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Initial value problems of p-Laplacian with a strong singular indefinite weight

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Abstract. In this paper, we present some existence and uniqueness theorems for initial value problems of p-Laplacian with a strong singular indefinite weight which are related to singular p-Laplacian eigenvalue problems. Our results improve and generalize some recent results.

1. Introduction

In this paper, we establish some existence and uniqueness results for the following initial value problem of p-Laplacian with a strong singular indefinite weight:

$$\begin{cases} \varphi_p(u'(t))' + h(t)f(u(t)) = 0, & \text{a.e. } \ t \in (0,1), \\ u(t_0) = 0, \ u'(t_0) = a, & t_0 \in [0,1], \ a \in \mathbb{R}, \end{cases} \tag{IVP$_{t_0}$}$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $f \in C(\mathbb{R},\mathbb{R})$, $h \in C((0,1),[0,\infty))$ may be singular at $t = 0$ and/or $t = 1$.

Problems (IVP$_{t_0}$) is related to the singular boundary value problem

$$\begin{cases} \varphi_p(u'(t))' + h(t)f(u(t)) = 0, & \text{a.e. } \ t \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{BVP}$$

In particular, it is helpful to find sign-changing solutions for problem (BVP) (see e.g. [6], [8]).

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The studies on the singular initial and boundary value problems with sign-changing nonlinearity are much recent. For the studies on the first-order cases, one may refer to AGARWAL, O'REGAN [1], [2] and AGARWAL, O'REGAN, LAKSHMIKANTHAM and LEELA [3]. For second-order initial value problems, by using the Gronwall inequality, YANG [9] proved the existence and uniqueness of a solution of the following initial value problem:

$$\begin{cases} \varphi_p(u'(t))' + f(t, u(t)) = 0, \\ u(0) = 0, \ u'(0) = a > 0, \end{cases}$$

where $|f(t,u)| \le ch(t)|u|^{p-1}$ with $h \in L^q(0,1)$, $q > 1$.

For $h \in L^1(0,1)$, ZHANG [10] showed the existence and uniqueness of a solution for (IVP_{t_0}) with $f(u) = \varphi_p(u)$, $t_0 = 0$ and $a = 1$ by transforming to a system and applying Sturmian comparison. One should also notice that GARCÍA-HUIDOBRO, MANÁSEVICH and ÔTANI [5] gave an existence and uniqueness result for initial value problem (IVP_{t_0}) with $t \in \mathbb{R}$, $h \in L^1_{loc}(\mathbb{R})$, $f(u) = \varphi_p(u)$.

For $h \in \mathcal{A}$ with $\mathcal{A}$ defined by the set

$$\left\{ h \in C((0,1),\ [0,\infty)) : \int_0^1 s^\alpha (1-s)^\beta h(s) ds < \infty \text{ for some } \alpha, \beta \in (0, p-1) \right\},$$

some existence and uniqueness results were proved by LEE and SIM [7] for three special cases of initial problem (IVP_{t_0}): $t_0 = 0$, $a = 1$; $t_0 = 1$, $a = -1$ and $a = 0$.

Denote

$$\mathcal{B} = \left\{ h \in C((0,1),\ [0,\infty)) : \int_0^1 (s(1-s))^{p-1} h(s) ds < \infty \right\}.$$

It is clear that $L^1(0,1) \subset \mathcal{A} \subset \mathcal{B}$. For more properties of the classes of singular indefinite weights $\mathcal{A}$ and $\mathcal{B}$, one may refer to [4], [6]. For $h \in \mathcal{B}$, KAJIKIYA, LEE and SIM [6] obtained some existence and uniqueness results for (IVP_{t_0}) under assumption that $f(t, u(t)) = \lambda \varphi_p(u(t))$ with λ a positive real parameter. Moreover, these results were applied in the study of some eigenvalue problems for p-Laplacian.

The aim of this paper is to present some existence and uniqueness results for problem (IVP_{t_0}) with $h \in \mathcal{B}$, $t_0 \in [0,1]$, $a \in \mathbb{R}$ and $f \in C(\mathbb{R}, \mathbb{R})$ by using Schauder's fixed point theorem (see Theorem 2.1, 2.2 and Corollary 2.3). Our results improve and generalize some results in [6], [7] (see Remark 2.4).

2. Statements of the main results

Recall that a function u is said to be a solution of (IVP_{t_0}), if $u \in C^1(0,1) \cap C[0,1]$ and $\varphi_p(u')$ is absolutely continuous in any compact subinterval of $(0,1)$ and u satisfies (IVP_{t_0}).

Let us give the following assumptions on f:

(H_1) $\exists C > 0$ such that $|f(u)| \le C|\varphi_p(u)|$ for $u \in \mathbb{R}$.

(H_2) $\exists C > 0$ such that $|f(u) - f(v)| \le C|\varphi_p(u) - \varphi_p(v)|$ for $u, v \in \mathbb{R}$.

(H_{2+}) $\forall \Gamma > 0$, $\exists C > 0$ such that $|f(u) - f(v)| \le C|\varphi_p(u) - \varphi_p(v)|$ for $u, v \in [0, \Gamma]$.

(H_{2-}) $\forall \Gamma > 0$, $\exists C > 0$ such that $|f(u) - f(v)| \le C|\varphi_p(u) - \varphi_p(v)|$ for $u, v \in [-\Gamma, 0]$.

The main results of this paper are as follows.

Theorem 2.1. *Assume $h \in \mathcal{B}$ and (H_1). Then problem (IVP_{t_0}) has at least one solution. Especially, if $a = 0$, problem (IVP_{t_0}) has only a trivial solution.*

Theorem 2.2. *Assume $h \in \mathcal{B}$. The following statements are true.*

(i) *Suppose that (H_{2+}) holds. If $a > 0$, $t_0 \in [0,1)$ or $a < 0$, $t_0 \in (0,1]$, then problem (IVP_{t_0}) has at most one solution.*

(ii) *Suppose that (H_{2-}) holds. If $a < 0$, $t_0 \in [0,1)$ or $a > 0$, $t_0 \in (0,1]$, then problem (IVP_{t_0}) has at most one solution.*

It is clear that (H_2) implies (H_1), (H_{2+}) and (H_{2-}). Then by Theorem 2.1 and 2.2, we can get the following result immediately.

Corollary 2.3. *Assume $h \in \mathcal{B}$ and (H_2). Then problem (IVP_{t_0}) has a unique solution.*

Remark 2.4. (a) Theorem 2.1 generalizes Theorem 1.1 and 1.3 in [7] by extending the class of singular indefinite weights from $h \in \mathcal{A}$ to $h \in \mathcal{B}$.

(b) Theorem 2.2 (i) improves Theorem 1.2 in [7]. In fact, it is clear that assumption (H_{2+}) is weaker than the assumption

(P) $\exists C > 0$ such that $|f(u) - f(v)| \le C|\varphi_p(u) - \varphi_p(v)|$ for $u, v \in [0, \infty)$.

For example, $f(u) = (\varphi_p(u))^2$ does not satisfy (P), but satisfies (H_{2+}) and (H_{2-}) with $C = 2\varphi_p(\Gamma)$. Then Theorem 2.2 (i) is the same as Theorem 1.2 in [7] if replace "$h \in \mathcal{B}$ and (H_{2+})" by "$h \in \mathcal{A}$ and (P)" and consider the two special cases $t_0 = 0$, $a = 1$ and $t_0 = 1$, $a = -1$.

(c) Corollary 2.3 generalizes Theorem 2.2 and 2.3 in [6] by considering the general function $f \in C(\mathbb{R}, \mathbb{R})$ instead of $\lambda\varphi_p(u)$ with λ a positive real parameter.

3. Proofs of the main results

In the sequel, we always assume that $t_0 \in [0,1)$ and $h \in \mathcal{B}$. For the case when $t_0 \in (0,1]$, we can analyze exactly the same way and we omit the details.

Let us start with some lemmas which will be used in the proofs of our main results.

Lemma 3.1. *Assume $h \in \mathcal{B}$ and* (H_1)*. If $a = 0$, problem* (IVP_{t_0}) *has only a trivial solution.*

Proof. Clearly 0 is a solution of (IVP_{t_0}) if $a = 0$. Let u be a solution of (IVP_{t_0}). It is enough to prove that $u = 0$ for $t \in [0,1]$. There are two cases to be considered: $t_0 \in (0,1)$ and $t_0 = 0$.

Case 1. $t_0 \in (0,1)$. For any $t_1 \in (t_0,1)$, we have $u \in C^1[t_0,t_1]$ and $h \in C[t_0,t_1]$. For $t \in [t_0,t_1]$, it follows from (IVP_{t_0}) that

$$|u(t)| = \left|\int_{t_0}^{t} \varphi_p^{-1}\left(\int_{t_0}^{s} h(\tau)f(u(\tau))d\tau\right)ds\right| \leq \varphi_p^{-1}\left(\int_{t_0}^{t} h(s)|f(u(s))|ds\right).$$

Then, by (H_1) we have, for $t \in [t_0,t_1]$,

$$\varphi_p(|u(t)|) \leq \int_{t_0}^{t} h(s)|f(u(s))|ds \leq C\int_{t_0}^{t} h(s)\varphi_p(|u(s)|)ds.$$

By the Gronwall inequality we have $\varphi_p(|u(t)|) = 0$ for $t \in [t_0,t_1]$, that is $u(t) = 0$, $t \in [t_0,t_1]$. This implies that $u(t) = 0$ for $t \in [t_0,1]$ since t_1 is arbitrary in $(t_0,1)$ and u is continuous in $[0,1]$. Similarly, we can prove that $u(t) = 0$ for $t \in [0,t_0]$ and then $u(t) = 0$ for all $t \in [0,1]$.

Case 2. $t_0 = 0$. For $t_1 \in (0,1)$, we have $u \in C^1[0,t_1]$. Let $v(0) = 0$ and $v(t) = u(t)/t$ for $t \in (0,t_1]$. Then $v \in C[0,t_1]$. For $t \in (0,t_1]$, from (IVP_{t_0}) we have

$$|v(t)| = \left|\frac{1}{t}\int_{0}^{t} \varphi_p^{-1}\left(\int_{0}^{s} h(\tau)f(u(\tau))d\tau\right)ds\right| \leq \varphi_p^{-1}\left(\int_{0}^{t} h(s)|f(u(s))|ds\right).$$

Thus for $t \in [0,t_1]$, by (H_1) we get

$$\varphi_p(|v(t)|) \leq \int_0^t h(s)|f(u(s))|ds \leq \int_0^t h(s)C|\varphi_p(u(s))|ds$$
$$= C\int_0^t h(s)s^{p-1}\varphi_p(|u(s)/s|)ds = C\int_0^t h(s)s^{p-1}\varphi_p(|v(s)|)ds.$$

By the Gronwall inequality we have $\varphi_p(|v(t)|) = 0$ for $t \in [0,t_1]$. That is $u(t) = 0$, $t \in [0,t_1]$. Therefore, it follows from the arbitrariness of t_1 in $(0,1)$ and the continuity of u that $u(t) = 0$ for $t \in [0,1]$. This completes the proof. $\square$

By the Mean Value Theorem and a fundamental calculation, it is easy to get the following inequality which will be used later:

$$|\varphi_p(x) - \varphi_p(y)| \le (p-1)z^{p-2}|x-y|, \quad \forall x, y \in \mathbb{R}, \tag{3.1}$$

where $z = \max\{|x|, |y|\}$.

Now we assume that $a \neq 0$. Let $K > |a|$ be a constant. Since $h \in \mathcal{B}$, there exists $\beta \in (t_0, 1)$ such that

$$\int_{t_0}^{\beta} h(s)(s-t_0)^{p-1} ds \le \min\left\{\frac{1-\varphi_p(|a|/K)}{C}, \frac{\varphi_p(|a|/K)}{2C}\right\}, \tag{3.2}$$

where C is the same constant as in (H_1). Set

$$C_0[t_0,\beta] = \{u \in C[t_0,\beta] : u(t_0) = 0\} \quad \text{with } \|u\| = \max_{t\in[t_0,\beta]} |u(t)| \text{ for } u \in C_0[t_0,\beta],$$

$$C_0^1[t_0,\beta] = C_0[t_0,\beta] \cap C^1[t_0,\beta] \qquad \text{with } \|u\|_1 = \max_{t\in[t_0,\beta]} |u'(t)| \text{ for } u \in C_0^1[t_0,\beta].$$

Clearly, $(C_0[t_0,\beta], \|\cdot\|)$ and $(C_0^1[t_0,\beta], \|\cdot\|_1)$ are Banach spaces and $\|u\| \le \|u\|_1$ for $u \in C_0^1[t_0,\beta]$. Let

$$M = \{u \in C_0^1[t_0,\beta] : \|u\|_1 \le K\}.$$

Define $G : C_0^1[t_0,\beta] \to C_0^1[t_0,\beta]$ by

$$G(u)(t) = \int_{t_0}^{t} \varphi_p^{-1}\left(\varphi_p(a) - \int_{t_0}^{s} h(\tau) f(u(\tau)) d\tau\right) ds, \quad \text{for } t \in [t_0,\beta].$$

For any $u \in C_0^1[t_0,\beta]$, we have $|u(t)| \le \|u\|_1 (t-t_0)$ for $t \in [t_0,\beta]$. Then by (H_1) we have, for $t \in [t_0,\beta]$,

$$\begin{aligned}\left|\int_{t_0}^{t} h(\tau) f(u(\tau)) d\tau\right| &\le \int_{t_0}^{t} h(\tau) C |u(\tau)|^{p-1} d\tau \\ &\le C\|u\|_1^{p-1} \int_{t_0}^{t} h(\tau)(\tau-t_0)^{p-1} d\tau < \infty.\end{aligned} \tag{3.3}$$

So G is well defined. In addition, noticing that

$$(G(u))'(t) = \varphi_p^{-1}\left(\varphi_p(a) - \int_{t_0}^{t} h(\tau) f(u(\tau)) d\tau\right),$$

(3.3) implies that G is bounded, i.e., send bounded subsets of $C_0^1[t_0,\beta]$ into bounded subsets of $C_0^1[t_0,\beta]$. Furthermore, it is easy to see that $u(t)$ is a local solution of problem (IVP_{t_0}) for $t \in [t_0,\beta]$ if and only if u is a fixed point of G in $C_0^1[t_0,\beta]$.

Lemma 3.2. *Assume $h \in \mathcal{B}$ and* (H$_1$). *Then $G(M) \subset M$ and $G : M \to M$ is continuous.*

Proof. By (H$_1$) and (3.2), for $u \in M, t \in [t_0, \beta]$,

$$\begin{aligned}|(G(u))'(t)| &= \left|\varphi_p^{-1}\left(\varphi_p(a) - \int_{t_0}^t h(\tau)f(u(\tau))d\tau\right)\right| \\ &\le \varphi_p^{-1}\left(\varphi_p(|a|) + \int_{t_0}^t h(\tau)C|\varphi_p(u(\tau))|d\tau\right) \\ &\le \varphi_p^{-1}\left(\varphi_p(|a|) + C\|u\|_1^{p-1}\int_{t_0}^t h(\tau)(\tau - t_0)^{p-1}d\tau\right) \\ &\le \varphi_p^{-1}\left(\varphi_p(|a|) + CK^{p-1}\frac{1-\varphi_p(|a|/K)}{C}\right) = K.\end{aligned}$$

That is $\|G(u)\|_1 \le K$, and then $G(M) \subset M$.

Now we prove the continuity of G. For $u \in M$, $t \in [t_0, \beta]$, by (H$_1$) and (3.2) we have

$$\begin{aligned}\left|\int_{t_0}^t h(\tau)f(u(\tau))d\tau\right| &\le \int_{t_0}^t h(\tau)C|\varphi_p(u(\tau))|d\tau \le C\|u\|_1^{p-1}\int_{t_0}^t h(\tau)(\tau - t_0)^{p-1}d\tau \\ &\le CK^{p-1}\frac{\varphi_p(|a|/K)}{2C} = \varphi_p(|a|)/2.\end{aligned}$$

Then

$$\left|\varphi_p(a) - \int_{t_0}^t h(\tau)f(u(\tau))d\tau\right| \le 3\varphi_p(|a|)/2 \quad \text{for } t \in [t_0, \beta]. \tag{3.4}$$

Let $q = p/(p-1)$, then $\varphi_p^{-1} = \varphi_q$. Denote

$$C_1 = \frac{1}{2(q-1)(3\varphi_p(|a|)/2)^{q-2}}. \tag{3.5}$$

Given $\varepsilon > 0$, there exists $\eta \in (t_0, \beta)$ such that

$$\int_{t_0}^\eta h(s)(s - t_0)^{p-1}ds \le \frac{C_1\varepsilon}{2CK^{p-1}}$$

since $h \in \mathcal{B}$. Then for $u \in M$, by (H$_1$),

$$\begin{aligned}\int_{t_0}^\eta h(s)|f(u(s))|ds &\le \int_{t_0}^\eta h(\tau)C|\varphi_p(u(\tau))|d\tau \le C\|u\|_1^{p-1}\int_{t_0}^\eta h(s)(s - t_0)^{p-1}ds \\ &\le CK^{p-1}\frac{C_1\varepsilon}{2CK^{p-1}} = \frac{C_1\varepsilon}{2}.\end{aligned} \tag{3.6}$$

Let $\{u_n\} \subset M$ such that $u_n \to u_0$ in M as $n \to \infty$. Then for $t \in [t_0, \beta]$, by (3.1), (3.4) and (3.5) we get

$$\begin{aligned}
&|(G(u_n))'(t) - (G(u_0))'(t)| \\
&\quad = \left| \varphi_p^{-1}\left(\varphi_p(a) - \int_{t_0}^{t} h(\tau) f(u_n(\tau)) d\tau\right) - \varphi_p^{-1}\left(\varphi_p(a) - \int_{t_0}^{t} h(\tau) f(u_0(\tau)) d\tau\right)\right| \\
&\quad \le (q-1)(3\varphi_p(|a|)/2)^{q-2} \int_{t_0}^{t} h(\tau)|f(u_n(\tau)) - f(u_0(\tau))| d\tau \\
&\quad = \frac{1}{2C_1} \int_{t_0}^{t} h(\tau)|f(u_n(\tau)) - f(u_0(\tau))| d\tau. \qquad (3.7)
\end{aligned}$$

Now we have two cases to be considered.

Case 1. Suppose $\int_\eta^\beta h(s)ds = 0$. Since $f \in C(\mathbb{R}, \mathbb{R})$, we have $|f(u)| \le C_2$ for $u \in [-K, K]$ and some $C_2 > 0$. Then by (3.6) and (3.7) we get

$$\begin{aligned}
|(G(u_n))'(t) - (G(u_0))'(t)| &\le \frac{1}{2C_1}\left(\int_{t_0}^{\eta} + \int_{\eta}^{\beta}\right) h(\tau)(|f(u_n(\tau))| + |f(u_0(\tau))|) d\tau \\
&\le \frac{1}{2C_1}\left(C_1\varepsilon + 2C_2 \int_\eta^\beta h(\tau) d\tau\right) = \frac{\varepsilon}{2}.
\end{aligned}$$

This implies that $G : M \to M$ is continuous.

Case 2. Suppose $\int_\eta^\beta h(s)ds \ne 0$. Since f is uniformly continuous in $[-K, K]$, there exists $\rho > 0$ such that $u, v \in [-K, K]$, $|u - v| < \rho$ implies

$$|f(u) - f(v)| \le C_1\varepsilon\left(\int_\eta^\beta h(s)ds\right)^{-1}.$$

Meanwhile, there exists $N > 0$ such that $|u_n(t) - u_0(t)| < \rho$ for $t \in [t_0, \beta]$, $n > N$. Thus

$$|f(u_n(t)) - f(u_0(t))| \le C_1\varepsilon\left(\int_\eta^\beta h(s)ds\right)^{-1} \quad \text{for } n > N,\ t \in [t_0, \beta]. \qquad (3.8)$$

Now, for $t \in [t_0, \beta]$ and $n > N$, by (3.6) – (3.8) we have

$$\begin{aligned}
&|(G(u_n))'(t) - (G(u_0))'(t)| \le \frac{1}{2C_1}\left(\int_{t_0}^{\eta} + \int_\eta^\beta\right) h(\tau)|f(u_n(\tau)) - f(u_0(\tau))| d\tau \\
&\le \frac{1}{2C_1}\left(\int_{t_0}^{\eta} h(\tau)(|f(u_n(\tau))| + |f(u_0(\tau))|) d\tau + \int_\eta^\beta h(\tau)(|f(u_n(\tau)) - f(u_0(\tau))|) d\tau\right) \\
&\le \frac{1}{2C_1}\left(C_1\varepsilon + \int_\eta^\beta h(\tau) C_1\varepsilon \left(\int_\eta^\beta h(s)ds\right)^{-1} d\tau\right) = \varepsilon,
\end{aligned}$$

which implies that $G : M \to M$ is continuous. The proof is complete. □

Lemma 3.3. *Assume $h \in \mathcal{B}$ and* (H_1). *Then $G : M \to M$ is compact.*

PROOF. Suppose $\{u_n\} \subset M$ is bounded, then $\{u_n\}$ and $\{G(u_n)\}$ are bounded in M. By the Arzela Ascoli Theorem, $\{u_n\}$ and $\{G(u_n)\}$ has a subsequence (denote again by $\{u_n\}$ and $\{G(u_n)\}$, respectively) converging to some u and v in $C_0[t_0, \beta]$, respectively. By (H_1) we have

$$\begin{aligned}|h(t)f(u_n(t))| \le Ch(t)|\varphi_p(u_n(t))| &\le Ch(t)\varphi_p(\|u_n\|_1(t-t_0))\\ &\le CK^{p-1}h(t)(t-t_0)^{p-1}.\end{aligned}$$

for $t \in [t_0, \beta]$. So by the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}G(u_n)(t) = \int_{t_0}^{t} \varphi_p^{-1}\left(\varphi_p(a) - \int_{t_0}^{s} h(\tau)f(u_n(\tau))d\tau\right) ds\\ \to \int_{0}^{t} \varphi_p^{-1}\left(\varphi_p(a) - \int_{t_0}^{s} h(\tau)f(u(\tau))d\tau\right) ds = v(t),\end{aligned}$$

uniformly in $t \in [t_0, \beta]$, and

$$\begin{aligned}(G(u_n))'(t) = \varphi_p^{-1}\left(\varphi_p(a) - \int_{t_0}^{t} h(\tau)f(u_n(\tau))d\tau\right)\\ \to \varphi_p^{-1}\left(\varphi_p(a) - \int_{t_0}^{t} h(\tau)f(u(\tau))d\tau\right),\end{aligned}$$

uniformly in $t \in [t_0, \beta]$. So $v \in C_0^1[0,1]$ and

$$v'(t) = \varphi_p^{-1}\left(\varphi_p(a) - \int_{t_0}^{t} h(\tau)f(u(\tau))d\tau\right).$$

Clearly, $\|v\|_1 \le K$. Therefore, $G : M \to M$ is compact. The proof is complete. $\square$

Now we are in a position to give the proofs of the main results.

PROOF OF THEOREM 2.1. Lemma 3.1 leads to the conclusion for the special case $a = 0$. Now we assume that $a \neq 0$. It follows from Lemma 3.2 and 3.3 that $G : M \to M$ is completely continuous. Then by Schauder's Fixed Point Theorem, G has a fixed point in M. That is problem (IVP_{t_0}) has a local solution $u \in M$.

Now we prove the global existence of solutions of problem (IVP_{t_0}). Let $[t_0, T)$ be the right maximal interval of existence for solution u. It is enough to show that $T = 1$. Suppose on the contrary that $T < 1$. Then by (H_1), for $t \in [t_0, T)$ we have

$$\begin{aligned}|\varphi_p(u'(t))| &= \left|\varphi_p(a) - \int_{t_0}^t h(\tau) f(u(\tau)) d\tau\right| \le \varphi_p(|a|) + C\int_{t_0}^t h(\tau)|u(\tau)|^{p-1} d\tau \\ &\le \varphi_p(|a|) + C\int_{t_0}^t h(\tau)(\tau - t_0)^{p-1} \max_{s\in[t_0,\tau]} |u'(s)|^{p-1} d\tau\end{aligned}$$

So we have

$$\max_{r\in[t_0,t]} |u'(r)|^{p-1} \le \varphi_p(|a|) + C\int_{t_0}^t (\tau)(\tau - t_0)^{p-1} \max_{s\in[t_0,\tau]} |u'(s)|^{p-1} d\tau.$$

By the Gronwall inequality, we obtain

$$\begin{aligned}\max_{r\in[t_0,t]} |u'(r)|^{p-1} &\le \varphi_p(|a|) \exp\left(C\int_{t_0}^t h(\tau)(\tau - t_0)^{p-1} d\tau\right) \\ &\le \varphi_p(|a|) \exp\left(C\int_{t_0}^T h(\tau)(\tau - t_0)^{p-1} d\tau\right),\end{aligned}$$

which implies that u' is bounded in $[t_0, T)$, and consequently u is bounded in $[t_0, T)$. This contradicts the fact that $[0, T)$ with $T < 1$ is the maximal existence interval for solution u. The proof is complete. □

Proof of Theorem 2.2. We prove statement (i). By a similar argument we can prove statement (ii) and we omit the details. Moreover, we only prove statement (i) for the case $a > 0$ and $t_0 \in [0, 1)$. The case $a < 0$ and $t_0 \in (0, 1]$ can be proved similarly and we also omit the details.

Suppose u, v are two solutions of problem (IVP_{t_0}). It suffices to prove that $u(t) = v(t)$ for $t \in [t_0, \beta]$ with some $\beta \in (t_0, 1)$ which will be determined later.

Let $K > a$ such that $\max\{|u'(t)|, |v'(t)|\} \le K$ for all $t \in [t_0, (1 + t_0)/2]$. Since $a > 0$, we can choose $\beta_1 \in (t_0, (1 + t_0)/2)$ such that $u(t), v(t) > 0$ for all $t \in (t_0, \beta_1]$. By (H_{2+}), there exists some $C > 0$ such that

$$|f(u)| \le C\varphi_p(u), \quad |f(u) - f(v)| \le C|\varphi_p(u) - \varphi_p(v)| \quad \text{for } u, v \in [0, K]. \tag{3.9}$$

Then

$$|f(u(t))| \le C\varphi_p(u(t)), \quad |f(v(t))| \le C\varphi_p(v(t)) \quad \text{and} \tag{3.10}$$

$$|f(u(t) - f(v(t))| \le C|\varphi_p(u(t)) - \varphi_p(v(t))| \quad \text{for } t \in [t_0, \beta_1]. \tag{3.11}$$

Since $h \in \mathcal{B}$, we can choose $\beta \in (t_0, \beta_1)$ such that

$$\int_{t_0}^{\beta} h(s)(s-t_0)^{p-1} ds < \min\left\{\frac{\varphi_p(a/K)}{2C}, \frac{1}{(3\varphi_p(a)/2)^{q-2}K^{p-2}C}\right\}. \tag{3.12}$$

where $q = p/(1-p)$, K and C are the same constants as in (3.9). It is obvious that

$$u, v \in M = \{w \in C_0^1[t_0, \beta] : \|w\|_1 \leq K\}.$$

Then by (3.10), (3.12) and the same way to get (3.4), we can obtain that (3.4) also holds for these u, v. That is

$$\left|\varphi_p(a) - \int_{t_0}^{t} h(\tau) f(x(\tau)) d\tau\right| \leq 3\varphi_p(a)/2 \quad \text{for } t \in [t_0, \beta], \tag{3.13}$$

where $x = u, v$. Meanwhile, by the Mean Value Theorem, for $t \in [t_0, \beta]$, there exist some $\theta_1, \theta_2, \theta_3 \in (t_0, t)$ such that

$$\left|\frac{u(t)}{t-t_0}\right| = |u'(\theta_1)| \leq \|u\|_1 \leq K, \tag{3.14}$$

$$\left|\frac{v(t)}{t-t_0}\right| = |v'(\theta_2)| \leq \|v\|_1 \leq K, \tag{3.15}$$

$$\left|\frac{u(t)-v(t)}{t-t_0}\right| = |u'(\theta_3) - v'(\theta_3)| \leq \|u-v\|_1. \tag{3.16}$$

Notice that $\varphi_p^{-1} = \varphi_q$ and $(p-1)(q-1) = 1$. If $\|u-v\|_1 > 0$, then by (3.1), (3.11)–(3.16), for $t \in [t_0, \beta]$,

$$\begin{aligned}
&|u'(t) - v'(t)| \\
&= \left|\varphi_p^{-1}\left(\varphi_p(a) - \int_{t_0}^{t} h(\tau) f(u(\tau)) d\tau\right) - \varphi_p^{-1}\left(\varphi_p(a) - \int_{t_0}^{t} h(\tau) f(v(\tau)) d\tau\right)\right| \\
&\leq (q-1)(3\varphi_p(a)/2)^{q-2} \int_{t_0}^{t} h(\tau)|f(u(\tau)) - f(v(\tau))| d\tau \\
&\leq (q-1)(3\varphi_p(a)/2)^{q-2} C \int_{t_0}^{t} h(\tau)|\varphi_p(u(\tau)) - \varphi_p(v(\tau))| d\tau \\
&= (q-1)(3\varphi_p(a)/2)^{q-2} C \int_{t_0}^{t} h(\tau)(\tau-t_0)^{p-1}\left|\varphi_p\left(\frac{u(\tau)}{\tau-t_0}\right) - \varphi_p\left(\frac{v(\tau)}{\tau-t_0}\right)\right| d\tau \\
&\leq (q-1)(3\varphi_p(a)/2)^{q-2} C \int_{t_0}^{t} h(\tau)(\tau-t_0)^{p-1}(p-1)K^{p-2}\left|\frac{u(\tau)-v(\tau)}{\tau-t_0}\right| d\tau
\end{aligned}$$

$$\le (3\varphi_p(a)/2)^{q-2} K^{p-2} C \int_{t_0}^{\beta} h(\tau)(\tau - t_0)^{p-1} d\tau \|u - v\|_1 < \|u - v\|_1,$$

which is impossible, and then $\|u - v\|_1 = 0$, i.e. $u(t) = v(t)$ for $t \in [t_0, \beta]$. This completes the proof. □

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