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Criteria for laws between infinite subsets of infinite groups

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Abstract. A theorem of B. H. Neumman shows that infinite group in which every two infinite subsets there exist two commuting elements, is abelian.

In this paper, we prove that if in an infinite group G, every two infinite subsets X and Y, there exist $a \in X$ and $b \in Y$ such that $[a^{n_1}, b^{n_2}] = 1$, then G satisfies the law $[x^{n_1}, y^{n_2}] = 1$, where $n_2 \equiv 0[n_1]$ and $n_2 \in \{3, 6, 2^k/k \in \mathbb{N}^*\}$.

Moreover, and using this result, we also prove that an infinite group satisfies the law $(x_1^{n_1} x_2^{n_2} \dots x_r^{n_r})^2 = 1$ if and only if in any r infinite subsets $X_1, \dots, X_r$, of G there exist $a_i \in X_i (i = 1, \dots, r)$ such that $(a_1^{n_1} \dots a_r^{n_r})^2 = 1$, where $n_1, \dots, n_r \in \{2^k/k \in \mathbb{N}^*\}$ and $r \geq 2$.

1. Introduction and results

Let $w(x_1, \dots, x_r)$ be a word in a free group of rank $r \geq 1$ and let $\mathcal{V}(w)$ be the variety of groups defined by the law $w(x_1, \dots, x_r) = 1$. We define the class of groups $\mathcal{V}(w^*)$ as follows:

$$\text{A group } G \text{ belongs to } \mathcal{V}(w^*) \text{ if and only if in every infinite subsets}$$
$$X_1, \dots, X_r \text{ there exist } a_1 \in X_1, \dots, a_r \in X_r, \text{ that } w(a_1, \dots, a_r) = 1$$

In [7], P. Longobardi et al. posed the question of whether $\mathcal{F} \cup \mathcal{V}(w) = \mathcal{V}(w^*)$ is true; $\mathcal{F}$ being the class of finite groups. As an immediate consequence of the answer of B. H Neumman to the question of P. Erdős [9], we have $\mathcal{F} \cup \mathcal{V}(w) = \mathcal{V}(w^*)$, where $w(x, y) = [x, y]$. Further questions of similar nature, with slightly different aspects, have been considered by many authors (see, for example, [1], [2], [3], [4], [13], [11], [12], [7], [8]).

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In [2], A. Abdollahi and B. Taeri proved that an infinite group G satisfies the law $[x, y^n] = 1$ if and only if in any pair X, Y of infinite subsets of G there exist $a \in X$ and $b \in Y$ such that $[a, b^n] = 1$, where $n \in \{2^k/k \in \mathbb{N}^*\}$. Later A. Boukaroura [4] proved that an infinite group G satisfies the law $[x^n, y^n] = 1$, if and only if in any pair X, Y of infinite subsets of G there exist $a \in X$ and $b \in Y$ such that $[a^n, b^n] = 1$, where $n \in \{2^k/k \in \mathbb{N}^*\}$.

In this note, we consider the words:

$$u_{n_1,n_2}(x, y) = [x^{n_1}, y^{n_2}] \quad \text{and} \quad w_{n,n_1,\dots,n_r}(x_1, \dots, x_r) = (x_1^{n_1} \dots x_r^{n_r})^n.$$

The following theorem generalizes Theorem 1 of [1] and Theorem 1.1 of [4].

Theorem 1. *If $n_2 \equiv 0[n_1]$ and $n_2 \in \{3, 6, 2^k/k \in \mathbb{N}^*\}$, then every infinite $\mathcal{V}(u_{n_1,n_2}(x, y)^*)$-group is a $\mathcal{V}(u_{n_1,n_2}(x, y))$-group.*

Using Theorem 1, we also prove the following result:

Theorem 2. *If $n_1, \dots, , n_r \in \{2^k/k \in \mathbb{N}^*\}$, then every infinite $\mathcal{V}(w_{2,n_1,\dots,n_r}(x_1, \dots, x_r)^*)$-group is a $\mathcal{V}(w_{2,n_1,\dots,n_r}(x_1, \dots, x_r))$-group.*

2. Notations

Our notations and terminologies are usual and can be found in [10]. We give a partial list for the convenience of the reader.

$\langle X \rangle$	the subgroup generated by X
x^y	the conjugate $y^{-1}xy$
X^y	the set $\{x^y/x \in X\}$
$[x, y]$	the commutator $x^{-1}y^{-1}xy$
$\mathcal{B}_e$	the Burnside variety of exponent dividing e

$Z(G)$ and $C_G(x)$ denote respectively the centre of the group G and the centralizer of x in G. The following commutator identities will be used frequently without special reference. $[xy, z] = [x, z]^y[y, z]$, $[x, yz] = [x, z][x, y]^z$,

3. Proofs

Lemma 3 ([6], Lemma 3)**.** *Let w be a word in a free group and let G be an infinite $\mathcal{V}(w^*)$-group. If G has an infinite normal abelian subgroup, then G is a $\mathcal{V}(w)$-group.*

Lemma 4. *Let n_1, n_2 be any integer such that $n_2 \equiv 0\,[n_1]$. If G is an infinite $\mathcal{V}(u_{n_1,n_2}(x,y)^*)$-group, then G satisfies the following properties:*

i) *$C_G(g^{n_2})$ is infinite, for any $g \in G$.*

ii) *If $n_2 \in \{3, 6, 2^k/k \in \mathbb{N}^*\}$, G has an infinite abelian subgroup.*

Proof. Let n_1, n_2 be any integer such that $n_2 \equiv 0[n_1]$ and let G be an infinite $\mathcal{V}(u_{n_1,n_2}(x,y)^*)$-group.

i) Let g be any element of G. If X and Y are two infinite subsets of G, then there exist $a \in X$ and $b \in Y$ such that $[a^{n_1}, b^{n_2}] = 1$. Clearly, we may assume that $n_2 = kn_1$ ($k \in \mathbb{N}$), so $[a^{n_2}, b^{n_2}] = [(a^{n_1})^k, b^{n_2}] = 1$. Therefore, G is an infinite $\mathcal{V}([x^{n_2}, y^{n_2}]^*)$-group. Thus, by Lemma 2.1 of [4], $C_G(g^{n_2})$ is infinite for any $g \in G$.

ii) As in the proof of i), G is a $\mathcal{V}([x^{n_2}, y^{n_2}]^*)$-group. Then, by Lemma 2.3 of [4], G has an infinite abelian subgroup. $\square$

The next lemma is a special case of Lemma 1 of [2].

Lemma 5. *If G be an infinite $\mathcal{V}(u_{n_1,n_2}(x,y)^*)$-group and A be an infinite abelian subgroup of G, then for any $g \in G$,*

$$A(g) = \{a \in A/[g^{n_1}, a^{n_2}] = 1\}$$

is infinite.

Proof of Theorem 1. Let G be an infinite group such that, for every pair X, Y of infinite subsets of G, there exist a in X, b in Y satisfying $u_{n_1,n_2}(a,b) = 1$. Let $g_1, g_2 \in G$ be arbitrary two elements of G. By i) and ii) in Lemma 4, $C_G(g_2^{n_2})$ is an infinite $\mathcal{V}(u_{n_1,n_2}(x,y)^*)$-group and contains an infinite abelian subgroup noted A. In order to show that $[g_1^{n_1}, g_2^{n_2}] = 1$, we consider two cases:

Case 1. $C_A(g_1)$ is infinite. If $C = C_A(g_2)$ is infinite, by Lemma 5, $C(g_1) = \{a \in C/[g_1^{n_1}, a^{n_2}] = 1\}$ is infinite. Consider the infinite subsets $g_1C_A(g_1)$ and $g_2C(g_1)$. Then there exist $a_1 \in C_A(g_1)$ and $a_2 \in C(g_1)$ such that $[(g_1a_1)^{n_1}, (g_2a_2)^{n_2}] = 1$. But

$$[(g_1a_1)^{n_1}, (g_2a_2)^{n_2}] = [g_1^{n_1}a_1^{n_1}, g_2^{n_2}a_2^{n_2}] = [g_1^{n_1}, {a_2}^{n_2}]\,[g_1^{n_1}, g_2^{n_2}]^{{a_2}^{n_2}} = [g_1^{n_1}, g_2^{n_2}]^{{a_2}^{n_2}}$$

Then $[g_1^{n_1}, g_2^{n_2}] = 1$. If $C = C_A(g_2)$ is finite, then $g_2^A = \{g_2^a/a \in A\}$ is infinite. Consider the infinite subsets $g_1C_A(g_1)$ and $g_2^A = \{g_2^a/a \in A\}$, then there exist $a_1 \in C_A(g_1)$ and $a_2 \in A$ such that $[(g_1a_1)^{n_1}, (g_2^{a_2})^{n_2}] = 1$. But

$$[(g_1a_1)^{n_1}, (g_2^{a_2})^{n_2}] = [g_1^{n_1}a_1^{n_1}, (g_2^{n_2})^{a_2}] = [g_1^{n_1}, g_2^{n_2}]^{a_1^{n_1}} = 1$$

Then $[g_1^{n_1}, g_2^{n_2}] = 1$.

Case 2. $C_A(g_1)$ is finite. If $C = C_A(g_2)$ is infinite, by Lemma 5, $C(g_1) = \{a \in C/[g_1^{n_1}, a^{n_2}] = 1\}$ is infinite. Consider the infinite subsets $g_1^A = \{g_1^a/a \in A\}$ and $g_2C(g_1)$. Then there exist $a_1 \in A$ and $a_2 \in C(g_1)$ such that $[(g_1^{a_1})^{n_1}, (g_2a_2)^{n_2}] = 1$. But

$$[(g_1^{a_1})^{n_1}, (g_2a_2)^{n_2}] = [(g_1^{n_1})^{a_1}, g_2^{n_2}{a_2}^{n_2}] = [g_1^{n_1}, {a_2}^{n_2}]\,[g_1^{n_1}, g_2^{n_2}]^{{a_2}^{n_2}} = [g_1^{n_1}, g_2^{n_2}]^{{a_2}^{n_2}}$$

Then $[g_1^{n_1}, g_2^{n_2}] = 1$. If $C = C_A(g_2)$ is finite. Consider the infinite subsets $g_1^A = \{g_1^a/a \in A\}$ and $g_2^A = \{g_2^a/a \in A\}$. Then there exist $a_1, a_2 \in A$ such that $[(g_1^{a_1})^{n_1}, (g_2^{a_2})^{n_2}] = 1$. But

$$[(g_1^{a_1})^{n_1}, (g_2^{a_2})^{n_2}] = [(g_1^{n_1})^{a_1}, (g_2^{n_2})^{a_2}] = [(g_1^{n_1})^{a_1}, g_2^{n_2}] = [g_1^{n_1}, g_2^{n_2}]^{a_1}$$

Then $[g_1^{n_1}, g_2^{n_2}] = 1$. $\square$

Lemma 6 ([3])**.** *If $w(x_1, \ldots, x_r) = x_1^{n_1} \ldots x_r^{n_r}$, where each n_i is a non-zero integer, then every infinite $\mathcal{V}(w(x_1, \ldots, x_r)^*)$-group belongs to the variety $\mathcal{V}(w(x_1, \ldots, x_r)) = \mathcal{B}_d$, where $d = \gcd_{1\leq i\leq r}(n_i)$.*

Lemma 7. *If $d \equiv 0[n]$, where $d = \gcd_{1\leq i\leq r}(n_i)$. Then every infinite $\mathcal{V}(w_{n,n_1,\ldots,n_r}(x_1, \ldots, x_r)^*)$-group is a $\mathcal{B}_{nd}$-group.*

Proof. Let G be an infinite $\mathcal{V}(w_{n,n_1,\ldots,n_r}(x_1, \ldots, x_r)^*)$-group and let $d = \gcd_{1\leq i\leq r}(n_i)$ and suppose that $d \equiv 0[n]$. Put $X = \{g \in G/g^d = 1\}$.

If X is infinite. For any $t(1 \leq t \leq r)$, let Y be an infinite subset of G and consider the infinite subsets $X_1 = \cdots = X_{t-1} = X$, $X_t = Y, X_{t+1} = \cdots = X_r = X$. Then there exist $a \in Y$ such that $a^{nn_t} = 1$. So G is $\mathcal{V}((x^{nn_t})^*)$-group, and by Lemma 6, G is $\mathcal{B}_{nn_t}$-group. Therefore G is $\mathcal{B}_{nd}$-group, where $d = \gcd_{1\leq i\leq r}(n_i)$.

If X is finite, by Dicman's Lemma (see Lemma 14.5.7 of [10]), $N = \langle X\rangle$ is a finite normal subgroup of G, and $\frac{G}{N}$ is $\mathcal{V}((x_1^{n_1} \ldots x_r^{n_r})^*)$-group. Hence $\frac{G}{N}$ is $\mathcal{B}_d$-group so that $G^d \leq N$, and therefore G^d is finite. Let $C = C_G(G^d)$, obviously $C_{1\leq i\leq r} = \cap C_G(G^{n_i})$ and C is infinite. If $C_1, C_2, \ldots, C_r$ are infinite subsets of C, then there exist $c_i \in C_i$ such that $(c_1^{n_1} \ldots c_r^{n_r})^n = c_1^{nn_1} \ldots c_r^{nn_r} = 1$. Therefore C is $\mathcal{B}_{nd}$-group. For any $t(1 \leq t \leq r)$, let Y be an infinite subset of G and consider the infinite subsets $X_1 = \cdots = X_{t-1} = C$, $X_t = Y, X_{t+1} = \cdots = X_r = C$. Then there exist $a \in Y$ such that $a^{nn_t} = 1$. So G is a $\mathcal{V}((x^{nn_t})^*)$-group, and by Lemma 6, G is $\mathcal{B}_{nn_t}$-group. Therefore G is a $\mathcal{B}_{nd}$-group, where $d = \gcd_{1\leq i\leq r}(n_i)$. $\square$

Lemma 8. *If $n_1, \ldots, n_r \in \{2^k/k \in \mathbb{N}^*\}$. Then every infinite $\mathcal{V}(w_{2,n_1,\ldots,n_r}(x_1, \ldots, x_r)^*)$-group is a $\mathcal{V}([x^{n_k}, y^{n_l}])$-group, for any $k, l(1 \leq k \leq r, 1 \leq l \leq r)$.*

Proof. Let k and l be a number such that $1 \leq k \leq l \leq r$ and put $X = \{g \in G/g^d = 1\}$, where $d = \gcd_{1 \leq i \leq r}(n_i) = \mathrm{Min}_{1 \leq i \leq r}(n_i)$.

If X is infinite, let U and V be two infinite subsets of G, and consider the infinite subsets $X = X_1 = \cdots = X_{k-1}, U, X = X_{k+1} = \cdots = X_{l-1}, V, X = X_{l+1} = \cdots = X_r$. Then there exist u in U and v in V such that $(u^{n_k} v^{n_l})^2 = 1$. It follows that $[u^{n_k}, v^{n_l}] = 1$ since G is $\mathcal{B}_{2n_k} \cap \mathcal{B}_{2n_l}$-group. Thus, G is $\mathcal{V}([x^{n_i}, y^{n_j}]^*)$-group, and by Theorem 1, G is $\mathcal{V}([x^{n_i}, y^{n_j}])$-group.

If X is finite, let $C_{1 \leq i \leq r} = \cap C_G(G^{n_i})$, by proof of Lemma 7, C is infinite. Consider now two arbitrary infinite subsets U and V of G and let $C = X_1 = \cdots = X_{k-1}, U, C = X_{k+1} = \cdots = X_{l-1}, V, C = X_{l+1} = \cdots = X_r$. Then there exist $c_1, \ldots, c_{k-1}, c_{k+1}, \ldots, c_{l-1}, c_{l+1}, \ldots, c_r \in C$ and $u \in U$, $v \in V$ such that

$$\left(c_1^{n_1} \ldots c_{k-1}^{n_{k-1}} u^{n_k} c_{k+1}^{n_k+1} \ldots c_{l-1}^{n_{l-1}} v^{n_l} c_{l+1}^{n_{l+1}} \ldots c_r^{n_r}\right)^2 = 1$$

It follows that

$$\begin{aligned} c_1^{n_1} \ldots c_{k-1}^{n_{k-1}} u^{n_k} c_{k+1}^{n_{k+1}} \ldots c_{l-1}^{n_{l-1}} v^{n_l} c_{l+1}^{n_{l+1}} \ldots c_r^{n_r} \\ = c_r^{n_r} \ldots c_{l+1}^{n_{l+1}} v^{n_l} c_{l-1}^{n_{l-1}} \ldots c_{k+1}^{n_{k+1}} u^{n_k} c_{k-1}^{n_{k-1}} \ldots c_1^{n_1} \end{aligned}$$

Then $u^{n_k} v^{n_l} = v^{n_l} u^{n_k}$ and G is $\mathcal{V}([x^{n_k}, y^{n_j}]^*)$-group, and by Theorem 1, G is $\mathcal{V}([x^{n_k}, y^{n_l}])$-group. □

Proof of Theorem 2. Let G be an infinite $\mathcal{V}(w_{2,n_1,\ldots,n_r}(x_1, \ldots, x_r)^*)$-group and let $a_1, \ldots, a_r$ be any element of G. By Lemma 8, G is a $\mathcal{V}([x^{n_k}, y^{n_l}])$-group for any $k, l (1 \leq k \leq r,\ 1 \leq l \leq r)$, so $(a_1^{n_1} \ldots a_r^{n_r})^2 = a_1^{2n_1} \ldots a_r^{2n_r}$. And, by Lemma 7, G is a $\mathcal{B}_{2n_i}$-group, for any i; $1 \leq i \leq r$. Hence $(a_1^{n_1} \ldots a_r^{n_r})^2 = 1$, and G is a $\mathcal{V}(w_{2,n_1,\ldots,n_r}(x_1, \ldots, x_r))$-group. □

References

[1] A. Abdollahi, A characterization of 3-abelian groups, *Arch. Math. (Basel)* **73** (1999), 104–108.

[2] A. Abdollahi and B. Taeri, Some conditions on infinite subsets of infinite groups, *Bull. Malaysian Math. Soc.* (2) **22**, no. 1 (1999), 87–93.

[3] A. Abdollahi and and B. Taeri, A condition on a certain variety of groups, *Rend. Sem. Mat. Univ. Padova* **104** (2000), 129–134.

[4] A. Boukaroura, Two conditions for infinite groups to satisfy certain law, *Algebra Colloq.* **10**, no. 1 (2003), 75–80.

[5] G. Endimioni, Groups in which certain equations have many solutions, *Rend. Sem. Mat. Univ. Padova* **106** (2001), 77–82.

[6] G. Endimioni, A Combinatorial problem in varieties of groups, *Comm. in Algebra* **23** (1995), 5297–5307.

[7] P. Longobardi, M. Maj and A. H. Rhemtulla, Infinite groups in a given variety and Ramsey's theorem, *Comm. in Algebra* **20** (1992), 127–139.

[8] P. Longobardi and M. Maj, A finiteness condition concerning commutators in groups, *Houston J. Math.* **19** (1993), 505–512.

[9] B. H. Neumann, A problem of Paul Erdős on groups, *J. Austral. Math. Soc.* (Ser. A) **21** (1976), 467–472.

[10] D. J. S. Robinson, A Course in the Theory of Groups, *Springer-Verlag, Berlin, Heidelberg, New York*, 1982.

[11] L. S. Spiezia, A characterization of the third Engel groups, *Arch. Math. (Basel)* **64** (1995), 369–373.

[12] L. S. Spiezia, A Property of the variety of 2-Engel groups, *Rend. Sem. Mat. Univ. Padova* **91** (1994), 225–228.

[13] B. Taeri, A combinatorial condition on a certain variety of groups, *Arch. Math. (Basel)* **77** (2001), 456–460.

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