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A monotonicity property of M–matrices

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Abstract. In this paper we show that if $A \leq B$, A and B are M-matrices, A is nonsingular or irreducible singular and B is nonsingular, then $L_A \leq L_B, D_A \leq D_B$ and $U_A \leq U_B$ also hold, where $X = L_X D_X U_X$ denotes the LDU-factorization of the matrix X.

1. Introduction

For two matrices A and B ($A, B \in \mathbb{R}^{m\times m}$) the inequality $A \leq B$ holds if $a_{ij} \leq b_{ij}$ $(i, j = 1, \dots, m)$. A matrix A ($A \in \mathbb{R}^{m\times m}$) is said to be an M-matrix if $A = sI - B$, where $s > 0$, $B \geq 0$, I is the $m \times m$ unit matrix and $s \geq \varrho(B)$, where $\varrho(B)$ denotes the spectral radius of B. We shall prove the following monotonicity theorem on the LDU-factorization of two M-matrices A and B satisfying $A \leq$B.

Theorem 1. *Let A and B be two M-matrices such that $A \leq B$ and assume that A is nonsingular or irreducible singular and B is nonsingular. Let $A = L_A V_A$ and $B = L_B V_B$ be the LU-factorizations of A and B such that both L_A and L_B are unit lower triangular. Then*

$$L_A \leq L_B, \quad V_A \leq V_B.$$

In addition, if $V_A = D_A U_A$ and $V_B = D_B U_B$, where U_A and U_B are unit upper triangular and D_A and D_B are diagonal matrices, then

$$D_A \leq D_B, \quad U_A \leq U_B.$$

The result is related to the fact ([1],[2],[5]) that a nonsingular matrix A is an M-matrix if and only if there exist lower and upper triangular M-matrices R and S, respectively, such that $A = RS$.

Theorem 1 is not true for general matrices. Using the results of JAIN and SNYDER [3] we define A and B as follows:

$$A = \begin{bmatrix} 1 & -4 & 9 & -12 \\ -4 & 17 & -40 & 57 \\ 9 & -40 & 98 & -148 \\ -12 & 57 & -148 & 242 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -4 & 9 & -12 \\ -4 & 17 & -40 & 57 \\ 9 & -40 & 98 & -148 \\ -12 & 57 & -148 & 242 \end{bmatrix}.$$

These matrices are monotone, $A \le B$ and

$$L_A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 9 & -4 & 1 & 0 \\ -12 & 9 & -4 & 1 \end{bmatrix} \qquad L_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ \frac{9}{2} & -\frac{22}{9} & 1 & 0 \\ -6 & \frac{11}{3} & -\frac{240}{67} & 1 \end{bmatrix}.$$

L_A and L_B are not comparable implying that Theorem 1 does not hold for monotone matrices. The reverse of Theorem 1 also fails to be true. Let

$$L_A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ -9 & -4 & 1 & 0 \\ -12 & -9 & -4 & 1 \end{bmatrix} \qquad L_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & -4 & 1 \end{bmatrix}.$$

The matrices L_A and L_B are M-matrices and $L_A \le L_B$. The product matrix $L_A L_A^T$ is not an M-matrix. Furthermore $L_A L_A^T$ and $L_B L_B^T$ are not comparable. We notice however that if L and R are lower triangular M-matrices such that $LL^T \le RR^T$ holds then $L \le R$ also holds. This result is due to SCHMIDT and PATZKE [4].

2. Proof of theorem 1

We prove the result by induction. Assume that the $k \times k$ matrices A and B are nonsingular and such that $L_A \le L_B$ and $V_A \le V_B$ hold. Let

$$A' = \begin{bmatrix} A & c \\ r^T & a \end{bmatrix} \qquad B' = \begin{bmatrix} B & p \\ q^T & b \end{bmatrix} \qquad (c, r, p, q \in R^k)$$

Assume that A' and B' are also nonsingular M-matrices. They have the LU-factorizations

$$A' = \begin{bmatrix} L_A & 0 \\ r^T V_A^{-1} & 1 \end{bmatrix} \begin{bmatrix} V_A & L_A^{-1} c \\ 0 & a - r^T A^{-1} c \end{bmatrix}$$

$$B' = \begin{bmatrix} L_B & 0 \\ q^T V_B^{-1} & 1 \end{bmatrix} \begin{bmatrix} V_B & L_B^{-1} p \\ 0 & b - q^T B^{-1} p \end{bmatrix}$$

By assumption $L_A \le L_B$, $V_A \le V_B$, $c \le p \le 0$, $r \le q \le 0$ and $0 < a \le b$. The relation $L_A^{-1} \ge L_B^{-1} \ge 0$ implies $L_A^{-1} c \le L_B^{-1} p \le 0$. The inequality

$V_A^{-1} \geq V_B^{-1} \geq 0$ implies $r^T V_A^{-1} \leq q^T V_B^{-1} \leq 0$. Notice that $a - r^T A^{-1} c = \det(A')/\det(A) > 0$ and $b - q^T B^{-1} p = \det(B')/\det(B) > 0$. Finally $a - r^T A^{-1} c \leq b - q^T B^{-1} p$ follows from $A^{-1} \geq B^{-1} \geq 0$. Thus we have proved the first part of the theorem for the nonsingular case. If A' is an irreducible singular matrix of order m, then by Theorem 4.16 of Berman and Plemmons [1, p. 156] A' has rank $m-1$, each principal submatrix of A' other than A' itself is a nonsingular M-matrix, $a = r^T A^{-1} c$ and A' has the LU-factorization

$$A' = \begin{bmatrix} L_A & 0 \\ r^T V_A^{-1} & 1 \end{bmatrix} \begin{bmatrix} V_A & L_A^{-1} c \\ 0 & 0 \end{bmatrix}$$

with singular upper triangular matrix $V_{A'}$. As $0 \leq b - q^T B^{-1} p$ the theorem also holds in this case. Let $V_A = D_A U_A$ and $V_B = D_B U_B$. As A^T and B^T are also M-matrices which satisfy $A^T \leq B^T$ we have the LU-factorizations $A^T = U_A^T (D_A L_A^T)$ and $B^T = U_B^T (D_B L_B^T)$ with $U_A^T \leq U_B^T$ and $D_A L_A^T \leq D_B L_B^T$. This implies $U_A \leq U_B$ and $0 \leq D_A \leq D_B$. The same reasoning applies to A' and B' if they are nonsingular. If A' is irreducible singular, then

$$D_{A'} U_{A'} = \begin{bmatrix} D_A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_A & D_A{}^{-1} L_A^{-1} c \\ 0 & 0 \end{bmatrix}$$

$$D_{B'} U_{B'} = \begin{bmatrix} D_B & 0 \\ 0 b - q^T B^{-1} p & \end{bmatrix} \begin{bmatrix} U_B & D_B{}^{-1} L_B^{-1} p \\ 0 & 1 \end{bmatrix},$$

from which $0 \leq D_{A'} \leq D_{B'}$ immediately follows. As $D_A{}^{-1} L_A^{-1} c \leq D_B{}^{-1} L_B^{-1} p \leq 0$ the rest of the theorem also follows.

Remark. From Theorem 4.16 of Berman and Plemmons [1, p. 156] it also follows that Theorem 1 does not hold if B is irreducible singular and $A \neq B$.

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