that on an arbitrary locally compact Abelian group in the class of infinitely divisible distributions a Gaussian measure has only Gaussian factors ([5, Remark 4.8]). Thus, for the characteristic function $\widehat{\mu}(y)$ we get the representation similar to (32), and the desired statement follows from this.

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