Publl. Math. Debrecen
77/3-4 (2010), 427–442

Some rigidity results for Dirac-harmonic maps

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Abstract. Let (ϕ,ψ) be a Dirac-harmonic maps from a Riemannian manifold into another Riemannian manifold. We call (ϕ,ψ) *trivial* if ϕ is harmonic. By using Bochner-type formula and extending Chen–Jost–Li–Wang' result, we give some sufficient conditions for a Dirac-harmonic map (ϕ,ψ) to be trivial. We also give a structure theorem of Dirac-harmonic maps from a Riemann surface generalizing result previously only known in the case when source manifold is a two sphere.

1. Introduction

Dirac-harmonic maps are a generalization and combination of harmonic maps and harmonic spinors while preserving the essential properties of the former. They arise from the supersymmetric nonlinear sigma model of quantum field theory [6].

Obviously, there are two types of basic examples, a harmonic map together with a vanishing spinor and a constant map together with a harmonic spinor. In [4], the authors constructed Dirac-harmonic maps (ϕ,ψ) from S^2 to S^2, where ϕ is a harmonic map (or equivalently, a (possible branched) conformal map), ψ could be written in the form

$$\psi=\Sigma_\alpha\epsilon_\alpha\cdot\Psi\otimes\phi_*(\epsilon_\alpha) \tag{1.1}$$

where ϵ_α $(\alpha=1,2)$ is a local orthonormal basis of S^2, and Ψ is a twistor spinor.

In the spirit of Chen–Jost–Li–Wang, recently, Jost–Mo–Zhu constructed explicit examples of Dirac-harmonic maps (ϕ,ψ) from an Euclidean space to a hyperbolic space which are non-trivial in the sense that ϕ is not harmonic [4],

Mathematics Subject Classification: 58E20.
Key words and phrases: Dirac-harmonic map, harmonic spinor, surface, degree.

[10]. Precisely, in their examples, $\phi : \mathbb{R}^n \to H^{n+1}$ is an isometric immersion where $n \geq 3$ and ψ could be written in the form

$$\psi^T = \Sigma_\alpha \epsilon_\alpha \cdot \Psi \otimes \phi_*(\epsilon_\alpha)$$

where $(\dots)^T$ denotes the orthogonal projection into the subbundle $\Sigma\mathbb{R}^n \otimes T\mathbb{R}^n$ and ϵ_α $(\alpha = 1, 2)$ is an orthonormal basis of $\mathbb{R}^n$. A natural question then is whether there exist non-trivial Dirac-harmonic maps for hypersphere in a hyperbolic space in this form.

In this paper, we will first give the following negative answer.

Theorem 1.1. *Let M^n be a compact positive scalar curved spinor manifold immersed in a non-positively constantly curved manifold N. Then there is no non-vanishing harmonic spinor ψ along this immersion with $\psi^T = \Sigma_\alpha \epsilon_\alpha \cdot \Psi \otimes \phi_*(\epsilon_\alpha)$, therefore, there is no non-trivial Dirac-harmonic map (ϕ, ψ) from M into N with $\psi^T = \Sigma_\alpha \epsilon_\alpha \cdot \Psi \otimes \phi_*(\epsilon_\alpha)$.*

For definitions of harmonic spinor and Dirac-harmonic map see Section 2 and Section 3. In the two-dimensional case we have the following:

Proposition 1.2. *Let $\phi : M \hookrightarrow N$ be a surface in a Riemannian manifold of constant curvature c with flat normal bundle. Then there is no non-trivial Dirac-harmonic map (ϕ, ψ) from M into N with $\psi^T = \Sigma_\alpha \epsilon_\alpha \cdot \Psi \otimes \phi_*(\epsilon_\alpha)$.*

Then we generalize Chen–Jost–Li–Wang' construction [4] as following:

Proposition 1.3. *Let M be a Riemann surface and N a Riemannian manifold. Let $\psi_{\phi,\Psi}$ be defined by $\psi_{\phi,\Psi} = \Sigma_\alpha \epsilon_\alpha \cdot \Psi \otimes \phi_*(\epsilon_\alpha)$ from a nonconstant conformal map $\phi : M \to N$ and a spinor $\Psi \in \Gamma(\Sigma M)$. If $(\phi, \psi_{\phi,\Psi})$ is a Dirac-harmonic map then ϕ is a branched minimal immersion and Ψ is a twistor spinor.*

See Section 2 for definition of twistor spinor. Using Proposition 1.3, we obtain the following structure theorem of Dirac-harmonic maps from Riemann surfaces:

Theorem 1.4. *Let (ϕ, ψ) is a non-constant Dirac-harmonic map from a compact Riemann surface M_g of genus g to the sphere M_0 with $|\deg \phi| > g - 1$. Then ϕ is $\pm$ holomorphic, and ψ could be written in the form*

$$\psi = \Sigma_\alpha \epsilon_\alpha \cdot \Psi \otimes \phi_*(\epsilon_\alpha)$$

where ϵ_α $(\alpha = 1, 2)$ is a local orthonormal basis of M_g, and Ψ is a twistor spinor.

It is worth mentioning the condition that $\deg \phi > g - 1$ is sharp. If $g \geq 1$ and $0 \leq d \leq g - 1$, Lemaire has constructed Riemann surface M_g and harmonic non$\pm$holomorphic maps $\phi : M_g \to M_0$ of degree d. Thus $(\phi, 0)$ is a Dirac-harmonic map from M_g into M_0.

2. Dirac-harmonic maps

In this section, we recall the basic definitions and introduce our notation. Let (N, h) be a Riemannian manifold of dimension n'. This will be our target manifold. Likewise, let (M, g), our domain manifold, be an n-dimensional Riemannian manifold with fixed spin structure. By ΣM, we denote its spinor bundle, on which we have a Hermitian metric $\langle \cdot\,, \cdot\rangle$ induced by the Riemannian metric $g(\cdot\,, \cdot)$ of M. Let ϕ be a smooth map from (M, g) to (N, h) and $\phi^{-1}TN$ the pull-back bundle of TN by ϕ. On the twisted bundle $\Sigma M \otimes \phi^{-1}TN$ there is a metric (still denoted by $\langle \cdot\,, \cdot\rangle$) induced from the metrics on ΣM and $\phi^{-1}TN$. There is also a natural connection $\tilde{\nabla}$ on $\Sigma M \otimes \phi^{-1}TN$ induced from those on ΣM and $\phi^{-1}TN$ (which in turn come from the Levi–Civita connections of (M, g) and (N, h), resp.).

We have the Clifford product $X \cdot \Phi$ of $X \in \Gamma(TM)$, $\Phi \in \Gamma(\Sigma M)$. This Clifford product satisfies the skew-symmetry relation

$$\langle X \cdot \Phi, \Psi\rangle = -\langle \Phi, X \cdot \Psi\rangle \tag{2.1}$$

as well as the Clifford relations

$$X \cdot Y \cdot \Phi + Y \cdot X \cdot \Phi = -2g(X, Y)\Phi$$

for $X, Y \in \Gamma(TM)$, $\Phi, \Psi \in \Gamma(\Sigma M)$.

We are now prepared to introduce an operator that couples the geometries of M and N via the map ϕ. Let ψ be a section of the bundle $\Sigma M \otimes \phi^{-1}TN$. The *Dirac operator along the map* ϕ is defined as

$$\not{D}\psi := \epsilon_\alpha \cdot \tilde{\nabla}_{\epsilon_\alpha}\psi$$

where ϵ_α is a local orthonormal basis of M. For background material about the spinor bundle and the Dirac operator, we refer to [9], [12].

We consider the space

$$\chi := \{(\phi, \psi) | \phi \in C^\infty(M, N) \text{ and } \psi \in C^\infty(\Sigma M \otimes \phi^{-1}TN)\}$$

of mappings and sections along those mappings. On χ, we have the functional

$$L(\phi, \psi) := \frac{1}{2}\int_M \left[|d\phi|^2 + \langle \psi, \not{D}\psi\rangle\right] {}^*1_M.$$

This functional couples the two fields ϕ and ψ because the operator $\not{D}$ depends on the map ϕ. The Euler–Lagrange equations of $L(\phi, \psi)$ then also couple the two fields; they are:

$$\tau(\phi) = \mathcal{R}(\phi, \psi) \tag{2.2}$$

and

$$\not{D}\psi = 0 \tag{2.3}$$

where $\tau(\phi)$ is the tension field of the map ϕ (the natural version of the Laplace operator for maps between manifolds) and the curvature term $\mathcal{R}(\phi,\psi)$ is defined by

$$\mathcal{R}(\phi,\psi) = \frac{1}{2} R^i{}_{jkl} \langle \psi^k, \nabla\phi^j \cdot \psi^l \rangle \frac{\partial}{\partial y^i},$$

where

$$\psi = \psi^i \otimes \frac{\partial}{\partial y^i}, \qquad (d\phi)^\sharp = \nabla\phi^i \otimes \frac{\partial}{\partial y^i},$$

$$R^{\phi^{-1}TN}\left(\frac{\partial}{\partial y^k}, \frac{\partial}{\partial y^l}\right)\frac{\partial}{\partial y^j} = R^i{}_{jkl}\frac{\partial}{\partial y^i}$$

where $\sharp : T^*M \otimes \phi^{-1}TN \to TM \otimes \phi^{-1}TN$ is the standard ("musical") isomorphism obtained from the Riemannian metric g.

Solutions (ϕ,ψ) to (2.2) and (2.3) are called *Dirac-harmonic maps* from M into N [4].

We now start with some differential geometric identities: Let ϵ_α be a local orthonormal basis of M. By using the Clifford relations we have

$$\epsilon_\alpha \cdot \epsilon_\beta \cdot \Psi = (-1)^{\delta_{\alpha\beta}+1} \epsilon_\beta \cdot \epsilon_\alpha \cdot \Psi = \begin{cases} -\Psi, & \alpha = \beta \\ -\epsilon_\beta \cdot \epsilon_\alpha \cdot \Psi, & \alpha \neq \beta \end{cases} \tag{2.4}$$

for $\Psi \in \Gamma(\Sigma M)$.

Lemma 2.1. *$\mathcal{R}(\phi,\psi) \in \Gamma(\phi^{-1}TN)$; in particular, it is real.*

Proof. For any (not necessarily orthonormal) frame $\{\epsilon_i\}$ on $\phi^{-1}TN$, we put

$$\psi = \psi^a \otimes \epsilon_a, \tag{2.5}$$

$$(d\phi)^\sharp = \nabla\phi^a \otimes \epsilon_a, \qquad R^{\phi^{-1}TN}(\epsilon_a, \epsilon_b)\epsilon_c = R^d{}_{abc}\epsilon_d \tag{2.6}$$

where $\sharp : T^*M \otimes \phi^{-1}TN \to TM \otimes \phi^{-1}TN$ is the musical isomorphism as before. Take

$$\epsilon_a = u_a^i \frac{\partial}{\partial y^i},$$

then

$$\psi^i = u_a^i \psi^a, \qquad \nabla\phi^i = u_a^i \nabla\phi^a, \qquad u_a^j u_b^k u_c^l R^i{}_{jkl} = R^d{}_{abc} u_d^i.$$

A simple calculation gives following

$$R^i{}_{jkl}\langle\psi^k, \nabla\phi^j \cdot \psi^l\rangle \frac{\partial}{\partial y^i} = R^a{}_{bcd}\left(\phi(x)\right)\langle\psi^c, \nabla\phi^b \cdot \psi^d\rangle\epsilon_a\left(\phi(x)\right). \tag{2.7}$$

It follows that the definition of $\mathcal{R}(\phi,\psi)$ is independent of the choice of frame. It is then well-defined vector field on $\phi^{-1}TN$. On the other hand, from the skew-symmetry of $R^i{}_{jkl}$ with respect to the induces k and l, we have

$$\begin{aligned}\overline{\frac{1}{2}R^i{}_{jkl}\langle\psi^k, \nabla\phi^j \cdot \psi^l\rangle} &= \frac{1}{2}R^i{}_{jkl}\langle\nabla\phi^j \cdot \psi^l, \psi^k\rangle = \frac{1}{2}R^i{}_{jlk}\langle\nabla\phi^j \cdot \psi^k, \psi^l\rangle \\ &= -\frac{1}{2}R^i{}_{jkl}\langle\nabla\phi^j \cdot \psi^k, \psi^l\rangle = \frac{1}{2}R^i{}_{jkl}\langle\psi^k, \nabla\phi^j \cdot \psi^l\rangle.\end{aligned}$$

It follows that $\mathcal{R}(\phi,\psi) \in \Gamma(\phi^{-1}TN)$. □

A spinor (field) $\Psi \in \Gamma(\Sigma M)$ is called a *twistor spinor* if Ψ belongs to the kernel of the twistor operator, equivalently,

$$\nabla_X\Psi + \frac{1}{n}X \cdot \partial\!\!\!/\Psi = 0, \quad \forall X \in \Gamma(TM)$$

where we recall that n is the dimension of the Riemannian manifold M, ΣM is the associated spinor bundle of M and $\partial\!\!\!/$ is the usual Dirac operator (cf. [1], [8], [11]).

In fact the concept of a twistor spinor (in particular, a Killing spinor) is motivated by theories from physics, like general relativity, 11-dimensional (resp. 10-dimensional) supergravity theory, supersymmetry (see, for example [2], [3], [5]).

We establish the following Lemma 2.2 required in the proof of Proposition 1.3.

Lemma 2.2. *Let $\Phi \in \Gamma(\Sigma M)$. Then Φ is a twistor spinor if and only if*

$$\epsilon_1 \cdot \nabla_{\epsilon_1}\Phi = \cdots = \epsilon_n \cdot \nabla_{\epsilon_n}\Phi \tag{2.8}$$

for some orthonormal frame field ϵ_α of M.

Proof. Let us assume that (2.8) holds for some orthonormal frame field of ϵ_α. Hence we may set

$$\Psi := \epsilon_\alpha \cdot \nabla_{\epsilon_\alpha}\Phi. \tag{2.9}$$

Then

$$\partial\!\!\!/\Phi = \sum_\alpha \epsilon_\alpha \cdot \nabla_{\epsilon_\alpha}\Phi = \sum_\alpha \Psi = n\Psi$$

where $n = \dim M$. Together with (2.4) and (2.9) we have

$$\nabla_{\epsilon_\beta}\Phi = -\epsilon_\beta \cdot \epsilon_\beta \cdot \nabla_{\epsilon_\beta}\Phi = -\epsilon_\beta \cdot \Psi = -\epsilon_\beta \cdot \left(\frac{1}{n}\partial\!\!\!/\Phi\right) = -\frac{1}{n}\epsilon_\beta \cdot \partial\!\!\!/\Phi.$$

It follows that

$$\nabla_X \Phi + \frac{1}{n} X \cdot \partial\!\!\!/ \Phi = \nabla_{X^\alpha \epsilon_\alpha} \Phi + \frac{1}{n} \left(X^\alpha \epsilon_\alpha \cdot \partial\!\!\!/ \Phi \right) = X^\alpha \nabla_{\epsilon_\alpha} \Phi + \frac{1}{n} X^\alpha \left(\epsilon_\alpha \cdot \partial\!\!\!/ \Phi \right)$$
$$= X^\alpha \left(\nabla_{\epsilon_\alpha} \Phi + \frac{1}{n} \epsilon_\alpha \cdot \partial\!\!\!/ \Phi \right) = 0$$

for arbitrary $X = X^\alpha \epsilon_\alpha \in \Gamma(TM)$. Thus we see that Φ is a twistor spinor.

Conversely, if Φ is a twistor spinor, then the spinor field

$$X \cdot \nabla_X \Phi$$

does not depend on the unit vector field X [1, page 23, Theorem 2]. □

3. Dirac-harmonic maps along an isometric immersion

In this section, we are going to give some sufficient conditions for a Dirac-harmonic map along an isometric immersion to be trivial.

Let $\psi \in \Gamma(\Sigma M \otimes \phi^{-1}TN)$ be a spinor field along $\phi : M \to N$. We call ψ to be *harmonic* if $D\!\!\!/ \psi = 0$ [4].

Let $\phi : M \hookrightarrow N$ be an isometric immersion. This means that the Riemannian metric on M induced from the ambient space N coincides with the original one on M. We identify M with its immersed image in N. For each $x \in M$ the tangent space T_xN can be decomposed into a direct sum of T_xM and its orthogonal complement $T_x^\perp M$. Such a decomposition is differentiable. Thus, we have an orthogonal decomposition of the tangent bundle TN along M

$$TN|_M = \phi^{-1}TN = TM \oplus T^\perp M.$$

Let $(\dots)^T$ denote the orthogonal projection into the subbundle $\Sigma M \otimes TM$ from the twsited bundle $\Sigma M \otimes \phi^{-1}TN$.

For a global section $\mathcal{R}(\phi, \psi)$ on $\phi^{-1}TN$ (see Lemma 2.1), we have

$$\mathcal{R}(\phi, \psi) = \mathcal{R}^T(\phi, \psi) + \mathcal{R}^N(\phi, \psi)$$

where

$$\mathcal{R}^T(\phi, \psi) \in \Gamma(TM), \quad \mathcal{R}^N(\phi, \psi) \in \Gamma(T^\perp M).$$

Similarly, for $D\!\!\!/ \psi \in \Gamma(\Sigma M \otimes \phi^{-1}TN)$, we have

$$D\!\!\!/ \psi = D\!\!\!/^T \psi + D\!\!\!/^N \psi$$

where

$$\not{D}^T\psi \in \Gamma(\Sigma M \otimes TM), \qquad \not{D}^N\psi \in \Gamma(\Sigma M \otimes T^{\perp}M).$$

The mean curvature vector of M in N is

$$H = \frac{1}{n}\tau(\phi) \in \Gamma(T^{\perp}M)$$

where $\tau(\phi)$ is the tension field of the map ϕ. Hence we have the following:

Lemma 3.1. *Let $\phi : M \hookrightarrow N$ be an isometric immersion with the mean curvature vector ξ and $\psi \in \Gamma(\Sigma M \otimes \phi^{-1}TN)$. Then (ϕ, ψ) is a Dirac-harmonic map from M into N if and only if*

(i) $\mathcal{R}^T(\phi, \psi) = 0$;

(ii) $\mathcal{R}^N(\phi, \psi) = n\xi$ where $n = \dim M$;

(iii) $\not{D}^T\psi = 0$;

(iv) $\not{D}^N\psi = 0$.

We shall be using the following ranges of indices:

$$1 \le \alpha, \beta, \cdots \le n, \qquad n+1 \le s, t, \cdots \le n', \quad 1 \le i, j, \cdots \le n'.$$

Choose a local orthonormal frame field $\{\epsilon_i\}$ of $\phi^{-1}TN$ such that $\{\epsilon_\alpha\}$ lies in the tangent bundle TM and $\{\epsilon_s\}$ in the normal bundle $T^{\perp}M$ of M. We put

$$(d\phi)^\sharp = \nabla\phi^i \otimes \epsilon_i \tag{3.1}$$

where $^\sharp : T^*M \otimes \phi^{-1}TN \to TM \otimes \phi^{-1}TN$ is the musical isomorphism. By using (3.1) we have

$$\nabla\phi^i = \sum \delta^i_\alpha \epsilon_\alpha. \tag{3.2}$$

Now we assume that $N = N(c)$ is a Riemannian manifold of constant curvature c. Then the components of the Riemannian curvature tensor of N satisfy

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \tag{3.3}$$

Proof of Theorem 1.1. Let $\psi \in \Gamma(\Sigma M \otimes \phi^{-1}TN)$ be a spinor field along the isometric immersion ϕ and $\psi = \psi^i\epsilon_i$. From (3.2) and (3.3) we obtain

$$\begin{aligned} R_{ijkl}\langle \nabla\phi^k \cdot \psi^i, \nabla\phi^l \cdot \psi^j\rangle &= c\left[\langle \nabla\phi^i \cdot \psi^i, \nabla\phi^i \cdot \psi^j\rangle - \langle \nabla\phi^j \cdot \psi^i, \nabla\phi^i \cdot \psi^j\rangle\right] \\ &= c\left[\langle \epsilon_\alpha \cdot \psi^\alpha, \epsilon_\beta \cdot \psi^\beta\rangle - \langle \epsilon_\beta \cdot \psi^\alpha, \epsilon_\alpha \cdot \psi^\beta\rangle\right] \\ &= c\sum_{\alpha \ne \beta}\left[\langle \epsilon_\alpha \cdot \psi^\alpha, \epsilon_\beta \cdot \psi^\beta\rangle - \langle \epsilon_\beta \cdot \psi^\alpha, \epsilon_\alpha \cdot \psi^\beta\rangle\right]. \end{aligned} \tag{3.4}$$

By using the skew-symmetry relation of the Clifford product and the Clifford relation we have

$$\begin{aligned}\sum_{\alpha\neq\beta}\langle\epsilon_\beta\cdot\psi^\alpha,\epsilon_\alpha\cdot\psi^\beta\rangle &= -\sum_{\alpha\neq\beta}\langle\psi^\alpha,\epsilon_\beta\cdot\epsilon_\alpha\cdot\psi^\beta\rangle = \sum_{\alpha\neq\beta}\langle\psi^\alpha,\epsilon_\alpha\cdot\epsilon_\beta\cdot\psi^\beta\rangle \\ &= -\sum_{\alpha\neq\beta}\langle\epsilon_\alpha\cdot\psi^\alpha,\epsilon_\beta\cdot\psi^\beta\rangle.\end{aligned}$$

Plugging this into (3.4) yields

$$R_{ijkl}\langle\nabla\phi^k\cdot\psi^i,\nabla\phi^l\cdot\psi^j\rangle = 2c\sum_{\alpha\neq\beta}\langle\epsilon_\alpha\cdot\psi^\alpha,\epsilon_\beta\cdot\psi^\beta\rangle.$$

Now we assume

$$\psi^T = \Sigma_\alpha\epsilon_\alpha\cdot\Psi\otimes\phi_*(\epsilon_\alpha).$$

where $\Psi\in\Gamma(\Sigma M)$ is a spinor (field). It follows that $\psi^\alpha = \epsilon_\alpha\cdot\Psi$, and therefore

$$\begin{aligned}R_{ijkl}\langle\nabla\phi^k\cdot\psi^i,\nabla\phi^l\cdot\psi^j\rangle &= 2c\sum_{\alpha\neq\beta}\langle\epsilon_\alpha\cdot\epsilon_\alpha\cdot\Psi,\epsilon_\beta\cdot\epsilon_\beta\cdot\Psi\rangle \\ &= 2c\sum_{\alpha\neq\beta}\langle\Psi,\Psi\rangle = 2(n-1)nc|\Psi|^2.\end{aligned}\tag{3.5}$$

Assume that ψ is a harmonic spinor field along the isometric immersion ϕ. From Proposition 3.4 in [4], we have the following Bochner-type formula

$$\frac{1}{2}\Delta|\psi|^2 = |\tilde{\nabla}\psi|^2 + \frac{1}{4}R|\psi|^2 - \frac{1}{2}R_{ijkl}\langle\nabla\phi^k\cdot\psi^i,\nabla\phi^l\cdot\psi^j\rangle\tag{3.6}$$

where R is the scalar curvature of M. Substituting (3.5) into (3.6) yields

$$\frac{1}{2}\Delta|\psi|^2 = |\tilde{\nabla}\psi|^2 + \frac{1}{4}R|\psi|^2 - 2(n-1)nc|\Psi|^2.\tag{3.7}$$

Therefore, under the assumption $R>0$ and $c\leq 0$, (3.7) shows that $|\psi|^2$ is subharmonic on M. By the Hope maximum principle, we see that this function must be a constant and the right hand side of (3.7) must be zero. In particular $|\psi| = 0$. □

Proof of Proposition 1.2. Plugging (3.2) into (2.7) yields

$$\mathcal{R}(\phi,\psi) = \frac{1}{2}R^i{}_{\alpha kl}\,(x)\,\langle\psi^k,\epsilon_\alpha\cdot\psi^l\rangle\epsilon_i\,(x)\,.\tag{3.8}$$

From which together with (3.3) we obtain

$$\begin{aligned}\mathcal{R}(\phi,\psi) &= c(\delta^i{}_k\delta_{\alpha l} - \delta^i{}_l\delta_{\alpha k})\,\mathrm{Re}\langle\psi^k, \epsilon_\alpha\cdot\psi^l\rangle\epsilon_i\\ &= c\left[\mathrm{Re}\langle\psi^i,\epsilon_\alpha\cdot\psi^\alpha\rangle - \mathrm{Re}\langle\psi^\alpha,\epsilon_\alpha\cdot\psi^i\rangle\right]\epsilon_i = 2c\,\mathrm{Re}\langle\psi^i,\epsilon_\alpha\cdot\psi^\alpha\rangle\epsilon_i.\end{aligned}$$

It follows that

$$\begin{aligned}\mathcal{R}^N(\phi,\psi) &= 2c\,\mathrm{Re}\langle\psi^s,\epsilon_\alpha\cdot\psi^\alpha\rangle\epsilon_s = 2c\,\mathrm{Re}\langle\psi^s,\epsilon_\alpha\cdot\epsilon_\alpha\cdot\Psi\rangle\epsilon_s\\ &= -2nc\,\mathrm{Re}\langle\psi^s,\Psi\rangle\epsilon_s.\end{aligned}$$

Together with (ii) of Lemma 3.1, we obtain

$$-2c\,\mathrm{Re}\langle\psi^{n+1},\Psi\rangle = \xi \tag{3.9}$$

where ξ is the mean curvature of ϕ. Choose a local orthonormal frame field $\{\epsilon_\alpha\}$ near $x\in M$ with $\nabla_{\epsilon_\alpha}\epsilon_\beta|_x = 0$. By (2.5) we have

$$\begin{aligned}\not{D}\psi = \not{D}(\psi^i\otimes\epsilon_i) &= \epsilon_\alpha\cdot\tilde{\nabla}_{\epsilon_\alpha}(\psi^i\otimes\epsilon_i)\\ &= \epsilon_\alpha\cdot\left[(\nabla_{\epsilon_\alpha}\psi^i)\otimes\epsilon_i + \psi^i\otimes\nabla_{\epsilon_\alpha}\epsilon_i\right]\\ &= (\epsilon_\alpha\cdot\nabla_{\epsilon_\alpha}\psi^i)\otimes\epsilon_i + \epsilon_\alpha\cdot\left[\psi^\beta\otimes\nabla_{\epsilon_\alpha}\epsilon_\beta + \psi^s\otimes\nabla_{\epsilon_\alpha}\epsilon_s\right]\\ &= \not{\partial}\psi^i\otimes\epsilon_i + \epsilon_\alpha\cdot\psi^s\otimes\nabla_{\epsilon_\alpha}\epsilon_s\end{aligned}\tag{3.10}$$

at x.

Let A_ν be the shape operator and $\nabla^\perp_X$ the normal connection of M in N where X denotes a tangent vector of M and ν a normal vector to M. Then

$$\nabla_{\epsilon_\alpha}\epsilon_s = -A_{\epsilon_s}\epsilon_\alpha + \nabla^\perp_{\epsilon_\alpha}\epsilon_s. \tag{3.11}$$

Let B be the second fundamental form of M in N. Then B satisfies the Weingarten equation

$$\langle B(X,Y),\nu\rangle = \langle A_\nu(X),Y\rangle \tag{3.12}$$

where $X,Y\in\Gamma(TM)$. By using (3.11) and (3.12) we have

$$\nabla_{\epsilon_\alpha}\epsilon_s = -\langle B(\epsilon_\alpha,\epsilon_\beta),\epsilon_s\rangle\epsilon_\beta + \nabla^\perp_{\epsilon_\alpha}\epsilon_s. \tag{3.13}$$

By plugging (3.13) into (3.10) we obtain

$$\not{D}\psi = \not{\partial}\psi^i\otimes\epsilon_i - \langle B(\epsilon_\alpha,\epsilon_\beta),\epsilon_s\rangle\epsilon_\alpha\cdot\psi^s\otimes\epsilon_\beta + \epsilon_\alpha\cdot\psi^s\otimes\nabla^\perp_{\epsilon_\alpha}\epsilon_s. \tag{3.14}$$

Let ψ^T be defined by

$$\psi^T = \Sigma_\alpha \epsilon_\alpha \cdot \Psi \otimes \phi_*(\epsilon_\alpha).$$

Choose a local orthonormal frame field $\{\epsilon_\alpha\}$ near $x \in M$ with $\nabla_{\epsilon_\alpha}\epsilon_\beta|_x = 0$.

$$\begin{aligned}\partial\!\!\!/ \psi^\alpha &= \partial\!\!\!/(\epsilon_\alpha \cdot \Psi) = \epsilon_\beta \cdot \nabla_{\epsilon_\beta}(\epsilon_\alpha \cdot \Psi) = \epsilon_\beta \left[(\nabla_{\epsilon_\beta}\epsilon_\alpha)\cdot\Psi + \epsilon_\alpha \cdot \nabla_{\epsilon_\beta}\Psi\right] \\ &= \epsilon_\beta \cdot \epsilon_\alpha \cdot \nabla_{\epsilon_\beta}\Psi = -\nabla_{\epsilon_\alpha}\Psi - \sum_{\beta\neq\alpha} \epsilon_\alpha \cdot \epsilon_\beta \cdot \nabla_{\epsilon_\beta}\Psi = -2\nabla_{\epsilon_\alpha}\Psi - \epsilon_\alpha \cdot \partial\!\!\!/\Psi. \quad (3.15)\end{aligned}$$

Substituting (3.15) into (3.14) and taking the tangent projection yield

$$D\!\!\!\!/^T \psi = -\left[2\nabla_{\epsilon_\beta}\Psi + \epsilon_\beta \cdot \partial\!\!\!/\Psi + \langle B(\epsilon_\alpha, \epsilon_\beta), \epsilon_s\rangle \epsilon_\alpha \cdot \psi^s\right] \otimes \epsilon_\beta. \quad (3.16)$$

It is easy to see that

$$\langle B(\epsilon_\alpha, \epsilon_\beta), \epsilon_s\rangle \epsilon_\alpha \cdot \psi^s \otimes \epsilon_\beta$$

does not depend on the choice of $\{\epsilon_\alpha\}$. Since the normal bundle of M is flat, we choose $\{\epsilon_\alpha\}$ such that

$$\langle B(\epsilon_\alpha, \epsilon_\beta), \epsilon_s\rangle = \lambda^s_\alpha \delta_{\alpha\beta}. \quad (3.17)$$

Therefore we have

$$\sum_\beta \lambda^{n+1}_\beta = n\xi, \text{ and } \sum_\beta \lambda^s_\beta = 0 \quad \text{for } s \neq n+1. \quad (3.18)$$

Plugging (3.17) into (3.16) yields

$$D\!\!\!\!/^T \psi = -\sum_\beta \left[2\nabla_{\epsilon_\beta}\Psi + \epsilon_\beta \cdot \partial\!\!\!/\Psi + \sum_{\alpha,A} \lambda^A_\beta \epsilon_\beta \cdot \psi^A\right] \otimes \epsilon_\beta. \quad (3.19)$$

Thus $D\!\!\!\!/^T \psi = 0$ if and only if

$$2\nabla_{\epsilon_\beta}\Psi + \epsilon_\beta \cdot \partial\!\!\!/\Psi + \sum_A \lambda^A_\beta \epsilon_\beta \cdot \psi^A = 0 \quad (3.20)$$

for all β. By (2.4), (3.20) holds if and only if

$$2\epsilon_\beta \cdot \nabla_{\epsilon_\beta}\Psi - \partial\!\!\!/\Psi = \sum_s \lambda^s_\beta \psi^s. \quad (3.21)$$

Summing on β and using (3.18) we have

$$(2-n)\partial\!\!\!/\Psi = n\xi\psi^{n+1}.$$

Note that $n = \dim M = 2$. It follows that

$$\xi\psi^{n+1} = 0. \quad (3.22)$$

Suppose that $\xi(x) \neq 0$ for some $x \in M$, then (3.22) implies that $\psi^{n+1}(x) = 0$. Plugging this into (3.9) yields $\xi(x) = 0$ which is a contradiction and therefore $\xi \equiv 0$. □

Corollary 3.2. *Let $\phi : M \hookrightarrow N$ be an $n(\geq 3)$-dimensional submanifold in a Riemannian manifold of constant curvature c with flat normal bundle and let (ϕ, ψ) be a Dirac-harmonic map where*

$$\psi^T = \sum_{\alpha} \epsilon_\alpha \cdot \Psi \otimes \phi_*(\epsilon_\alpha)$$

for some $\Psi \in \Gamma(\Sigma M)$. Then ϕ is minimal if and only if Ψ is harmonic.

4. Dirac-harmonic maps from a Riemann surface

In this section, we extend Chen–Jost–Li–Wang' result and give a structure theorem of Dirac-harmonic maps from a Riemann surface.

Proof of Proposition 1.3. We claim that

$$\mathcal{R}(\phi, \psi_{\phi,\Psi}) \equiv 0, \qquad D\!\!\!/\psi_{\phi,\Psi} = -\Psi\otimes\tau(\phi) - 2\left(\nabla_{\epsilon_\alpha}\Psi + \frac{1}{2}\epsilon_\alpha \cdot \partial\!\!\!/\Psi\right)\otimes\phi_*(\epsilon_\alpha) \quad (4.1)$$

where ϵ_α $(\alpha = 1, 2)$, as always, is a local orthonormal basis of M.

In fact, we define local vector fields $\nabla\phi^i$ on M by

$$\nabla\phi^i := (d\phi)^\sharp(dy^i)$$

where $\{dy^i\}$ is the natural local dual basis on N. By using (1.1), we have

$$\psi^i := \psi_{\phi,\Psi}(dy^i) = \nabla\phi^i \cdot \Psi.$$

Set $d\phi = \phi^i_\alpha \theta^\alpha \otimes \frac{\partial}{\partial y^i}$ where θ^α is the dual basis for ϵ_α. Then $\nabla\phi^i = \sum \phi^i_\alpha \epsilon_\alpha$ and

$$\langle\psi^k, \nabla\phi^j \cdot \psi^l\rangle = \phi^k_\alpha \phi^j_\beta \phi^l_\gamma \langle\epsilon_\alpha \cdot \Psi, \epsilon_\beta \cdot \epsilon_\gamma \cdot \Psi\rangle.$$

Note that $\mathrm{Re}\langle\epsilon_\alpha \cdot \Psi, \epsilon_\beta \cdot \epsilon_\gamma \cdot \Psi\rangle = 0$ [10, Lemma 3.1]. We conclude that $R^i{}_{jkl}\langle\psi^k, \nabla\phi^j \cdot \psi^l\rangle$ is purely imaginary. On the other hand, from the proof of Lemma 2.1, $R^i{}_{jkl}\langle\psi^k, \nabla\phi^j \cdot \psi^l\rangle$ must be real, and hence

$$\mathcal{R}(\phi, \psi_{\phi,\Psi}) \equiv \frac{1}{2} R^i{}_{jkl}\langle\psi^k, \nabla\phi^j \cdot \psi^l\rangle \frac{\partial}{\partial y^i} \equiv 0.$$

By using (2.4) we have

$$\nabla_{\epsilon_\alpha}\Psi + \frac{1}{2}\epsilon_\alpha \cdot \partial\!\!\!/\Psi = \nabla_{\epsilon_\alpha}\Psi + \frac{1}{2}\epsilon_\alpha \cdot \left[\Sigma\epsilon_\beta \cdot \nabla_{\epsilon_\beta}\Psi\right]$$

$$= \begin{cases} \frac{1}{2}(\nabla_{\epsilon_1}\Psi + \epsilon_1 \cdot \epsilon_2 \cdot \nabla_{\epsilon_2}\Psi), & \alpha = 1 \\ \frac{1}{2}(\nabla_{\epsilon_2}\Psi - \epsilon_1 \cdot \epsilon_2 \cdot \nabla_{\epsilon_1}\Psi), & \alpha = 2 \end{cases} \tag{4.2}$$

We choose a local orthonormal frame field ϵ_α such that $\nabla_{\epsilon_\alpha}\epsilon_\beta = 0$ at $x \in M$. Then

$$\begin{aligned} \not{D}\psi_{\phi,\Psi} &= \epsilon_\beta \cdot \tilde{\nabla}_{\epsilon_\beta}\psi_{\phi,\Psi} = \epsilon_\beta \cdot \tilde{\nabla}_{\epsilon_\beta}\left(\epsilon_\alpha \cdot \Psi \otimes \phi_*(\epsilon_\alpha)\right) \\ &= \epsilon_\beta \cdot \left[\nabla_{\epsilon_\beta}(\epsilon_\alpha \cdot \Psi) \otimes \phi_*(\epsilon_\alpha) + \epsilon_\alpha \cdot \Psi \otimes \nabla_{\epsilon_\beta}(\phi_*(\epsilon_\alpha))\right] \\ &= \epsilon_\beta \cdot \left[((\nabla_{\epsilon_\beta}(\epsilon_\alpha) \cdot \Psi + \epsilon_\alpha \cdot \nabla_{\epsilon_\beta}\Psi) \otimes \phi_*(\epsilon_\alpha) + \epsilon_\alpha \cdot \Psi \otimes \nabla_{\epsilon_\beta}(\phi_*(\epsilon_\alpha))\right] \\ &= \epsilon_\beta \cdot \epsilon_\alpha \cdot \left\{\nabla_{\epsilon_\beta}\Psi \otimes \phi_*(\epsilon_\alpha) + \Psi \otimes \nabla_{\epsilon_\beta}(\phi_*(\epsilon_\alpha))\right\} \\ &= (\Sigma_{\alpha=\beta} + \Sigma_{\alpha\neq\beta})\epsilon_\beta \cdot \epsilon_\alpha \cdot \left\{\nabla_{\epsilon_\beta}\Psi \otimes \phi_*(\epsilon_\alpha) + \Psi \otimes \nabla_{\epsilon_\beta}(\phi_*(\epsilon_\alpha))\right\} \\ &= (I) + (II) \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} (I) &= \epsilon_\alpha \cdot \epsilon_\alpha \cdot \{\nabla_{\epsilon_\alpha}\Psi \otimes \phi_*(\epsilon_\alpha) + \Psi \otimes \nabla_{\epsilon_\alpha}(\phi_*(\epsilon_\alpha))\} \\ &= -\{\nabla_{\epsilon_\alpha}\Psi \otimes \phi_*(\epsilon_\alpha) + \Psi \otimes [\nabla_{\epsilon_\alpha}(\phi_*(\epsilon_\alpha)) - \phi_*(\nabla_{\epsilon_\alpha}(\phi_*(\epsilon_\alpha))]\} \\ &= -\{\nabla_{\epsilon_\alpha}\Psi \otimes \phi_*(\epsilon_\alpha) + \Psi \otimes \tau(\phi)\} \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} (II) &= \epsilon_1 \cdot \epsilon_2 \cdot \left\{\nabla_{\epsilon_1}\Psi \otimes \phi_*(\epsilon_2) + \Psi \otimes \nabla_{\epsilon_1}(\phi_*(\epsilon_2))\right\} \\ &\quad + \epsilon_2 \cdot \epsilon_1 \cdot \left\{\nabla_{\epsilon_2}\Psi \otimes \phi_*(\epsilon_1) + \Psi \otimes \nabla_{\epsilon_2}(\phi_*(\epsilon_1))\right\} \\ &= \epsilon_1 \cdot \epsilon_2 \cdot \left\{\nabla_{\epsilon_1}\Psi \otimes \phi_*(\epsilon_2) - \nabla_{\epsilon_2}\Psi \otimes \phi_*(\epsilon_1) + \Psi \otimes \nabla_{\epsilon_1}(\phi_*(\epsilon_2)) \right. \\ &\quad \left. - \Psi \otimes \nabla_{\epsilon_2}(\phi_*(\epsilon_1))\right\} \\ &= \epsilon_1 \cdot \epsilon_2 \cdot \left\{\nabla_{\epsilon_1}\Psi \otimes \phi_*(\epsilon_2) - \nabla_{\epsilon_2}\Psi \otimes \phi_*(\epsilon_1)\right\} \end{aligned} \tag{4.5}$$

here we have used the following

$$\nabla_{\epsilon_1}(\phi_*(\epsilon_2)) = (\nabla_{\epsilon_1}\phi_*)(\epsilon_2) = (\nabla_{\epsilon_2}\phi_*)(\epsilon_1) = \nabla_{\epsilon_2}(\phi_*(\epsilon_1)).$$

Substituting (4.4) and (4.5) into (4.3) yields

$$\begin{aligned} \not{D}\psi_{\phi,\Psi} &= -\{\nabla_{\epsilon_\alpha}\Psi \otimes \phi_*(\epsilon_\alpha) + \Psi \otimes \tau(\phi)\} \\ &\quad + \epsilon_1 \cdot \epsilon_2 \cdot \{\nabla_{\epsilon_1}\Psi \otimes \phi_*(\epsilon_2) - \nabla_{\epsilon_2}\Psi \otimes \phi_*(\epsilon_1)\} \\ &= -\Psi \otimes \tau(\phi) - (\nabla_{\epsilon_1}\Psi + \epsilon_1 \cdot \epsilon_2 \cdot \nabla_{\epsilon_2}\Psi) \otimes \phi_*(\epsilon_1) \\ &\quad + (\epsilon_1 \cdot \epsilon_2 \cdot \nabla_{\epsilon_1}\Psi - \nabla_{\epsilon_2}\Psi) \otimes \phi_*(\epsilon_2). \end{aligned} \tag{4.6}$$

Plugging (4.2) into (4.6) yields the second equation of (4.1).

By using the Clifford relation, ones obtain

$$\nabla_{\epsilon_1}\Psi + \frac{1}{2}\epsilon_1 \cdot \partial\!\!\!/\Psi = \frac{1}{2}(\nabla_{\epsilon_1}\Psi + \epsilon_1 \cdot \epsilon_2 \cdot \nabla_{\epsilon_2}\Psi) = \frac{1}{2}\epsilon_1 \cdot \Phi \tag{4.7}$$

where

$$\Phi := -\epsilon_1 \cdot \nabla_{\epsilon_1}\Psi + \epsilon_2 \cdot \nabla_{\epsilon_2}\Psi.$$

We recall that $\Phi = 0$ if and only if Ψ is a twistor spinor, equivalently, Ψ belongs to the kernel of the twistor operator (cf. Lemma 2.2). Similarly, we have

$$\nabla_{\epsilon_2}\Psi + \frac{1}{2}\epsilon_2 \cdot \partial\!\!\!/\Psi = \frac{1}{2}(\nabla_{\epsilon_2}\Psi + \epsilon_2 \cdot \epsilon_1 \cdot \nabla_{\epsilon_1}\Psi) = -\frac{1}{2}\epsilon_2 \cdot \Phi. \tag{4.8}$$

Plugging (4.7) and (4.8) into (4.1) yields

$$D\!\!\!/\psi_{\phi,\Psi} = -\Psi \otimes \tau(\phi) + \frac{1}{2}\epsilon_1 \cdot \Phi \otimes \phi_*(\epsilon_1) - \frac{1}{2}\epsilon_2 \cdot \Phi \otimes \phi_*(\epsilon_2). \tag{4.9}$$

Note that $(\phi, \psi_{\phi,\Psi})$ is a Dirac-harmonic map, i.e.

$$\tau(\phi) = \mathcal{R}(\phi, \psi_{\phi,\Psi}), \tag{4.10}$$

$$D\!\!\!/\psi_{\phi,\Psi} = 0. \tag{4.11}$$

(4.1) and (4.10) imply that

$$\tau(\phi) = 0. \tag{4.12}$$

Hence ϕ is a harmonic map, equivalently, it is a branched minimal immersion. Substituting (4.12) into (4.9) and using (4.11) yield

$$\epsilon_1 \cdot \Phi \otimes \phi_*(\epsilon_1) - \epsilon_2 \cdot \Phi \otimes \phi_*(\epsilon_2) = 0. \tag{4.13}$$

Since $\phi : (M, g) \to (N, h)$ is conformal, we can assume that $\phi^* h = e^{\lambda} g$. It follows that

$$h(\phi_*(\epsilon_\alpha), \phi_*(\epsilon_\beta)) = \delta_{\alpha\beta} e^{\lambda}. \tag{4.14}$$

Note that ϕ is non-constant, there exists an α such that $\phi_*(\epsilon_\alpha) \neq 0$. Without loss of generality, we assume $\phi_*(\epsilon_1) \neq 0$. From (4.13) and (4.14) we have $\epsilon_1 \cdot \Phi = 0$. It follows that

$$\Phi = -\epsilon_1 \cdot (\epsilon_1 \cdot \Phi) = 0.$$

Thus Ψ is a twistor spinor. □

Remark. Note that the Dirac-harmonicity of $(\phi, \psi_{\phi,\Psi})$ implies the harmonicity of ϕ and any harmonic map from a sphere is conformal. Hence Proposition 1.3 is a natural generalization of Proposition 2.2 of [4].

Proof of Theorem 1.4. We only consider the case that $g > 0$ and $\deg \phi > g-1$ where g is the genus of compact Riemann surface M. Let (ϕ, ψ) is a Dirac-harmonic map from a compact Riemann surface M_g of genus g to the sphere M_0 and ϕ is non-constant. By using Theorem 1.1 in [15], ϕ is a harmonic map. Note that M_0 is homeomorphic to $S^2 = \mathbf{C}\mathrm{P}^1$ and ϕ is a non-constant map. Hence ϕ is linearly full into $\mathbf{C}\mathrm{P}^1$. By using Liao's result ϕ is isotropic [13, Corollary 1]. Recall that isotropic harmonic maps are generated from holomorphic maps by a process of taking derivatives. Therefore ϕ is $\pm$holomorphic for $n = 1$. Consider the Fubini–Study metric on $\mathbf{C}\mathrm{P}^1$ with the constant holomorphic sectional curvature 4. The degree of ϕ can be computed as follows [7], [14]

$$\deg(\phi) = \frac{1}{\pi}\left[E'(\phi) - E''(\phi)\right]$$

where $E'(\phi)$ (resp. $E''(\phi)$) is the holomorphic (resp. anti-holomorphic) energy of ϕ. If ϕ is anti-holomorphic, then $E'(\phi) = 0$. It follows that

$$0 \le g-1 < \deg(\phi) = -\frac{E''(\phi)}{\pi}$$

Thus $E''(\phi) \le 0$. Hence ϕ is also holomorphic. We conclude that ϕ is constant which is a contradiction.

The twisted bundle $\Sigma M_g \otimes \phi^{-1}TM_0$ can be divided into the following

$$\begin{aligned}\Sigma M_g \otimes \phi^{-1}TM_0 &= \Sigma M_g \otimes (\phi^{-1}TM_0)^{\mathbb{C}} \\ &= (\Sigma^+ M_g \otimes \phi^{-1}T'M_0) \oplus (\Sigma^+ M_g \otimes \phi^{-1}T''M_0) \\ &\quad \oplus (\Sigma^- M_g \otimes \phi^{-1}T'M_0) \oplus (\Sigma^- M_g \otimes \phi^{-1}T''M_0) \qquad (4.15)\end{aligned}$$

where

$$\Sigma^{\pm} M_g := \left\{\Psi \in \Sigma M_g | \sqrt{-1}\epsilon_1 \cdot \epsilon_2 \cdot \Psi = \pm\Psi\right\} \qquad (4.16)$$

for some orthonormal frame field ϵ_α of M_g and $T'M_0$ (resp. $T''M_0$) denote the tangent bundle of M_0 of type $(1,0)$ (resp. (0,1)). Denote by π^+ (resp. π^-) the projection of the twisted bundle $\Sigma M_g \otimes \phi^{-1}TM_0$ onto the subbundle $\Sigma^+ M_g \otimes \phi^{-1}T''M_0$ (resp. $\Sigma^- M_g \otimes \phi^{-1}T'M_0$). Let m denote the sum of the multiplicaties of the zeros of the function $|\pi^+(\psi)|$. If $|\pi^+(\psi)|$ is not identically zero, then (cf. [15, Theorem 4.2])

$$m = g - 1 - 2\deg(\phi).$$

Note that

$$g - 1 - 2\deg(\phi) < g - 1 - 2(g-1) = 1 - g \le 0.$$

It follows that $m \le -1$ which is a contradiction, therefore

$$|\pi^{+}(\psi)| \equiv 0. \tag{4.17}$$

Similarly we have

$$|\pi^{-}(\psi)| \equiv 0. \tag{4.18}$$

For non-constant holomorphic map ϕ

$$\phi^{-1}T'M_0 = \mathrm{Span}\{\phi_*(\epsilon_1) - \sqrt{-1}\phi_*(\epsilon_2)\},$$

$$\phi^{-1}T''M_0 = \mathrm{Span}\{\phi_*(\epsilon_1) + \sqrt{-1}\phi_*(\epsilon_2)\}.$$

We write

$$\Sigma^{\pm}M_g = \mathrm{Span}\{\psi^{\pm}\}$$

where

$$\psi^{-} = \epsilon_1 \cdot \psi^{+}. \tag{4.19}$$

By using (4.15), (4.17) and (4.18), we have

$$\psi = f\psi^{+} \otimes \left[\phi_*(\epsilon_1) - \sqrt{-1}\phi_*(\epsilon_2)\right] + g\psi^{-} \otimes \left[\phi_*(\epsilon_1) + \sqrt{-1}\phi_*(\epsilon_2)\right]. \tag{4.20}$$

From (4.16), (4.19) and the Clifford relation ones obtain

$$\psi^{+} = -\epsilon_1 \cdot \psi^{-} = -\sqrt{-1}\epsilon_2 \cdot \psi^{-}, \tag{4.21}$$

$$\psi^{-} = \epsilon_1 \cdot \psi^{+} = -\sqrt{-1}\epsilon_2 \cdot \psi^{+}. \tag{4.22}$$

Plugging (4.21) and (4.22) into (4.20) yields

$$\psi = \Sigma_\alpha \epsilon_\alpha \cdot \Psi \otimes \phi_*(\epsilon_\alpha)$$

where $\Psi = g\psi^{+} - f\psi^{-}$. Note that arbitrary isotropic harmonic map is conformal. From Proposition 1.3, Ψ is a twistor spinor. □

In particular, we have the following

Corollary 4.1. *Let (ϕ, ψ) is a non-constant Dirac-harmonic map from a torus T^2 to a sphere with non-zero degree. Then ϕ is $\pm$ holomorphic, and ψ could be written in the form*

$$\psi = \Sigma_\alpha \epsilon_\alpha \cdot \Psi \otimes \phi_*(\epsilon_\alpha)$$

where ϵ_α $(\alpha = 1, 2)$ is a local orthonormal basis of T^2, and Ψ is a twistor spinor.

We have several special case of Theorem 1.4.

(1) When $g = 0$, our corollary have been given by Yang Ling [15];

(2) When $\psi = 0$, our result is reduced to Liao's isotropy work [13, Corollary 1].

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(Received July 21, 2009; revised March 4, 2010)