Publ. Math. Debrecen
77/3-4 (2010), 499–516

On binomial Thue equations and ternary equations with S-unit coefficients

By ANDRÁS BAZSÓ (Debrecen)

Abstract. In this paper we obtain some new results for a collection of equations of the form (2) $Ax^n - By^n = \pm 1$ resp. (3) $Ax^n - By^n = z^m$ with $m \in \{3, n\}$, where x, y, z, A, B, n are unknown nonzero integers such that $n \geq 3$ is a prime and AB is composed of two fixed primes. We prove among other things that under certain conditions formulated in Section 2, equations (3) have no solutions with $|xy| > 1$, Ax, By and z coprime and $n > 13$ (cf. Theorems 2 to 4). Combining this with some other results and techniques, we establish a similar result for equations (2) (cf. Theorem 1).

1. Introduction and results on binomial Thue equations

In case of many number theoretical problems one has to deal with Diophantine equations of the form

$$Ax^n - By^n = C \tag{1}$$

where A, B, C, n are nonzero integers and $n \geq 3$. We may assume that $1 \leq A < B$ and $\gcd(A, B) = 1$. If the exponent n were fixed, equation (1) would be a binomial Thue equation, and we keep this name in our terminology also in the case of unknown n. Thue equations and generalized Thue equations have many applications in number theory, see e.g. [21], [25], [4], [19], [6], [8], [15], [7], [10], [2], [17] and the references given there. By a classical theorem of Thue [29], for fixed n, equation (1) has at most finitely many solutions in integers x, y. The first

---

Mathematics Subject Classification: Primary: 11D45, 11D61; Secondary: 11D41, 11J82, 11J86.
Key words and phrases: Diophantine equations, binomial Thue equations, exponential equations, resolution of equations.
Research was supported by the Hungarian Academy of Sciences, and by the OTKA grant K75566.

effective upper bounds for the size of the solutions of (1) are due to Baker [1] for n fixed. For n also unknown, Tijdeman [30] proved that $\max\{|x|, |y|, n\}$ can be still effectively bounded for every integer solution (x, y, n) of (1) with $|xy| > 1$. This effective finiteness result is extended in [14] by Győry, Pink and Pintér to the case when the numbers A, B, C are taken to be unknown S-units (i.e., all their prime factors lie in S, where S is a finite set of primes).

Using Baker's theory of linear forms in logarithms, the results of [1] and [30] have been improved several times, but even the best known upper bounds are too large for finding the solutions of (1) in concrete cases.

In [16], Győry and Pintér studied equation (1) for bounded positive integer coefficients A, B and C. They first derived, for concrete values of $A, B, C \leq 100$, a relatively small upper bound for n, provided that (1) has no solutions with $|xy| \leq 1$. Moreover, they explicitly solved (1) for $\max\{A, B, C\} \leq 10$, for $C = 1$ and $\max\{A, B\} \leq 20$ and for $A = C = 1$ and $B \leq 70$, respectively. The latter results were recently generalized by Bazsó, Bérczes, Győry and Pintér [3] for the cases $C = 1$ and $\max\{A, B\} \leq 50$ and for $A = C = 1$ and $B < 235$. Further related results can also be found in [3] concerning (1) with bounded coefficients.

Apart from the above mentioned results, equation (1) was solved in only a few instances, in each case with $C = \pm 1$, including the cases when $B = A + 1$ (cf. [4]) or when for a finite set of primes S (with $|S| = 1, 2$), the coefficients A, B were unknown S-units. In the sequel we also restrict our attention to the equation

$$Ax^n - By^n = \pm 1 \tag{2}$$

in unknown S-units $A, B \in \mathbb{Z}$, and unknown integers x, y, n with $|xy| \geq 1$ and $n \geq 3$. For $S = \{p\}$ with a prime $p \in \{3, 5, 7, 11, 13, 17, 19, 23, 29, 53, 59\}$, it follows from the work of Wiles [31], Darmon and Merel [13] and Ribet [23] on Fermat-type equations that (2) has no solutions with $|xy| > 1$ and $n \geq 3$. For $S = \{2, 3\}$, (2) was solved by Bennett [6]. His result was extended by Bennett, Győry, Mignotte and Pintér [7] to the case when $S = \{p, q\}$ with primes $2 \leq p, q \leq 13$. Independently, Bugeaud, Mignotte and Siksek [12] solved (2) in the case when, in (2), $A = 2^\alpha$, $B = q^\beta$ with a prime $3 \leq q < 100$, or $A = p^\alpha$, $B = q^\beta$ with primes $3 \leq p < q \leq 31$, and in both cases α, β are nonnegative integers. Recently, Győry and Pintér [17] generalized the results of [7] to the case when $S = \{p, q\}$ with primes $2 \leq p, q \leq 29$.

In the present paper we extend the above results by studying the solutions of equation (2) in the case when $S = \{p, q\}$ with primes $2 \leq p, q \leq 71$. Although our Theorem 1 does not give the resolution of equation (2), we give reasonable upper bounds for n which may be useful if someone needs to solve concrete binomial Thue equations of such type.

Our main result is the following.

Theorem 1. *Let $n \geq 3$ be a prime, $S = \{p, q\}$ with primes $2 \leq p, q \leq 71$ and let A, B be coprime integer S-units with $A < B$. If $(A, p, q) \neq (1, 2, 31)$ and*

$$(p, q) \notin \{(23, 41), (17, 47), (29, 61), (61, 67), (17, 71)\},$$

then for every integer solution (x, y, A, B, n) of equation (2) with $|xy| > 1$ we have $n \leq 31$.

Moreover,

(i) *if $A = 1$ and $(p, q, n) \notin \{(47, q, 23), (59, q, 29), (2, 61, 31), (17, 61, 31), (43, 61, 31), (53, 67, 17)\}$, then for every integer solution (x, y, A, B, n) of equation (2) with $|xy| > 1$ we have $n \leq 13$;*

(ii) *if $A > 1$ and $(p, q, n) \notin \{(3, 37, 19), (5, 37, 19), (3, 61, 31), (17, 61, 31), (43, 61, 31)\}$, then for every integer solution (x, y, A, B, n) of equation (2) with $|xy| > 1$ we have $n \leq 17$.*

For the exceptional (p, q, n), the methods used in the proof of Theorem 1 proved to be inefficient to solve equation (2) for arbitrary nonnegative integer exponents of the primes p, q. However, they work for several particular exponents. We further note that binomial Thue equations with degree at most 17 can be solved in most cases by using a powerful computer and the program packages MAGMA [11], PARI [22] or SAGE [28].

2. Results on ternary equations

Before proving Theorem 1, we first deal with more general Diophantine equations of the form

$$Ax^n - By^n = z^m \quad \text{with} \quad m \in \{3, n\}, \tag{3}$$

where A, B are given nonzero integers, $n \geq 3$ and x, y, z are unknown integers. Approaches to solve such equations, analogous to that employed by WILES [31] to prove Fermat's Last Theorem, are based on the connection between a putative integer solution (x, y, z) of ternary equations, Frey curves and certain modular forms. We note that the applicability of this "modular" approach depends only on the prime factors of the coefficients A, B. In this direction significant contributions can be found e.g. in [24], [23], [20], [13], [18], [9], [6], [7] and [17].

By means of the modular method we establish new results on the solutions of equation (3) both for $m = n$ and for $m = 3$. These results will be crucial in the proof of Theorem 1.

Theorem 2. *Let $AB = 2^\alpha q^\beta$ with a prime $3 \le q \le 151$, $q \ne 31, 127$ and with nonnegative integers α, β. If n is a prime, then for every integer solution (x, y, z, A, B, n) of the equation*

$$Ax^n - By^n = z^n \tag{4}$$

with $|xy| > 1$ and Ax, By and z pairwise coprime we have $n \le 53$.

Moreover, apart from 31 possible exceptions (q, n, α) given in Table 1 below, for every integer solution (x, y, z, A, B, n) of equation (4) with $|xy| > 1$ and Ax, By and z pairwise coprime we have $n \le 13$.

(q,n,α)	(q,n,α)	(q,n,α)	(q,n,α)	(q,n,α)
$(3,n,1)$	$(17,n,4)$	$(73,17,1)$	$(109,29,1)$	$(149,37,4)$
$(3,n,2)$	$(37,19,\alpha)$	$(73,37,\alpha)$	$(113,19,\alpha)$	$(149,41,1)$
$(3,n,3)$	$(47,23,4)$	$(83,41,4)$	$(137,17,4)$	$(151,19,\alpha)$
$(5,n,2)$	$(53,17,1)$	$(97,29,1)$	$(137,23,\alpha)$	
$(5,n,3)$	$(59,29,4)$	$(101,17,\alpha)$	$(137,29,1)$	
$(7,n,2)$	$(61,31,\alpha)$	$(103,17,4)$	$(139,23,4)$	
$(7,n,3)$	$(67,17,\alpha)$	$(107,53,4)$	$(149,17,1)$	

Table 1

For $q \le 13$, $n > 13$, this gives Theorem 2.2 of [7]; and for $q \le 29$, $n > 13$, this implies Theorem 3 of [17] (cf. Lemma 2). Further, our Theorem 2 can be compared with the corresponding results of [24], [31], [23] and [6].

Theorem 3. *Let $AB = p^\alpha q^\beta$ with primes $5 \le p, q \le 71$ and nonnegative integers α, β. If n is a prime, then apart from 28 possible exceptions (p, q, n) given in Table 2 below, for every integer solution (x, y, z, A, B, n) of (4) with $|xy| > 1$ and Ax, By and z pairwise coprime we have $n \le 13$.*

(p,q,n)	(p,q,n)	(p,q,n)	(p,q,n)	(p,q,n)
$(5,7,n)$	$(17,23,n)$	$(p,47,23)$	$(17,61,31)$	$(61,67,n)$
$(7,11,n)$	$(5,37,n)$	$(17,47,n)$	$(29,61,n)$	$(7,71,n)$
$(5,13,n)$	$(5,41,n)$	$(11,53,n)$	$(31,61,17)$	$(17,71,n)$

(p,q,n)	(p,q,n)	(p,q,n)	(p,q,n)	(p,q,n)
$(7,13,n)$	$(13,41,n)$	$(5,59,n)$	$(43,61,31)$	$(43,71,17)$
$(7,17,n)$	$(23,41,n)$	$(p,59,29)$	$(5,67,17)$	
$(13,19,n)$	$(11,43,n)$	$(5,61,n)$	$(53,67,17)$	

Table 2

This is a generalization of Theorem 4 of [17] (cf. Lemma 3). For $\max\{p,q\}\leq 29$, $n>13$ our result possesses two exceptions (p,q,n) fewer.

Theorem 4. *Let $AB=p^\alpha q^\beta$ with nonnegative integers α, β and primes $3\leq p<q\leq 71$ such that $pq\leq 583$. If n is a prime, then apart from 29 possible exceptions (p,q,n) given in Table 3 below, for every integer solution (x,y,z,A,B,n) of the equation*

$$Ax^n - By^n = z^3 \tag{5}$$

with $|xy|>1$, xy even and Ax, By and z pairwise coprime we have $n\leq 13$.

(p,q,n)	(p,q,n)	(p,q,n)	(p,q,n)	(p,q,n)
$(11,23,17)$	$(11,31,19)$	$(7,43,19)$	$(11,47,23)$	$(5,61,31)$
$(13,23,17)$	$(3,37,19)$	$(13,43,17)$	$(3,59,29)$	$(7,61,31)$
$(11,29,17)$	$(5,37,19)$	$(3,47,23)$	$(5,59,29)$	$(3,67,17)$
$(11,29,23)$	$(7,37,19)$	$(5,47,23)$	$(7,59,19)$	$(5,67,17)$
$(13,29,19)$	$(11,37,19)$	$(7,47,23)$	$(7,59,29)$	$(7,67,17)$
$(19,29,23)$	$(13,37,19)$	$(11,47,17)$	$(3,61,31)$	

Table 3

For $q\leq 13$, $n>13$, this gives Theorem 2.1 of [7]. Further, Theorem 4 is a considerable extension of Theorem 5 of [17] (cf. Lemma 4). Under the assumptions of Theorem 5 of [17] on p, q our result implies that if $n>13$ is a prime, then (5) has no solutions with xy even and $|xy|>1$, without any exception (p,q,n).

3. Auxiliary results

In the proofs of our Theorems 1–4 we apply the results of this section.

The following Lemma 1 summarizes some results obtained by Kraus [20], and Bennett, Vatsal and Yazdani [9] on ternary equations of the form (3).

For a given prime q and nonzero integer u, set

$$\mathrm{Rad}_q(u) := \prod_{\substack{p|u\\ p\neq q}} p,$$

where the product is taken over all positive primes p different from q and dividing u, and write $\mathrm{ord}_q(u)$ for the largest integer k with $q^k|u$. Suppose that for given A, B and $n \geq 3$, we have a solution (x, y, z) to (3) in nonzero integers.

If $m = 3$ (see [9]) we assume, without loss of generality, that $3 \nmid Ax$ and $By^n \not\equiv 2 \pmod 3$, and A and B are nth-power free. We consider the elliptic curve

$$E : Y^2 + 3zXY + By^nY = X^3,$$

and set

$$N_n(E) = \mathrm{Rad}_3(AB)\varepsilon_3,$$

where

$$\varepsilon_3 := \begin{cases} 1 & \text{if } \mathrm{ord}_3(B) = 3,\\ 3 & \text{if } \mathrm{ord}_3(By^n) > 3 \text{ and } \mathrm{ord}_3(B) \neq 3,\\ 3^2 & \text{if } 9 \mid (2 + By^n - 3z),\\ 3^3 & \text{if } 3 \| (2 + By^n - 3z) \text{ or } \mathrm{ord}_3(By^n) = 2,\\ 3^4 & \text{if } \mathrm{ord}_3(By^n) = 1,\end{cases}$$

If $m = n$ (see [20]), then we may assume without loss of generality that $Ax^n \equiv -1 \pmod 4$ and $By^n \equiv 0 \pmod 2$. The corresponding Frey curve is

$$E : Y^2 = X(X - Ax^n)(X + By^n).$$

Put

$$N_n(E) = \mathrm{Rad}_2(AB)\varepsilon_n,$$

where

$$\varepsilon_n := \begin{cases} 1 & \text{if } \mathrm{ord}_2(AB) = 4,\\ 2 & \text{if } \mathrm{ord}_2(AB) = 0 \text{ or } \mathrm{ord}_2(AB) \geq 5,\\ 2 & \text{if } 1 \leq \mathrm{ord}_2(B) \leq 3 \text{ and } xyz \text{ even},\\ 8 & \text{if } \mathrm{ord}_2(AB) = 2 \text{ or } 3 \text{ and } xyz \text{ odd},\\ 32 & \text{if } \mathrm{ord}_2(AB) = 1 \text{ and } xyz \text{ odd}.\end{cases}$$

We note that both for $m = 3$ and for $m = n$, the numbers $N_n(E)$ are closely related to the conductors of the above curves (cf. [9] and [20]).

Lemma 1. *Suppose that A, B, x, y and z are nonzero integers with Ax, By and z pairwise coprime, $xy \neq \pm 1$, satisfying equation* (3) *with prime $n \geq 5$*

and $n \nmid AB$. Then there exists a cuspidal newform $f = \sum_{r=1}^{\infty} c_r q^r$ ($q := e^{2\pi iz}$) of weight 2, trivial Nebentypus character and level $N_n(E)$ for $N_n(E)$ given as above. Moreover, if we write K_f for the field of definition of the Fourier coefficients c_r of the form f and suppose that p is a prime coprime to $nN_n(E)$, then

$$\mathrm{Norm}_{K_f/\mathbb{Q}}(c_p - a_p) \equiv 0 \pmod{n}$$

with $a_p = \pm(p+1)$ (if $p \mid xy$) or $a_p \in S_{p,m}$ (if $p \nmid xy$), where

$$S_{p,3} = \{u : |u| < 2\sqrt{p},\ u \equiv p+1 \pmod{3}\}$$

and

$$S_{p,n} = \{u : |u| < 2\sqrt{p},\ u \equiv p+1 \pmod{4}\}\,.$$

Proof. This deep result was proved in [9] (for $m = 3$) and [20] (for $m = n$). (For a survey on this topic, see [5], [26] or [27].) □

Lemma 2. *Suppose that $AB = 2^\alpha q^\beta$, where q is a prime with $3 \le q \le 29$ and α, β are nonnegative integers. If $n > 11$ is a prime, then equation (4) has no solutions in integers (x, y, z) with $|xy| > 1$ and Ax, By and z pairwise coprime, unless, possibly,*

$$(q, \alpha) \in \{(3,1),(3,2),(3,3),(5,2),(5,3),(7,2),(7,3),(17,4)\}$$

and xy is odd.

Proof. See Theorem 3 in [17]. □

Lemma 3. *Suppose that $AB = p^\alpha q^\beta$, where p, q are primes with $5 \le p < q \le 29$ and α, β are nonnegative integers. If $n > 11$ is a prime, then equation (4) has no solutions in integers (x, y, z) with $|xy| > 1$ and Ax, By and z pairwise coprime, unless, possibly $(p, q, n) = (19, 29, 13)$ or $(p, q) \in \{(5,7),(5,13),(7,11),(7,13),(7,17),(7,23),(13,17),(13,19),(17,23)\}$.*

Proof. See Theorem 4 in [17]. □

Lemma 4. *Suppose that $AB = p^\alpha q^\beta$, where α, β are nonnegative integers and p, q are primes with $3 \le p < q \le 29$ such that either $p \le 7$ or*

$$(p, q) \in \{(11,13),(11,17),(11,19),(13,17),(13,19),(17,23)\}\,.$$

If $n > 11$ is a prime, then equation (5) has no solutions in integers (x, y, z) with $|xy| > 1$, xy even, and Ax, By and z pairwise coprime, unless, possibly $(p, q, n) \in \{(3,23,13),(5,19,13),(5,23,23),(5,29,13),(5,29,23),(7,17,17),(7,17,19),(7,19,13),(11,13,13),(11,17,23),(11,19,13),(11,19,31),(13,17,17),(13,19,13\}$.

Proof. This is Theorem 5 in [17]. □

We recall that for a finite set of primes S, an integer u is an S-unit if all its prime factors lie in S. The following result is due to Bennett, Győry, Mignotte and Pintér [7] for $2 \leq p, q \leq 13$, and to Győry and Pintér [17] for $2 \leq p, q \leq 29$.

Lemma 5. *Let $S = \{p, q\}$ for p and q primes with $2 \leq p, q \leq 29$. If A, B, x, y and n are positive integers with A, B S-units, $A < B$ and $n \geq 3$, then the only solutions to equation (2) are those with*

$$n \geq 3, \quad A \in \{1, 2, 3, 4, 7, 8, 16\}, \quad x = y = 1$$

and

$$n = 3, \quad (A, x) = (1, 2), (1, 3), (1, 4), (1, 9), (1, 19), (1, 23), (3, 2), (5, 11),$$

$$n = 4, \quad (A, x) = (1, 2), (1, 3), (1, 5), (3, 2),$$

$$n = 5, \quad (A, x) = (1, 2), (1, 3),$$

$$n = 6, \quad (A, x) = (1, 2).$$

Proof. This is Theorem 1 in [17]; see also Theorem 1.1 in [7]. □

The following two lemmas are special cases of two theorems of Bugeaud, Mignotte and Siksek [12].

Lemma 6. *Suppose $3 \leq q < 100$ is a prime. The equation*

$$q^{\alpha} x^n - 2^{\beta} y^n = \pm 1$$

has no solutions in integers x, y, α, β, n with $x, y > 0$, $|xy| > 1$, $\alpha, \beta \geq 0$ and $n > 5$.

Proof. See Theorem 1.1 in [12]. □

Lemma 7. *Suppose $3 \leq p < q \leq 31$ are primes. The equation*

$$p^{\alpha} x^n - q^{\beta} y^n = \pm 1$$

has no solutions in integers x, y, α, β, n with $x, y > 0$, $\alpha, \beta \geq 0$ and $n > 5$.

Proof. See Theorem 1.2 in [12]. □

We note that in contrast with Lemma 5, Lemmas 6 and 7 cannot be applied to equations of the form (2) when $A = 1$ and B has two distinct prime factors. Further, in case $A = 1$ equation (2) cannot be solved by the methods used in [7], [17] and [12] when B is divisible by more than two distinct primes.

Let $\phi(B)$ denote Euler's function. The following result has recently been proved by Bazsó, Bérczes, Győry and Pintér [3].

Lemma 8. *Suppose that in the equation*

$$x^n - By^n = \pm 1 \tag{6}$$

n is a prime and that each of the following conditions holds:

(i) $n \geq 17$,

(ii) $B \leq \exp\{3000\}$,

(iii) $n \nmid B\phi(B)$,

(iv) $B^{n-1} \not\equiv 2^{n-1} \pmod{n^2}$,

(v) $r^{n-1} \not\equiv 1 \pmod{n^2}$ *for some divisor r of B.*

Then equation (6) *has no solutions in integers (x, y, n) with $|xy| > 1$.*

Proof. See Theorem 6 in [3]. □

4. Proofs

First we prove Theorems 2, 3 and 4.

Proof of Theorem 2. Suppose that for some prime $n > 13$ and for some A, B under consideration, equation (4) has a nontrivial solution (x, y, z, A, B, n) with Ax, By, and z coprime. By Lemma 2 we may assume that $30 < q \leq 151$. Further, we may assume that $\alpha > 0$ and $\beta > 0$, since otherwise the assertion of Theorem 2 follows from the results of [31], [23] and [13].

By Lemma 1, there exists a cuspidal newform f of level $N = 2^\gamma q$ with $\gamma \in \{0, 1, 3, 5\}$. Using the notation of Lemma 1 with $m = n$, set

$$\begin{aligned} A_{r,n} := {} & \mathrm{Norm}_{K_f/\mathbb{Q}}(c_r - (r+1)) \cdot \mathrm{Norm}_{K_f/\mathbb{Q}}(c_r + (r+1)) \\ & \cdot \prod_{a_r \in S_{r,n}} \mathrm{Norm}_{K_f/\mathbb{Q}}(c_r - a_r), \end{aligned}$$

where r is a prime, coprime to $2nq$. In fact, in $A_{r,n}$, the index n is used only to indicate that we are dealing with the case $m = n$. In view of Lemma 1, n

must be a divisor of $A_{r,n}$ for every prime r with $r \nmid 2nq$. In the following Table 4 we give the common prime divisors of the nonzero values of $A_{3,n}, A_{5,n}, \dots, A_{47,n}$ for every level N under consideration. There is "$\oslash$" in those cells for which all corresponding values of $A_{r,n}$ are equal to 0. One can see that in these cases $x = y = 1$ is a solution to (4) for every $n \geq 3$.

$q \backslash N$	q	$2q$	$8q$	$32q$
31	5	$\oslash$	2, 3	2, 3, 7
37	3	3, 19	2, 3, 5	2, 3, 5
41	2, 5	2, 3, 7	2, 3, 5	2, 3, 7, 13
43	3, 7	3, 5, 11	2, 3, 5	2, 3, 5, 11
47	3, 23	2, 3	2, 5	2, 3, 5
53	3, 13	3	2, 7	2, 3, 5, 17
59	29	3, 5	2, 3, 5, 7	2, 3, 5, 7
61	3, 5	3, 31	2, 5, 7	2, 3, 5, 13
67	3, 5, 11	3, 17	2, 3, 5	2, 3, 5, 17
71	3, 5, 7	2, 3, 5	2, 3, 5, 7	2, 3, 7
73	2, 3, 5	2, 3, 37	2	2, 3, 5, 13, 17
79	3, 5, 13	2, 3, 5	2, 3, 5	2, 3, 5
83	3, 41	3, 5, 7	2, 3, 5	2, 3, 5, 7
89	2, 3, 5, 11	2, 3, 5	2, 3, 5	2, 3, 5, 7
97	2	2, 3, 5, 7	2, 3, 5	2, 3, 5, 7, 29
101	3, 5	3, 7, 17	2, 3	2, 3, 5, 13
103	5, 17	2, 3, 5, 7, 13	2, 3, 5	2, 3, 5, 13
107	5, 53	3, 5	2, 3	2, 3, 5
109	3	3, 5, 11	2, 3	2, 3, 5, 13, 29
113	2, 3, 7	2, 3, 19	2, 3, 5	2, 3, 5
127	3, 7	$\oslash$	2, 3, 5	2, 3, 5
131	3, 5, 13	3, 5, 7, 11	2, 3, 5	2, 3, 5, 11
137	2, 7, 17	2, 3, 5, 23	2, 3, 5	2, 3, 5, 29
139	3, 7, 23	3, 5, 7	2, 3, 7	2, 3, 5, 7
149	3, 37	3, 5	2, 5	2, 3, 5, 17, 41
151	3, 5	2, 3, 5, 19	2, 3	2, 3, 5, 7, 19

Table 4

Now Table 4 shows that $n \leq 53$ for all (q, α) under consideration, and that nontrivial solutions with $n > 13$ may occur only in the cases (q, n, α) which are listed in Table 1. This completes the proof of Theorem 2. $\square$

Proof of Theorem 3. Suppose that for some prime $n > 13$ and for some A, B having the required properties, equation (4) has a nontrivial solution $(x, y, z,$

$A, B, n)$ with Ax, By, and z coprime. Again we may assume that, in $AB = p^\alpha q^\beta$, both α and β are positive. In view of Lemma 3 we may further assume that $30 < \max\{p, q\} \le 71$ or that

$$(p, q) \in \{(5, 7), (5, 13), (7, 11), (7, 13), (7, 17), (7, 23), (13, 17), (13, 19), (17, 23)\}\,.$$

As in the proof of Theorem 2, we apply Lemma 1 with $m = n$. Under the assumptions of Theorem 3 the level N of the corresponding modular forms is $2pq$. In Table 5, for all the 134 pairs (p, q) under consideration, we list the common prime divisors (briefly *CPD*'s) of $A_{r,n}$ (defined in the proof of Theorem 2) for primes $r \in \{3, 5, 7, \ldots, 47\}$ which are coprime to pq. Again "$\oslash$" indicates the case that the corresponding values of $A_{r,n}$ are all equal to 0.

(p,q)	CPD's	(p,q)	CPD's	(p,q)	CPD's
$(5,7)$	$\oslash$	$(37,43)$	$2,3,5,7$	$(17,61)$	$2,3,5,7,31$
$(7,11)$	$\oslash$	$(41,43)$	$2,3,5,7$	$(19,61)$	$2,3,5,7,11$
$(5,13)$	$\oslash$	$(5,47)$	$2,3,5,23$	$(23,61)$	$2,3,5,7,11$
$(7,13)$	$\oslash$	$(7,47)$	$2,3,5,23$	$(29,61)$	$\oslash$
$(7,17)$	$\oslash$	$(11,47)$	$2,3,5,7,23$	$(31,61)$	$2,3,5,7,17$
$(13,17)$	$2,3,5$	$(13,47)$	$2,3,5,7,23$	$(37,61)$	$2,3,5,7$
$(13,19)$	$\oslash$	$(17,47)$	$\oslash$	$(41,61)$	$2,3,5,7,11$
$(7,23)$	$2,3,5,11$	$(19,47)$	$2,3,5,23$	$(43,61)$	$2,3,5,7,13,31$
$(17,23)$	$\oslash$	$(23,47)$	$2,3,5,11,23$	$(47,61)$	$2,3,5,7,23$
$(5,31)$	$2,3,5$	$(29,47)$	$2,3,5,7,23$	$(53,61)$	$2,3,5,7,13$
$(7,31)$	$2,3,5,7$	$(31,47)$	$2,3,5,7,23$	$(59,61)$	$2,3,5,7,11,29$
$(11,31)$	$2,3,5,7,11$	$(37,47)$	$2,3,5,7,11,23$	$(5,67)$	$2,3,5,7,11,17$
$(13,31)$	$2,3,5,7$	$(41,47)$	$2,3,5,7,23$	$(7,67)$	$2,3,5,11$
$(17,31)$	$2,3,5$	$(43,47)$	$2,3,5,7,23$	$(11,67)$	$2,3,5,7,11$
$(19,31)$	$2,3,5$	$(5,53)$	$2,3,5,7,11,13$	$(13,67)$	$2,3,5,11$
$(23,31)$	$2,3,5,7,11$	$(7,53)$	$2,3,5,7,13$	$(17,67)$	$2,3,5,7,11$
$(29,31)$	$2,3,5,7$	$(11,53)$	$\oslash$	$(19,67)$	$2,3,5,7,11$
$(5,37)$	$\oslash$	$(13,53)$	$2,3,5,13$	$(23,67)$	$2,3,5,7,11$
$(7,37)$	$2,3,5,7$	$(17,53)$	$2,3,5,13$	$(29,67)$	$2,3,5,7,11$
$(11,37)$	$2,3,5,7$	$(19,53)$	$2,3,5,7,13$	$(31,67)$	$2,3,5,7,11$
$(13,37)$	$2,3,5$	$(23,53)$	$2,3,5,11,13$	$(37,67)$	$2,3,5,7,11$
$(17,37)$	$2,3,5,7$	$(29,53)$	$2,3,5,7,13$	$(41,67)$	$2,3,5,7,11,13$
$(19,37)$	$2,3,5,7$	$(31,53)$	$2,3,5,7,13$	$(43,67)$	$2,3,5,7,11$
$(23,37)$	$2,3,5,11$	$(37,53)$	$2,3,5,11,13$	$(47,67)$	$2,3,5,11,23$
$(29,37)$	$2,3,5,7$	$(41,53)$	$2,3,5,7,13$	$(53,67)$	$2,3,5,11,13,17$
$(31,37)$	$2,3,5,7,13$	$(43,53)$	$2,3,5,7,13$	$(59,67)$	$2,3,5,11,29$
$(5,41)$	$\oslash$	$(47,53)$	$2,3,5,7,13,23$	$(61,67)$	$\oslash$
$(7,41)$	$2,3,5,7$	$(5,59)$	$\oslash$	$(5,71)$	$2,3,5,7$

(p, q)	CPD's	(p, q)	CPD's	(p, q)	CPD's
$(11, 41)$	$2, 3, 5, 7$	$(7, 59)$	$2, 3, 5, 7, 29$	$(7, 71)$	$\emptyset$
$(13, 41)$	$\emptyset$	$(11, 59)$	$2, 3, 5, 13, 29$	$(11, 71)$	$2, 3, 5, 7$
$(17, 41)$	$2, 3, 5, 7$	$(13, 59)$	$2, 3, 5, 7, 29$	$(13, 71)$	$2, 3, 5, 7$
$(19, 41)$	$2, 3, 5, 7$	$(17, 59)$	$2, 3, 5, 7, 29$	$(17, 71)$	$\emptyset$
$(23, 41)$	$\emptyset$	$(19, 59)$	$2, 3, 5, 29$	$(19, 71)$	$2, 3, 5, 7$
$(29, 41)$	$2, 3, 5, 7$	$(23, 59)$	$2, 3, 5, 11, 29$	$(23, 71)$	$2, 3, 5, 7, 11$
$(31, 41)$	$2, 3, 5$	$(29, 59)$	$2, 3, 5, 7, 29$	$(29, 71)$	$2, 3, 5, 7$
$(37, 41)$	$2, 3, 5, 7$	$(31, 59)$	$2, 3, 5, 7, 29$	$(31, 71)$	$2, 3, 5, 7, 11$
$(5, 43)$	$2, 3, 5, 7, 11$	$(37, 59)$	$2, 3, 5, 7, 29$	$(37, 71)$	$2, 3, 5, 7$
$(7, 43)$	$2, 3, 5, 7$	$(41, 59)$	$2, 3, 5, 7, 29$	$(41, 71)$	$2, 3, 5, 7$
$(11, 43)$	$\emptyset$	$(43, 59)$	$2, 3, 5, 7, 29$	$(43, 71)$	$2, 3, 5, 7, 17$
$(13, 43)$	$2, 3, 5, 7, 11$	$(47, 59)$	$2, 3, 5, 7, 23, 29$	$(47, 71)$	$2, 3, 5, 7, 11, 23$
$(17, 43)$	$2, 3, 5, 7$	$(53, 59)$	$2, 3, 5, 13, 29$	$(53, 71)$	$2, 3, 5, 7, 11, 13$
$(19, 43)$	$2, 3, 5, 7, 11$	$(5, 61)$	$\emptyset$	$(59, 71)$	$2, 3, 5, 7, 29$
$(23, 43)$	$2, 3, 5, 7, 11$	$(7, 61)$	$2, 3, 5$	$(61, 71)$	$2, 3, 5, 7$
$(29, 43)$	$2, 3, 5, 7, 11$	$(11, 61)$	$2, 3, 5$	$(67, 71)$	$2, 3, 5, 7, 11$
$(31, 43)$	$2, 3, 5, 7, 13$	$(13, 61)$	$2, 3, 5, 11$		

Table 5

By Lemma 1, n must divide $A_{r,n}$ for each r in question. However, as is seen from Table 5, apart from the exceptions listed in Table 2, we get a contradiction since $n > 13$. Thus Theorem 3 is proved. □

Proof of Theorem 4. Suppose that for some A, B under consideration, equation (5) has a nontrivial solution (x, y, z, A, B, n) with xy even, Ax, By and z coprime, and with $n > 13$. Lemma 4 proves the assertion for those primes p, q for which either $p \le 7$ and $q \le 29$ or

$$(p, q) \in \{(11, 13), (11, 17), (11, 19), (13, 17), (13, 19), (17, 23)\},$$

unless

$$(p, q) \in \{(5, 23), (5, 29), (7, 17), (11, 17), (11, 19), (13, 17)\}.$$

We use again Lemma 1 but now with $m = 3$. First we study the case when, in $AB = p^\alpha q^\beta$, either $p = 3$, $\alpha > 0$, $q \in \{31, 37, 41, 43, 47, 53, 59, 61, 67, 71\}$ or $\alpha\beta = 0$. Then we have to consider modular forms f of level $N = 3^\gamma q$ with $\gamma \in \{0, 1, 2, 3, 4\}$. With the notation of Lemma 1, put

$$B_{r,3} := \mathrm{Norm}_{K_f/\mathbb{Q}}(c_r - (r+1)) \cdot \mathrm{Norm}_{K_f/\mathbb{Q}}(c_r + (r+1)).$$

Since xy is even, in the case $r = 2$, it is enough to consider $B_{2,3}$ instead of the product

$$A_{2,3} := \mathrm{Norm}_{K_f/\mathbb{Q}}(c_2 - 3) \cdot \mathrm{Norm}_{K_f/\mathbb{Q}}(c_2) \cdot \mathrm{Norm}_{K_f/\mathbb{Q}}(c_2 + 3).$$

Moreover, the Hasse–Weil (HW) bound yields $n \leq 2\sqrt{2}+3$ for all rational newforms f, so we deal only with the non-rational ones. We note that all the newforms of level $N=37$ are one dimensional. The following Table 6 contains the common prime divisors of $B_{2,3}$ and $A_{r,3}$ for primes $r \in \{5,7,\ldots,47\}$ different from q.

$q\backslash N$	$1q$	$3q$	$9q$	$27q$	$81q$
31	5	2, 7	2, 3, 5, 7	2, 3, 5	2, 3, 5, 7
37	HW	2, 19	2, 3, 5, 19	2, 3, 7	2, 3, 5
41	2, 5	2, 7	2, 5, 7	2, 3, 7	2, 3, 7, 11
43	7	2, 7, 11	2, 3, 7, 11	2, 3, 5	2, 3, 5, 7
47	23	2	2, 23	2, 3, 13	2, 3, 7
53	2, 5, 13	2, 3	2, 3, 5, 13	2, 3, 5	2, 3, 5, 13
59	2, 29	2, 5, 7	2, 5, 7, 29	2, 3, 5, 11	2, 3, 5, 7
61	2, 5	2, 5, 31	2, 3, 5, 31	2, 3, 5	3, 7
67	5, 11	2, 17	2, 3, 5, 11, 17	3, 5, 7, 11	2, 3, 7, 13
71	5, 7	2, 3, 5	2, 3, 5, 7	2, 3, 5, 7	2, 3, 5, 7

Table 6

In view of Lemma 1 Table 6 shows that we get a contradiction with $n>13$ unless $(p,q,n) \in \{(3,37,19),(3,47,23),(3,59,29),(3,61,31),(3,67,17)\}$.

In the remaining cases we have in $AB = p^\alpha q^\beta$ that $p \geq 5$ and $\alpha,\beta > 0$. By virtue of Lemma 4, it suffices to deal with the pairs (p,q) which are not considered there and with $(p,q) \in \{(5,23),(5,29),(7,17),(11,17),(11,19),(13,17)\}$. For each of the remaining pairs (p,q) we use again Lemma 1 with $m=3$, and collect the common prime divisors of $B_{2,3}$ and $A_{r,n}$ with primes $r \in \{5,7,11,\ldots,47\}$ for each occuring newforms of levels $N = 3pq$, $9pq$, $27pq$. To these computations we used MAGMA and its results are listed in the following Table 7.

(p,q)	$3pq$	$9pq$	$27pq$
(5, 23)	2, 3, 7, 11	2, 3, 7, 11	2, 3, 5, 7
(5, 29)	2, 5, 7	2, 3, 5, 7	2, 3, 5, 7
(5, 31)	2, 5, 7	2, 3, 5, 7	2, 3, 5
(5, 37)	2, 5, 11, 19	2, 3, 5, 7, 11, 19	2, 3, 5, 7, 11
(5, 41)	2, 3, 5	2, 3, 5, 7, 11	2, 3, 5, 7
(5, 43)	2, 5, 7	2, 3, 5, 7, 11	2, 3, 5, 7
(5, 47)	2, 5, 23	2, 3, 5, 7, 23	2, 3, 5, 13
(5, 53)	2, 5, 7, 13	2, 3, 5, 7, 13	2, 3, 5
(5, 59)	2, 3, 5, 29	2, 3, 5, 29	2, 3, 5, 7, 11

(p,q)	$3pq$	$9pq$	$27pq$
$(5,61)$	$2,3,5,7$	$2,3,5,7,31$	$2,3,5,7$
$(5,67)$	$2,5,7,11$	$2,3,5,7,11,13,17$	$2,3,5,7,11$
$(5,71)$	$2,3,5,7,13$	$2,3,5,7,13$	$2,3,5,7$
$(7,17)$	$2,7$	$2,3,5,7$	$2,3,5$
$(7,31)$	$2,5,7$	$2,3,5,7$	$2,3,5,7$
$(7,37)$	$2,3,7$	$2,3,5,7,19$	$2,3,5$
$(7,41)$	$2,5,7$	$2,3,5,7,11$	$2,3,5,7$
$(7,43)$	$2,5,7,11$	$2,3,5,7,11$	$2,3,5,11,19$
$(7,47)$	$2,5,7,13,23$	$2,3,5,7,13,23$	$2,3,5$
$(7,53)$	$2,5,7,13$	$2,3,5,7,13$	$2,3,5,7$
$(7,59)$	$2,5,7,11,19,29$	$2,3,5,7,11,19,29$	$2,3,5,7,11,13$
$(7,61)$	$2,3,5,7$	$2,3,5,7,13,31$	$2,3,5,13$
$(7,67)$	$2,3,11$	$2,3,5,11,17$	$2,3,5,7$
$(7,71)$	$2,3,5,7$	$2,3,5,7$	$2,3,5,7,11$
$(11,17)$	$2,3,5$	$2,3,5$	$2,3,5$
$(11,23)$	$2,3,5,7,11$	$2,3,5,7,11,17$	$2,3,5,7$
$(11,29)$	$2,3,5,7,13,17$	$2,3,5,7,13,17$	$2,3,23$
$(11,31)$	$2,5,7$	$2,3,5,7,19$	$2,3,5,13$
$(11,37)$	$2,3,5,7,13$	$2,3,5,7,11,13,19$	$2,3,5,7$
$(11,41)$	$2,3,5,7$	$2,3,5,7$	$2,3,5,7$
$(11,43)$	$2,5,7,11$	$2,3,5,7,11$	$2,3,5$
$(11,47)$	$2,3,5,7,17,23$	$2,3,5,7,17,23$	$2,3,5,7,13$
$(11,53)$	$2,3,5,7,13$	$2,3,5,7,13$	$2,3,5,7$
$(13,17)$	$2,5$	$2,3,5,7$	$3,5,7$
$(13,23)$	$2,5,11,13$	$2,3,5,7,11,13$	$2,3,5,11,17$
$(13,29)$	$2,5,7$	$2,3,5,7$	$2,3,5,7,19$
$(13,31)$	$2,3,5,7$	$2,3,5,7$	$2,3,5,7$
$(13,37)$	$2,3,5,7,19$	$2,3,5,7,19$	$2,3,5,7$
$(13,41)$	$2,5,7$	$2,3,5,7$	$2,3,5,7,11,13$
$(13,43)$	$2,3,5,7$	$2,3,5,7,11,17$	$2,3,5,7,13$
$(17,19)$	$2,3,5,7$	$2,3,5,7$	$2,3,5,7,13$
$(17,29)$	$2,3,5,7,11$	$2,3,5,7,11$	$2,3,5,7$
$(17,31)$	$2,5,11$	$2,3,5,11$	$2,3,5,7,11,13$
$(19,23)$	$2,3,5,11$	$2,3,5,7,11$	$2,3,5$
$(19,29)$	$2,3,5,7$	$2,3,5,7$	$2,3,5,7,11,23$

Table 7

Lemma 1 now implies that equation (5) has no solutions for those triples (p,q,n) for which n does not occur in Table 7 as a common prime divisor. It is seen from Tables 6 and 7 that there are 29 triples (p,q,n) with $n>13$ which are those listed in our Theorem 4 as possible exceptions. This proves Theorem 4. $\square$

Proof of Theorem 1. In view of Lemmas 7 and 5 it is enough to solve equation (2) for primes $31 \leq \max\{p,q\} \leq 71$. Let x,y, A, B, n be a solution of equation (2) with $|xy| > 1, n \geq 3$ and A, B coprime S-units. Then clearly, $(x, y, \pm 1)$ is a solution of the ternary equations (4) and (5) respectively. Then Theorems 2, 3 and 4 imply that $n \leq 31$ unless

$$(p,q) \in \{(2,31),(23,41),(17,47),(29,61),(61,67),(17,71)\}\,.$$

For $A > 1$ and $(p,q) = (2,31)$ one can apply Lemma 6 to obtain that $n < 6$ is true for all solutions of $2^\alpha x^n - 31^\beta y^n = \pm 1$, thus the first statement of the theorem is proved.

For the proof of the stronger statements (i) and (ii) of Theorem 1, by Theorems 2, 3 and 4 we have to consider the equation

$$Ax^n - By^n = \pm 1$$

for 50 cases of (p,q,n) which are listed in Table 8.

(p,q,n)	(p,q,n)	(p,q,n)	(p,q,n)	(p,q,n)
$(2,37,19)$	$(23,47,23)$	$(7,59,29)$	$(43,59,29)$	$(59,61,29)$
$(3,37,19)$	$(29,47,23)$	$(11,59,29)$	$(47,59,23)$	$(2,67,17)$
$(5,37,19)$	$(31,47,23)$	$(13,59,29)$	$(47,59,29)$	$(3,67,17)$
$(2,47,23)$	$(37,47,23)$	$(17,59,29)$	$(53,59,29)$	$(5,67,17)$
$(3,47,23)$	$(41,47,23)$	$(19,59,29)$	$(2,61,31)$	$(47,67,23)$
$(5,47,23)$	$(43,47,23)$	$(23,59,29)$	$(3,61,31)$	$(53,67,17)$
$(7,47,23)$	$(2,53,17)$	$(29,59,29)$	$(17,61,31)$	$(59,67,29)$
$(11,47,23)$	$(47,53,23)$	$(31,59,29)$	$(31,61,17)$	$(43,71,17)$
$(13,47,23)$	$(2,59,29)$	$(37,59,29)$	$(43,61,31)$	$(47,71,23)$
$(19,47,23)$	$(3,59,29)$	$(41,59,29)$	$(47,61,23)$	$(59,71,29)$

Table 8

For each such triple, we have to consider the equation for

$$A = 1,\ B = p^\alpha q^\beta; \quad \text{and for } A = p^\alpha,\ B = q^\beta$$

with every $(\alpha,\beta) \in \{1,\ldots,n-1\}^2$. For example, that means $2 \cdot 28^2 = 1568$ equations to solve when $n = 29$.

First, let $A = 1$. For $(p,q,n) \in \{(3,37,19),(5,37,19),(2,53,17),(3,61,31),$ $(31,61,17),(3,67,17),(43,71,17)\}$ we applied Lemma 8 combined with the modular method with signature (n,n,n) to exclude the solvability of all equations under consideration. To illustrate how this approach works we give the details for the case $(p,q,n)=(5,37,19)$. We checked that apart from the pairs (α,β) in Table 9 below, for each $(\alpha,\beta)\in\{1,\dots,18\}^2$ the equations

$$x^{19}-5^{\alpha}37^{\beta}y^{19}=\pm 1 \tag{7}$$

fullfill the conditions $(i)-(v)$ of Lemma 8, so they do not have nontrivial integer solutions. For each pair in Table 9, by local arguments we found two distinct primes p_1,p_2 which divide xy, where x, y is a putative nontrivial solution of the corresponding equation (7). These primes are also listed in Table 9.

(α,β)	p_1,p_2	(α,β)	p_1,p_2	(α,β)	p_1,p_2
$(1,5)$	$419,457$	$(8,1)$	$191,761$	$(14,3)$	$191,229$
$(2,18)$	$191,229$	$(9,14)$	$191,229$	$(15,16)$	$191,229$
$(3,12)$	$229,419$	$(10,8)$	$191,229$	$(16,10)$	$191,229$
$(4,6)$	$191,419$	$(11,2)$	$191,229$	$(17,4)$	$191,419$
$(6,13)$	$191,419$	$(12,15)$	$229,457$	$(18,17)$	$191,229$
$(7,7)$	$457,571$	$(13,9)$	$229,1483$		

Table 9

There are 16 cuspidal newforms f at level $2\cdot 5\cdot 37$. We recall that K_f denotes the number field generated by the Fourier coefficients c_r of the modular form f. Using the program package MAGMA for each pairs (α,β) of Table 9, we obtained that

$$19\nmid \mathrm{Norm}_{K_f/\mathbb{Q}}(c_{p_i}-(p_i+1))\cdot \mathrm{Norm}_{K_f/\mathbb{Q}}(c_{p_i}+(p_i+1))$$

with either $i=1$ or $i=2$ for all 16 newforms. Thus, Lemma 1 implies that the equations (7) corresponding to the pairs (α,β) in Table 9 have no solutions with $|xy|>1$.

In the case $(p,q,n)\in\{(2,37,19),(2,67,17),(5,67,17)\}$, we combined Lemma 8 with the routine of PARI for solving Thue equations of low degree. For example, Lemma 8 implies that the equation

$$x^{19}-2^{\alpha}37^{\beta}y^{19}=\pm 1$$

has no nontrivial solutions, unless $(\alpha,\beta)\in\{(3,16),(5,13),(6,2),(7,10),(8,18),(9,7),(10,15),(11,4),(12,12),(13,1),(14,9),(15,17),(16,6),(17,14),(18,13)\}$. We solved each equation corresponding to these pairs using PARI.

In the sequel let $A>1$. For $(p,q,n)\in\{(2,37,19),(2,47,23),(2,59,29),(2,61,31)\}$ we can apply again Lemma 6 to exclude the solvability of the corresponding equations.

For the remaining 46 triples of Table 8, and for each corresponding binomial Thue equation we used the following local method. Choose a small integer k such that $p = 2kn + 1$ is a prime. Then both x^n and y^n are either $2kth$ roots of unity (mod p) or zero. Thus we have to check the congruence

$$Ax^n - By^n \equiv \pm 1 \pmod p$$

only in $(2k+1)^2$ cases. Programmed in MAGMA, this method works very efficiently. (We note that it cannot be used when $A = 1$, because $x^n - By^n = 1$ always has the solution $(x, y) = (1, 0)$.) These computations proved the unsolvability of each binomial Thue equation under consideration, except the ones with

$$(p, q, n) \in \{(3, 37, 19), (5, 37, 19), (3, 61, 31), (17, 61, 31), (43, 61, 31)\}.$$

This completes the proof of Theorem 1. □

References

[1] A. Baker, Contributions to the theory of Diophantine equations, *Phil. Trans. Roy. Soc. London* **263** (968), 173–208.

[2] A. Bazsó, Further computational experiences on norm form equations with solutions forming arithmetic progressions, *Publ. Math. Debrecen* **71** (2007), 489–497.

[3] A. Bazsó, A. Bérczes, K. Győry and Á. Pintér, On the resolution of equations $Ax^n - By^n = C$ in integers x, y and $n \geq 3$, II, *Publ. Math. Debrecen* **76** (2010), 227–250.

[4] M. A. Bennett, Rational approximation to algebraic numbers of small height: the Diophantine equation $|ax^n - by^n| = 1$, *J. Reine Angew. Math.* **535** (2001), 1–49.

[5] M. A. Bennett, Recipes for ternary Diophantine equations of signature (p, p, k), *RIMS Kokyuroku (Kyoto)* **319** (2003), 51–55.

[6] M. A. Bennett, Products of consecutive integers, *Bull. London Math. Soc.* **36** (2004), 683–694.

[7] M. A. Bennett, K. Győry, M. Mignotte and Á. Pintér, Binomial Thue equations and polynomial powers, *Compos. Math.* **142** (2006), 1103–1121.

[8] M. A. Bennett, K. Győry and Á. Pintér, On the Diophantine equation $1^k + 2^k + \dots + x^k = y^n$, *Compos. Math.* **140** (2004), 1417–1431.

[9] M. A. Bennett, M. Vatsal and S. Yazdani, Ternary Diophantine equations of signature $(p, p, 3)$, *Compositio Math.* **140** (2004), 1399–1416.

[10] A. Bérczes and A. Pethő, Computational experiences on norm form equations with solutions from an arithmetic progression, *Glasnik Matematički. Serija III* **41**(61) (2006), 1–8.

[11] W. Bosma, J. Cannon and C. Playoust, The Magma algebra system. I. The user language, *J. Symbolic Comput.* **24** (1997), 235–265.

[12] Y. Bugeaud, M. Mignotte and S. Siksek, A multi-Frey approach to some multi-parameter families of Diophantine equations, *Canad. J. Math.* **60** (2008), 491–519.

[13] H. Darmon and L. Merel, Winding quotients and some variants of Fermat's last theorem, *J. Reine Angew. Math.* **490** (1997), 81–100.

[14] K. Győry, I. Pink and Á. Pintér, Power values of polynomials and binomial Thue-Mahler equations, *Publ. Math. Debrecen* **65** (2004), 341–362.

[15] K. Győry and Á. Pintér, Almost perfect powers in products of consecutive integers, *Monatsh. Math.* **145** (2005), 19–33.

[16] K. Győry and Á. Pintér, On the resolution of equations $Ax^n - By^n = C$ in integers x, y and $n \geq 3$, I, *Publ. Math. Debrecen* **70** (2007), 483–501.

[17] K. Győry and Á. Pintér, Binomial Thue equations, ternary equations and power values of polynomials, *Mathematical Notes* (*to appear*).

[18] E. Halberstadt and A. Kraus, Courbes de Fermat: résultats et problèmes, *J. Reine Angew. Math.* **548** (2002), 167–234.

[19] G. Hanrot, N. Saradha and T. N. Shorey, Almost perfect powers in consecutive integers, *Acta Arith.* **99** (2001), 13–25.

[20] A. Kraus, Majorations effectives pour l'équation de Fermat généralisée, Vol. 49, 1997, 1139–1161.

[21] L. J. Mordell, Diophantine equations, *Academic Press, London*, 1969.

[22] The PARI Group, *Bordeaux*, *PARI/GP, version* `2.1.5`, 2004, available from `http://pari.math.u-bordeaux.fr/`.

[23] K. A. Ribet, On the equation $a^p + 2^\alpha b^p + c^p = 0$, *Acta Arith.* **79** (1997), 7–16.

[24] J.-P. Serre, Sur les représentations modulaires de degré 2 de $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, *Duke Math. J.* **54** (1987), 179–230.

[25] T. N. Shorey and R. Tijdeman, Exponential Diophantine Equations, *Cambridge – New York*, 1986.

[26] S. Siksek, The Modular Approach to Diophantine Equations, in: H. Cohen: Number Theory, *Springler-Verlag*, *Berlin*, 2007, 1107–1138.

[27] W. Stein, Modular forms, a computational approach, vol. 79 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2007, with an appendix by Paul E. Gunnells.

[28] W. Stein et al., The Sage Development Team, Sage Mathematics Software (Version 3.4), 2009, available from `http://www.sagemath.org/`.

[29] A. Thue, Über Annäherungenswerte algebraischer Zahlen, *J. Reine Angew. Math.* **135** (1909), 289–305.

[30] R. Tijdeman, Some applications of Baker's sharpened bounds to Diophantine equations, in: *Séminaire Delange-Pisot-Poitou (16e année: 1974/75), Théorie des nombres, Fasc. 2, Exp. No. 24*, *Secrétariat Mathématique*, *Paris*, 1975, 7.

[31] A. Wiles, Modular elliptic curves and Fermat's last theorem, *Ann. of Math.* (2), **141** (1995), 443–551.

ANDRÁS BAZSÓ
INSTITUTE OF MATHEMATICS
NUMBER THEORY RESEARCH GROUP OF THE
HUNGARIAN ACADEMY OF SCIENCES
UNIVERSITY OF DEBRECEN
H-4010 DEBRECEN, P.O. BOX 12
HUNGARY

E-mail: bazsoa@math.unideb.hu

(Received May 26, 2010; revised August 10, 2010)