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Weak compactness in the space of operator valued measures

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Abstract. In this short paper we present a characterization of weakly compact sets in the space of finitely additive nuclear operator valued measures. The paper is concluded with a physical example involving feedback control of stochastic systems on Hilbert space perturbed by centered Poisson counting measures. Both existence of optimal policies and necessary conditions of optimality are presented.

1. Introduction

The question of compactness of subsets of the space of vector measures is very important in control theory and many other applications involving optimization. This has been studied extensively and the most celebrated results are those of Bartle–Dunford–Schwartz [1, Theorem 5, p. 105] and Brooks–Dinculeanu [1, Corollary 6, p. 106]. See also [3], [8] for more on weak compactness in the space of vector measures. In physical sciences and engineering, there are many problems found in the study of optimal control and optimization where one requires compactness (weak/strong) of a set of operator valued measures. The author is not aware of any result on compactness involving operator valued measures. This was the main motivation for this study. The rest of the paper is organized as follows: Some notations are introduced in Section 2. Main results are presented

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in Section 3. An application of these results to control theory is presented in Section 4. Both existence of optimal policies and necessary conditions of optimality are presented illustrating the usefulness of the compactness result.

2. Some notations and a basic result

Let $\{X, Y\}$ denote a pair of real separable Hilbert spaces with a corresponding pair of complete orthonormal basis denoted by $\{x_i, y_i\}$, $i \in N$. Let $\mathcal{L}(Y, X)$ denote the space of bounded linear operators from Y to X. Furnished with the uniform operator topology, this is a Banach space. Let $\mathcal{L}_1(X, Y)$ denote the space of nuclear operators from X to Y. This is furnished with the norm topology given by

$$\|L\|_1 \equiv \sum_{i=1}^{\infty} |(Lx_i, y_i)|,$$

whenever it is defined. It is easy to verify that this norm is independent of the choice of the orthonormal basis. With respect to this norm topology, $\mathcal{L}_1(X, Y)$ is a Banach space. In case $X = Y$ we use the notation $\mathcal{L}(X)$ for $\mathcal{L}(X, X)$, $\mathcal{L}_1(X)$ for $\mathcal{L}_1(X, X)$ and $\mathcal{L}_s^+(X)$ for bounded positive selfadjoint operators in X. Now we introduce the function spaces. Let I be a finite interval of the real line with $\Sigma \equiv \sigma(I)$ denoting the sigma algebra of subsets of the set I. Let $B_\infty(I, \mathcal{L}(Y, X))$ denote the vector space of all Σ measurable functions defined on I and taking values from the Banach space $\mathcal{L}(Y, X)$. In other words these functions are measurable with respect to the uniform operator (or norm) topology on $\mathcal{L}(Y, X)$. We furnish this space with the norm topology

$$\|B\|_\infty \equiv \sup\{\|B(t)\|_{\mathcal{L}(Y,X)},\ t \in I\}.$$

This is equivalent to the topology of uniform convergence on I in the uniform operator topology of $\mathcal{L}(Y, X)$. Let $M_{ba}(\Sigma)$ denote the class of (real) bounded finitely additive signed measures on I. It is well known [1], [2] that, furnished with the total variation norm, this is a Banach space.

For each $\mu \in M_{ba}(\Sigma)$, we let $|\mu|(\cdot) \in M_{ba}^+(\Sigma)$ denote the positive finitely additive measure induced by the variation of the measure μ.

Necessary and sufficient conditions for conditional weak compactness of subsets of $M_{ba}(\Sigma)$ are well known [3]. For convenience of reference we present it here.

Lemma 2.1. *A set $\Xi \subset M_{ba}(\Sigma)$ is conditionally weakly compact if and only if* (1)*: Ξ is bounded and* (2)*: the set $\{|\mu|,\ \mu \in \Xi\}$ is uniformly additive.*

In case Ξ is weakly compact, condition (2) is equivalent to (2'): there exists a $\nu \in M_{ba}^{+}(\Sigma)$ such that the set $\{|\mu|, \mu \in \Xi\}$ is uniformly absolutely continuous with respect to ν.

This result is a scalar version of the more general result due to BROOKS [3], [8] that holds for vector measures $M_{ba}(\Sigma, E)$ with E being a reflexive Banach space. For more on this topic and vector measures, the reader is referred to the excellent book of DIESTEL and UHL JR. [1].

We are interested in the class of finitely additive $\mathcal{L}_1(X,Y)$ valued vector measures defined on I. We denote this class by $M_{ba}(\Sigma, \mathcal{L}_1(X,Y))$, and furnish this with the norm topology given by

$$\|K\|_{M_{ba}(\Sigma,\mathcal{L}_1(X,Y))} \equiv \sup_{\pi} \sum_{\sigma\in\pi} \|K(\sigma)\|_1,$$

where the supremum is taken over all finite Σ measurable disjoint partitions of the interval I. With respect to this norm topology, $M_{ba}(\Sigma, \mathcal{L}_1(X,Y))$ is also a Banach space.

3. Weak compactness in $M_{ba}(\Sigma, \mathcal{L}_1(X,Y))$

Here we are interested in finding sufficient conditions for weak compactness of subsets of the space of operator valued measures $M_{ba}(\Sigma, \mathcal{L}_1(X,Y))$. Let Z be any real Banach space and $\ell_1(Z)$ denote the space of (infinite) sequences with values in Z. Furnished with the norm topology given by

$$\|z\|_{\ell_1(Z)} = \sum_{i=1}^{\infty} \|z_i\|_Z,$$

it is easy to see that $\ell_1(Z)$ is a Banach space.

Lemma 3.1. *For any pair of separable Hilbert spaces $\{X,Y\}$, the space $M_{ba}(\Sigma, \mathcal{L}_1(X,Y))$ is isometrically isomorphic to the space $\ell_1(M_{ba}(\Sigma))$ which we denote by*

$$M_{ba}(\Sigma, \mathcal{L}_1(X,Y)) \cong \ell_1(M_{ba}(\Sigma)). \tag{1}$$

PROOF. Let $\{x_i, y_i\}$ be any pair of orthonormal basis of the pair of Hilbert spaces $\{X,Y\}$ respectively. Given $\lambda \in \ell_1(M_{ba}(\Sigma))$, define the operator valued measure K by

$$K(\cdot) \equiv \sum_{i=1}^{\infty} \lambda_i(\cdot)(y_i \otimes x_i), \tag{2}$$

with $K(\sigma)x = \sum_{i=1}^{\infty} \lambda_i(\sigma)(x_i, x)y_i$ for every $x \in X$ and $\sigma \in \Sigma$. Clearly, it follows from this expression that, for each $i \in N$, $(K(\cdot)x_i, y_i)_Y = \lambda_i(\cdot)$. Hence,

$$\|\lambda\|_{\ell_1(M_{ba}(\Sigma))} = \sum_{i=1}^{\infty} \|\lambda_i\|_{M_{ba}(\Sigma)} = \sum_{i=1}^{\infty} \Big(\sup_{\pi} \sum_{\sigma \in \pi} |\lambda_i(\sigma)| \Big).$$

By definition of nuclear norm, we have

$$\begin{aligned} \sum_{i=1}^{\infty} \Big(\sup_{\pi} \sum_{\sigma \in \pi} |\lambda_i(\sigma)| \Big) &= \sum_{i=1}^{\infty} \Big(\sup_{\pi} \sum_{\sigma \in \pi} |(K(\sigma)x_i, y_i)| \Big) \\ &\geq \sup_{\pi} \sum_{\sigma \in \pi} \|K(\sigma)\|_{\mathcal{L}_1(X,Y)} \equiv \sup_{\pi} \sum_{\sigma \in \pi} \|K(\sigma)\|_1 = \|K\|_{M_{ba}(\Sigma, \mathcal{L}_1(X,Y))}, \end{aligned}$$

where the supremum is taken over all finite, mutually disjoint, Σ-measurable partitions of the interval I. Thus it follows from the above inequalities, that every $\lambda \in \ell_1(M_{ba}(\Sigma))$ determines a $K \in M_{ba}(\Sigma, \mathcal{L}_1(X,Y))$ through the expression (2). Conversely, given $K \in M_{ba}(\Sigma, \mathcal{L}_1(X,Y))$, we have $K(\sigma) \in \mathcal{L}_1(X,Y)$ for all $\sigma \in \Sigma$. Hence there exists a sequence of scalars $\{\mu_i\}$ dependent on $\sigma \in \Sigma$, such that $\sum_{i=1}^{\infty} |\mu_i(\sigma)| < \infty$ for every $\sigma \in \Sigma$ and that

$$K(\sigma) = \sum_{i=1}^{\infty} \mu_i(\sigma)(y_i \otimes x_i), \quad \text{for } \sigma \in \Sigma. \tag{3}$$

Thus it follows from the definition of nuclear norm that

$$\sup_{\pi} \sum_{\sigma \in \pi} \Big(\sum_{i=1}^{\infty} |\mu_i(\sigma)| \Big) = \sup_{\pi} \sum_{\sigma \in \pi} \|K(\sigma)\|_{\mathcal{L}_1(X,Y)} \equiv \|K\|_{M_{ba}(\Sigma, \mathcal{L}_1(X,Y))} < \infty, \tag{4}$$

where π denotes any finite disjoint Σ-measurable partition of the interval I. Let $\mathcal{F}(\Sigma)$ denote the class of all finite disjoint Σ-measurable partitions of the interval I. Clearly, it follows from the above inequality that for each $\pi \in \mathcal{F}(\Sigma)$ and every $n \in N$, we have,

$$\begin{aligned} \sum_{i=1}^{n} \Big(\sum_{\sigma \in \pi} |\mu_i(\sigma)| \Big) &= \sum_{\sigma \in \pi} \Big(\sum_{i=1}^{n} |\mu_i(\sigma)| \Big) \leq \sup_{\pi} \sum_{\sigma \in \pi} \Big(\sum_{i=1}^{\infty} |\mu_i(\sigma)| \Big) \\ &= \|K\|_{M_{ba}(\Sigma, \mathcal{L}_1(X,Y))} < \infty. \end{aligned}$$

Now taking the supremum with respect to $\pi \in \mathcal{F}(\Sigma)$, it follows from this and the inequality (4) that, for every $n \in N$,

$$\sum_{i=1}^{n} \|\mu_i\|_{M_{ba}(\Sigma)} \leq \|K\|_{M_{ba}(\Sigma, \mathcal{L}_1(X,Y))} < \infty.$$

Hence we conclude that for each $i \in N$, $\mu_i \in M_{ba}(\Sigma)$ and that

$$\|\mu\|_{\ell_1(M_{ba}(\Sigma))} \leq \|K\|_{M_{ba}(\Sigma,\mathcal{L}_1(X,Y))} < \infty,$$

and consequently μ, given by $\mu \equiv \{\mu_i\}_{i=1}^{\infty}$ determining K as in (3), belongs to $\ell_1(M_{ba}(\Sigma))$. Thus we have proved that for every $K \in M_{ba}(\Sigma, \mathcal{L}_1(X,Y))$ there exists a $\mu \in \ell_1(M_{ba}(\Sigma))$ such that K has the representation given by (3). Combining the above results we conclude that $M_{ba}(\Sigma, \mathcal{L}_1(X,Y))$ is isometrically isomorphic to $\ell_1(M_{ba}(\Sigma))$ proving (2). □

An interesting problem in the theory of vector measures is the characterization of its weakly compact sets [1, Theorem IV.5; Corollary IV.6, p. 105]. This has natural applications in optimization and optimal controls. Here, we are interested in the characterization of weakly compact sets in $M_{ba}(\Sigma, \mathcal{L}_1(X,Y))$. For convenience of notation we let P_i denote the projection (coordinate map) taking $\ell_1(M_{ba}(\Sigma))$ to its i-th coordinate, that is, for $\lambda \in \ell_1(M_{ba}(\Sigma))$, $P_i(\lambda) = \lambda_i$.

Theorem 3.2. *A set $\Gamma \subset M_{ba}(\Sigma, \mathcal{L}_1(X,Y))$ is relatively (or conditionally) weakly compact if, and only if, the following conditions are satisfied*

(c1): *] Γ is bounded,*

(c2): *for all $\sigma \in \Sigma$, the sum $\sum_{i=1}^{\infty} |(K(\sigma)x_i, y_i)_Y|$ is convergent uniformly in $K \in \Gamma$,*

(c3): *for each $i \in N$, the set of scalar valued measures $\{(K(\cdot)x_i, y_i), K \in \Gamma\}$ is a conditionally weakly compact subset of $M_{ba}(\Sigma)$.*

Proof. First we prove that these are necessary conditions. Suppose Γ is a relatively weakly compact subset of $M_{ba}(\Sigma, \mathcal{L}_1(X,Y))$. Since, in a Banach space, a weakly conditionally compact set is always bounded, Γ is bounded justifying (c1). By Lemma 3.1, $M_{ba}(\Sigma, \mathcal{L}_1(X,Y)) \cong \ell_1(M_{ba}(\Sigma))$. Let the map

$$\Psi : M_{ba}(\Sigma, \mathcal{L}_1(X,Y)) \longrightarrow \ell_1(M_{ba}(\Sigma))$$

denote the isometric isomorphism. Since compactness is preserved under isomorphism and Γ is relatively weakly compact, we have $\Psi(\Gamma) \equiv \Lambda$ a weakly relatively compact subset of $\ell_1(M_{ba}(\Sigma))$. Hence the sum $\sum_{i=1}^{\infty} |\lambda_i(\sigma)|$ must be convergent uniformly with respect to $\lambda \in \Lambda$. But, under the isomorphism, this is equivalent to the statement that the sum $\sum_{i=1}^{\infty} |(K(\sigma)x_i, y_i)|$ be convergent uniformly with respect to $K \in \Gamma$. Thus (c2) is necessary. Finally, since under the isomorphism, the set Λ is a relatively weakly compact subset of $\ell_1(M_{ba}(\sigma))$, every i-th section $\Lambda_i \equiv P_i(\Lambda)$ must be a relatively weakly compact subset of $M_{ba}(\Sigma)$. Hence the set

$$\{(K(\cdot)x_i, y_i),\ K \in \Gamma\} = \{\lambda_i \in M_{ba}(\Sigma) : \lambda \in \Lambda\} \equiv \Lambda_i$$

is a weakly relatively compact subset of $M_{ba}(\Sigma)$ justifying the necessity of condition (c3).

Next, we prove that these conditions are sufficient. We show that these conditions are equivalent to the conditions that guarantee relative weak compactness of the set Λ. Sufficient conditions for relative weak compactness for a set $\Lambda \subset \ell_1(M_{ba}(\Sigma))$ are:

(s1): Λ is bounded,

(s2): $\sum_{i=1}^{\infty} \|\lambda_i\|_{M_{ba}(\Sigma)}$ and consequently, $\sum_{i=1}^{\infty} |\lambda_i(\sigma)|$ for each $\sigma \in \Sigma$, is convergent uniformly with respect to $\lambda \in \Lambda$, and

(s3): for each $i \in N$, $P_i(\Lambda)$ is a conditionally weakly compact subset of $M_{ba}(\Sigma)$.

Since Γ is a bounded subset of $M_{ba}(\Sigma, \mathcal{L}_1(X,Y))$ and Ψ is continuous linear, it is clear that $\Lambda \equiv \Psi(\Gamma)$ is a bounded subset of $\ell_1(M_{ba}(\Sigma))$ proving (s1). For each $K \in \Gamma$, we have $\Psi(K) = \lambda \in \Lambda$. Since (c2) holds for Γ, the series

$$\sum_{i=1}^{\infty} |\lambda_i(\sigma)| = \sum_{i=1}^{\infty} |(K(\sigma)x_i, y_i)|$$

is convergent uniformly with respect to $\lambda \in \Lambda$ proving (s2). By virtue of condition (c3), for each $i \in N$, the set

$$\{(K(\cdot)x_i, y_i),\ K \in \Gamma\} = \{\lambda_i, \lambda \in \Lambda\} = P_i(\Lambda)$$

is a relatively weakly compact subset of $M_{ba}(\Sigma)$ verifying (s3). Thus Λ is a relatively weakly compact subset of $\ell_1(M_{ba}(\Sigma))$ and hence $\Gamma = \Psi^{-1}(\Lambda)$ is a relatively weakly compact subset of $M_{ba}(\Sigma, \mathcal{L}_1(X,Y))$. This completes the proof. $\square$

Remark 1. An open problem of significant interest in the study of operator valued measures is the search for necessary and sufficient conditions for weak compactness of a set $\Gamma \subset M_{ba}(\Sigma, \mathcal{L}(X,Y))$.

Remark 2. Similar questions arise in the study of operator valued multimeasures. An interesting example is the system model given by the following differential inclusion,

$$dx \in Ax(t)dt + \mathcal{C}(dt)x(t-), x(0) = x_0,$$

where $\mathcal{C} : \Sigma \longrightarrow cbc(\mathcal{L}(X))$, the class of nonempty closed (in the weak operator topology) bounded convex subsets of the space of bounded linear operators in X. Here, one is interested in the relative weak compactness of the measure selections $\mathcal{S}_{\mathcal{C}}$ of the operator valued multimeasure $\mathcal{C}$. This has interesting application in the study of systems determined by operator valued measures [5], [6]. For multimeasures see the excellent book due to Hu and Papageorgiou [4, p. 850].

4. An application

Consider the control system

$$dx = Ax(t)dt + u(dt) + \int_{Z_0} B(t)zq(dt \times dz), x(0) = x_0, \quad t \in I, \tag{5}$$

where A is the infinitesimal generator of a C_0-semigroup $\{S(t),\ t \geq 0\}$ on X and $u \in M_{ba}(\Sigma, X)$ is the control. The operator valued function B is a bounded Borel measurable function defined on I and taking values $B(t) \in \mathcal{L}(Z, X)$ where Z is another Hilbert space with $Z_0 \equiv Z \setminus \{0\}$. Let $\mathcal{B}(Z_0)$ denote the sigma algebra of Borel subsets of the set Z_0; and $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0} \subset \mathcal{F}, P)$ a complete filtered probability space with a nondecreasing current of subsigma algebras $\mathcal{F}_t$, $t \geq 0$, which are right continuous having left limits. The process $\{q\}$, defined on this probability space and taking values $q(J \times \Delta)$, $J \in \Sigma$, $\Delta \in \mathcal{B}(Z_0)$, is a centered random Poisson measure (counting measure) with mean $E\{q(J \times \Delta)\} = 0$ and variance $E\{q(J \times \Delta)\}^2 = \lambda(J)m(\Delta)$, where λ is the Lebesgue measure and m is a bounded countably additive positive Borel measure on $\mathcal{B}(Z_0)$ known as the Levy measure. For any $\Delta \in \mathcal{B}(Z_0)$, $m(\Delta)$ gives the expected (average) number of jumps per unit time hitting the set Δ. To understand the impact of the last term of equation (5) on the process $\{x\}$, let us consider the random process $\{\xi\}$ given by

$$d\xi(t) = \int_{Z_0} B(t)zq(dt \times dz), \xi(0) = 0, \quad t \geq 0.$$

This is a pure jump process with values in X. For example, if there is one and only one jump during the interval $[t_1, t_2)$ and it has size $z^* \in Z_0$ and occurs at time $t^* \in [t_1, t_2)$, and $B(t^*)$ is uniquely defined, we have $\xi(t) = \xi(t_1-)$ for all $t \in [t_1, t^*)$ and $\xi(t) = \xi(t_1-) + B(t^*)z^*$, for $t \in [t^*, t_2)$. The process $\{\xi\}$ is an $\mathcal{F}_t$ martingale and it evolves purely by jumps. It is easy to verify that

$$E\{\xi(t)\} = 0 \quad \text{and} \quad E\{(\xi(t), e)_X^2\} = \int_0^t \int_{Z_0} (z, B^*(s)e)^2 m(dz)ds,\ \forall\ e \in X.$$

For the uncontrolled system, $(u \equiv 0)$, its presence in equation (5) induces volatility (fluctuation) in the process $\{x\}$ of intensity which is dependent on the norm of B and the measure m through the covariance operator Q_m where

$$(Q_m \eta, \eta)_Z \equiv \int_{Z_0} (z, \eta)^2 m(dz), \quad \forall\, \eta \in Z.$$

We assume that $Q_m \in \mathcal{L}_s^+(Z)$, the space of positive selfadjoint bounded linear operators in Z. The objective is to choose a linear state feedback control law of the form

$$u(dt) \equiv K(dt)x(t-), \quad t \geq 0, \tag{6}$$

that minimizes this volatility. The feedback (closed loop) system is then given by

$$dx = Ax(t)dt + K(dt)x(t-) + \int_{Z_0} B(t)zq(dt \times dz),\ x(0) = x_0, \quad t \in I. \tag{7}$$

The volatility of this process at time t is given by the trace of its covariance operator $P(t) \in \mathcal{L}(X)$ where P is defined by

$$(P(t)\xi, \xi) \equiv E([x(t) - \bar{x}(t)], \xi)^2, \quad \forall\, \xi \in X,\ t \geq 0,$$

with $\bar{x}(t)$ denoting the mean of the process x. This is given by the mild solution of the deterministic evolution equation

$$d\bar{x}(t) = A\bar{x}(t)dt + K(dt)\bar{x}(t-),\ \bar{x}(0) = \bar{x}_0, \quad t \in I.$$

It can be shown that P satisfies the following differential equation on the Banach algebra $\mathcal{L}(X)$,

$$\begin{aligned} dP(t) &= (AP(t) + P(t)A^*)dt + (K(dt)P(t-) + P(t-)K^*(dt)) + \hat{Q}_m(t)dt, \quad t \in I, \\ P(0) &= P_0, \end{aligned} \tag{8}$$

where $\hat{Q}_m(t) \equiv B(t)Q_mB^*(t)$ with Q_m as defined above. We assume that, for each $t \geq 0$, $\hat{Q}_m(t) \in \mathcal{L}_1^+(X)$, the space of positive nuclear operators in X. Under this assumption, for each $t \geq 0$, the operator $P(t) \in \mathcal{L}_1^+(X)$ also. The cost functional, measured in terms of cumulative volatility of the process $\{x\}$, can be taken as the integral of the weighted trace of P given by

$$J(K) \equiv \int_I \mathrm{Tr}(G(t)P(t))dt, \tag{9}$$

where G is a positive self adjoint uniformly bounded operator valued function defined on I and P is the solution of equation (8) corresponding $K \in M_{ba}(\Sigma, \mathcal{L}_1(X))$. Regarding equation (8) as the dynamic system on $\mathcal{L}(X)$ and the expression (9) as the cost functional, and K as the control to be chosen, we have an optimal control problem with control constraint $K \in \Gamma \subset M_{ba}(\Sigma, \mathcal{L}_1(X))$. The problem is: find $K \in \Gamma$ that minimizes the functional (9) subject to the dynamic constraint (8). This is where weak compactness of the set Γ is used. We present the following results. Existence of optimal control law is given in Theorem 4.1 and the necessary conditions of optimality are presented in Theorem 4.2.

Theorem 4.1 (Existence). *Consider the system* (8) *with the objective functional given by* (9) *and suppose that A is the infinitesimal generator of a C_0-semigroup $S(t)$, $t \geq 0$, on X, $\hat{Q}_m \in L_1(I, \mathcal{L}_1^+(X))$, $P_0 \in \mathcal{L}_1^+(X)$, $G \in B_\infty(I, \mathcal{L}_s^+(X))$ and Γ is a weakly compact subset of $M_{ba}(\Sigma, \mathcal{L}_1(X))$. Then, there exists a $K_o \in \Gamma$ that minimizes the functional* (9).

PROOF. Since a weakly continuous functional attains its minimum on any weakly compact set, it suffices to verify that J is weakly continuous. By our assumption, Γ is weakly compact and hence weakly sequentially compact by Eberlein–Smulian theorem and therefore it suffices to verify that the functional (9) is weakly (sequentially) continuous. So we prove that $J(K_n) \longrightarrow J(K_o)$ whenever $K_n \xrightarrow{w} K_o$ in $M_{ba}(\Sigma, \mathcal{L}_1(X))$. First note that, for every $K \in M_{ba}(\Sigma, \mathcal{L}_1(X))$, the pair $\{A, K\}$ generates a unique strongly measurable bounded evolution operator $U_K(t,s)$, $0 \leq s \leq t < \infty$ on X solving the Cauchy problem,

$$dx = Axdt + K(dt)x(t-), x(s+0) = \xi \in X,$$

giving $x(t) \equiv U_K(t,s)\xi$ in the mild sense with $x \in B_\infty(I, X)$. Thus, for every such K, the mild solution of (8), denoted by P_K, is given by

$$P_K(t) = U_K(t,0)P_0U_K^*(t,0) + \int_0^t U_K(t,s)\hat{Q}_m(s)U_K^*(t,s)ds, t \in I.$$

Since P_0 is a positive (self adjoint) nuclear operator and $\hat{Q}_m(\cdot) \in L_1(I, \mathcal{L}_1^+(X))$, it follows from the above expression that $P_K(t)$ is a positive self adjoint bounded operator valued function on I. From this expression, it is easy to verify that

$$\mathrm{Tr}(P_K(t)) \leq M_K^2 \Big\{ \mathrm{Tr}(P_0) + \int_I \mathrm{Tr}(\hat{Q}_m(t))dt \Big\}, \tag{10}$$

where $M_K \equiv \sup\{\|U_K(t,s)\|_{\mathcal{L}(X)},\ 0 \leq s \leq t \leq T\} < \infty$. Hence $P_K \in B_\infty(I, \mathcal{L}_1^+(X))$. By assumption Γ is a bounded subset of $M_{ba}(\Sigma, \mathcal{L}_1(X))$, and hence, there exists a finite positive number $M \equiv \sup\{M_K, K \in \Gamma\}$ such that the set of mild solutions of equation (8) denoted by $\{P_K, K \in \Gamma\}$ is contained in a bounded subset of $B_\infty(I, \mathcal{L}_1^+(X))$. Now we prove the continuity. Let $K_n \in \Gamma$ and $K_n \xrightarrow{w} K_o$ in Γ and let P_n and P_o denote the corresponding mild solutions of equation (8) respectively for the same initial state $P(0) = P_0$. Then, using the semigroup $S(t)$, $t \in I$, and defining $R_n \equiv P_n - P_o$, it follows from straight forward computation involving equation (8) that

$$\begin{aligned}R_n(t) = &\int_0^t S(t-\theta)K_n(d\theta)R_n(\theta-)S^*(t-\theta)\\&+\int_0^t S(t-\theta)R_n(\theta-)K_n^*(d\theta)S^*(t-\theta)\\&+\int_0^t S(t-\theta)(K_n(d\theta)-K_o(d\theta))P_o(\theta-)S^*(t-\theta)\\&+\int_0^t S(t-\theta)P_o(\theta-)(K_n^*(d\theta)-K_o^*(d\theta))S^*(t-\theta).\end{aligned}\tag{11}$$

Recall that the norm of a bounded operator coincides with that of its adjoint. Hence the variation norm of a bounded operator valued measure coincides with that of its adjoint. Using this fact and taking the trace on either side of equation (11), we have

$$|\operatorname{Tr}(R_n(t))| \leq 2M_0^2\int_0^t |\operatorname{Tr}(R_n(\theta-)|\,\|K_n\|_v(d\theta)+|e_n(t)|\tag{12}$$

for all $t \in I$, where

$$\begin{aligned}e_n(t) \equiv &\int_0^t \operatorname{Tr}\big(S(t-\theta)\big(K_n(d\theta)-K_o(d\theta)\big)P_o(\theta-)S^*(t-\theta)\big)\\&+\int_0^t \operatorname{Tr}\big(S(t-\theta)P_o(\theta-)\big(K_n^*(d\theta)-K_o^*(d\theta)\big)S^*(t-\theta)\big),\end{aligned}\tag{13}$$

with $\|K_n\|_v(\cdot)$, denoting the measure induced by the variation of the operator valued measure K_n, and $M_0 \equiv \sup\{\|S(t)\|_{\mathcal{L}(X)},\ t \in I\}$. Since P_o is a nuclear operator valued function, and $S(t)$, $t \geq 0$, and $S^*(t)$, $t \geq 0$, are strongly continuous semigroups (because X is Hilbert), it is easy to verify that $e_n(t) \longrightarrow 0$ uniformly on the compact interval I. Define

$$\varphi_n(t) \equiv \sup\{|\operatorname{Tr}(R_n(s))|,\ 0 \leq s \leq t\},$$

for all $t \in I$. Then it follows from (12) that

$$\varphi_n(t) \leq \sup\{|e_n(s)|, s \in I\} + 2M_0^2\int_0^t \varphi_n(s)\|K_n\|_v(ds), \quad t \in I.\tag{14}$$

Hence, by virtue of generalized Gronwall inequality [7, Lemma 5, p. 268], it follows from this that

$$\varphi_n(t) \leq \sup\{|e_n(s)|,\ s \in I\}\exp\{2M_0^2\big(\|K_n\|_{M_{ba}(\Sigma,\mathcal{L}_1(X))}\big)\}\tag{15}$$

for all $t \in I$. Since the set Γ is bounded and $\{K_n\} \subset \Gamma$, there exists a finite positive number γ such that

$$\varphi_n(t) \leq \sup\{|e_n(s)|,\ s \in I\} \exp\{2M_0^2\gamma\} \quad \forall\, t \in I. \tag{16}$$

Recalling that e_n converges to zero uniformly on I, it follows from the above inequality that $\varphi_n(t) \to 0$ uniformly on I. Hence $\mathrm{Tr}(R_n(t)) \to 0$ uniformly on I implying that $P_n \xrightarrow{s} P_o$ in the topology of the Banach space $B_\infty(I, \mathcal{L}_1(X))$. Since $P_0 \in \mathcal{L}_1^+(X)$, $\hat{Q}_m \in L_1(I, \mathcal{L}_1^+(X))$ and Γ is bounded, it follows from (10) that

$$|\,\mathrm{Tr}(P_n(t))| \leq M^2\Big\{\mathrm{Tr}(P_0) + \int_I \mathrm{Tr}(\hat{Q}_m(t))dt\Big\} \leq C < \infty,$$

for all $t \in I$ and for all $n \in N$, where M is as defined following equation (10). By assumption, $G \in B_\infty(I, \mathcal{L}_s^+(X))$, and I is a finite interval, and hence $\mathrm{Tr}(GP_n) \in L_1(I)$. Thus, it follows from Lebesgue dominated convergence theorem and convergence of $P_n \xrightarrow{s} P_o$ in $B_\infty(I, \mathcal{L}_1(X))$ that

$$\int_I \mathrm{Tr}(GP_n)dt \longrightarrow \int_I \mathrm{Tr}(GP_o)dt$$

proving that $J(K_n) \longrightarrow J(K_o)$ whenever $K_n \xrightarrow{w} K_o$ in $M_{ba}(\Sigma, \mathcal{L}_1(X))$. This proves weak continuity. Since Γ is weakly compact, J attains its minimum on Γ proving existence. $\square$

Given the existence of an optimal policy, the next question is how we determine it. This is done by use of necessary conditions of optimality. These are conditions that characterize the optimal policy.

Theorem 4.2 (Necessary Conditions)**.** *Suppose the assumptions of Theorem 4.1 hold and further,* Γ *is a closed convex subset of* $M_{ba}(\Sigma, \mathcal{L}_1(X))$*. Let* P_o *be the solution of equation* (8) *corresponding to the control* $K_o \in \Gamma$*. Then, in order for* K_o *to be optimal, it is necessary that there exists a* $Q_o \in B_\infty(I, \mathcal{L}(X))$ *satisfying the inequality* (17) *and the evolution equations* (18)–(19) *as shown below:*

$$\begin{aligned} dJ(K_o, K - K_o) = \int_I \mathrm{Tr}\big((Q_o(t)(K - K_o)(dt)P_o \\ + P_o(t)(K^* - K_o^*)(dt)Q_o(t))\big) \geq 0,\ \forall\ K \in \Gamma. \end{aligned} \tag{17}$$

$$-dQ_o = (A^*Q_o + Q_oA)dt + (Q_o(t+)K_o(dt) + K_o^*(dt)Q_o(t+)) + G(t)dt, t \in I,$$
$$Q(T) = 0 \tag{18}$$

$$dP_o = (AP_o + P_oA^*)dt + (K_o(dt)P_o(t-) + P_o(t-)K_o(dt)) + \hat{Q}_m(t)dt, t \in I,$$
$$P_o(0) = P_0, \tag{19}$$

where $dJ(K_o, K - K_o)$ *denotes the Gateaux differential of* J *at* K_o *in the direction* $K - K_o$*.*

Proof. Let $K_o \in \Gamma$ be optimal and $P_o \in B_\infty(I, \mathcal{L}_1(X))$, the corresponding mild solution of the evolution equation (8)/(19) and $K \in \Gamma$ an arbitrary element. By (closed) convexity of Γ, it is clear that $K_o + \varepsilon(K - K_o) \in \Gamma$ for all $\varepsilon \in [0, 1]$. Let P_ε denote the (mild) solution of (8) corresponding to K_ε. Thus by optimality of K_o,

$$J(K_\varepsilon) = \int_I \mathrm{Tr}(GP_\varepsilon)dt \geq \int_I \mathrm{Tr}(GP_o)dt = J(K_o), \quad \forall\, \varepsilon \in [0, 1]$$

and hence

$$\int_I \mathrm{Tr}(G(P_\varepsilon - P_o))dt \geq 0, \quad \forall\, \varepsilon \in [0, 1].$$

From this it follows that the Gateaux differential of J at K_o in the direction $K - K_o$ must satisfy the following inequality,

$$dJ(K_o, K - K_o) = \int_I \mathrm{Tr}(G\tilde{P})dt \geq 0, \quad \forall\, K \in \Gamma, \tag{20}$$

where $\tilde{P} \in B_\infty(I, \mathcal{L}_1(X))$ is the mild solution of the variational equation given by

$$\begin{aligned} d\tilde{P} &= (A\tilde{P} + \tilde{P}A^*)dt + (K_o(dt)\tilde{P} + \tilde{P}K_o^*(dt)) \\ &\quad + (K - K_o)(dt)P_o + P_o(K^* - K_o^*)(dt), \tilde{P}(0) = 0. \end{aligned} \tag{21}$$

From this it is easy to verify that the map

$$(K - K_o)P_o + P_o(K^* - K_o^*) \longrightarrow \tilde{P}$$

is continuous and linear from $M_{ba}(I, \mathcal{L}_1(X))$ to $B_\infty(I, \mathcal{L}_1(X))$. Since $G \in B_\infty$ $(I, \mathcal{L}_s^+(X)) \subset B_\infty(I, \mathcal{L}(X))$, it is evident that the map

$$\tilde{P} \longrightarrow \int_I \mathrm{Tr}(G\tilde{P})dt$$

is a continuous linear functional on $B_\infty(I,\mathcal{L}_1(X)) \subset L_1(I,\mathcal{L}_1(X))$. Thus the composition map

$$(K-K_o)P_o + P_o(K^*-K_o^*) \longrightarrow \tilde{P} \longrightarrow \int_I \mathrm{Tr}(G\tilde{P})dt$$

is a continuous linear functional on $M_{ba}(\Sigma,\mathcal{L}_1(X))$. Hence, there exists a $Q_o \in B_\infty(I,\mathcal{L}_s^+(X))$ (symmetry and positivity is justified later in the proof) such that

$$\int_I \mathrm{Tr}(G\tilde{P})dt = \int_I \mathrm{Tr}\big(Q_o(K-K_o)(dt)P_o + P_o(K^*-K_o^*)(dt)Q_o\big). \tag{22}$$

From the expressions (20) and (22) we arrive at the necessary inequality,

$$\begin{aligned} &dJ(K_o, K-K_o) \\ &\quad = \int_I \mathrm{Tr}\big(Q_o(K-K_o)(dt)P_o + P_o(K^*-K_o^*)(dt)Q_o\big) \geq 0, \quad \forall\, K \in \Gamma, \end{aligned} \tag{23}$$

which is inequality (17) as stated in the theorem. Considering the variation of the trace $\mathrm{Tr}(Q_o\tilde{P})$ and integrating over the interval I and setting $Q_o(T)=0$, it follows from the variational equation (21) that

$$\begin{aligned} &\int_I \mathrm{Tr}\big(\tilde{P}\{dQ_o + (Q_oA + A^*Q_o)dt + (Q_oK_o(dt) + K_o^*(dt)Q_o)\}\big) \\ &\qquad + \int_I \mathrm{Tr}\big(Q_o(K-K_o)(dt)P_o + (P_o(K^*-K_o^*)(dt)Q_o\big) = 0. \end{aligned} \tag{24}$$

Now choosing Q_o, whose existence was announced above, as the mild solution of the evolution equation

$$dQ_o + (Q_oA + A^*Q_o)dt + (Q_oK_o(dt) + K_o^*(dt)Q_o) = -G(t)dt, Q_o(T) = 0, \tag{25}$$

it follows from equation (24) that

$$\begin{aligned} \int_I \mathrm{Tr}(\tilde{P}G)dt &= \int_I \mathrm{Tr}(G\tilde{P})dt \\ &= \int_I \mathrm{Tr}\big(Q_o(K-K_o)(dt)P_o + (P_o(K^*-K_o^*)(dt)Q_o\big). \end{aligned} \tag{26}$$

This is precisely the equation (22) as expected. Clearly, equation (25) gives the adjoint equation (18) as stated in the theorem. By our assumption, $G \in$

$B_\infty(I, \mathcal{L}_s^+(X))$ and hence the positivity and symmetry of Q_o follows from the closed form expression given by

$$Q_o(t) = \int_t^T U_o(s,t)G(s)U_o^*(s,t)ds, \quad t \in I, \tag{27}$$

where U_o is the strongly measurable bounded evolution operator corresponding to the pair $\{A, K_o\}$. Clearly, equation (19) is the state equation (8) corresponding to the optimal policy K_o and hence automatic. Thus, we have proved all the necessary conditions of optimality. This completes the proof. □

Remark 3. Recall that the solution of the state equation (19) is nuclear because both P_0 and $\hat{Q}_m(t), t \in I$, are nuclear. On the other hand, it follows from the expression (27) that, if G is nuclear then Q_o is also nuclear. Thus, if $G \in L_1(I, \mathcal{L}_1(X))$ (for Theorem 4.2), both the state equation (19) and the adjoint equation (18) will have mild solutions possessing identical regularities. Though not necessary, this is an interesting symmetry between the adjoint and the state equations.

Remark 4. The necessary conditions stated in Theorem 4.2, are also sufficient. In fact, we can show that, for every $K \in \Gamma$ with P being the corresponding solution of equation (8) and Q_o being the solution of the adjoint equation (18) corresponding to $K_o \in \Gamma$, the identity

$$\begin{aligned}\int_I &\mathrm{Tr}(G(t)(P(t) - P_o(t)))dt \\ &= \int_I \mathrm{Tr}\big((P(t)(K - K_o)(dt)Q_o(t) + Q_o(t)(K^* - K_o^*)(dt)P(t))\big)\end{aligned} \tag{28}$$

holds. From this one can easily deduce that the necessary conditions given by the above theorem are also sufficient.

Remark 5. If, in addition to the Poisson process $\{q\}$, there is also disturbance due to Brownian motion $\{W\}$, we have to deal with the system

$$dx = Ax(t)dt + K(dt)x(t-) + \int_{Z_0} B(t)zq(dt \times dz) + \sigma(t)dW, x(0) = x_0, \quad t \in I. \tag{29}$$

A standard assumption is that the random processes $\{x_0, q, W\}$ are mutually independent. In that case, the covariance equation (8) requires minor modification; $\hat{Q}_m(t)$ is replaced by $\hat{Q}(t) \equiv \hat{Q}_m(t) + \hat{Q}_W(t)$, where $\hat{Q}_W(t) = \sigma(t)Q\sigma^*(t)$

with Q being the incremental covariance operator, associated with the Brownian motion $W(t), t \geq 0$, taking values from a separable Hilbert space U, and $\sigma \in B_\infty(I, \mathcal{L}(U, X))$. For the process x, determined by the evolution equation (29), to have bounded second moments, it is essential that $\hat{Q}$ be nuclear, more precisely, $\hat{Q} \in L_1(I, \mathcal{L}_1^+(X))$. This is guaranteed if both $\hat{Q}_m$ and $\hat{Q}_W$ have this property.

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