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On the maximum principle for discrete inclusions with constraints

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Abstract. We consider an optimization problem given by a discrete inclusion, whose trajectories are constrained to closed sets. Necessary optimality conditions in the form of the maximum principle and in terms of local generalized derived cones of constraints are obtained.

1. Introduction

Consider the following problem

$$\text{minimize} \quad g(x_N) \tag{1.1}$$

over the solutions of the discrete inclusion

$$x_i \in F_i(x_{i-1}), \quad i = 1, 2, \dots, N,\ dx_0 \in K_0 \tag{1.2}$$

with state constraints of the form

$$x_i \in K_i, \quad i = 1, \dots, N, \tag{1.3}$$

where $F_i : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ are set-valued maps, $i = 1, 2, \dots, N$, $K_i \subset \mathbb{R}^n$, $i = 1, 2, \dots, N$ are closed sets and $g : \mathbb{R}^n \to \mathbb{R}$ is a given function.

Optimization problems given by discrete inclusions have been studied by many authors ([4], [15], [22], [24], [26], [28] etc.). In the framework of multivalued

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problems necessary optimality conditions for problems described by discrete inclusions are obtained in [6]–[9] and in [27]. All these approaches are presented in the absence of state constraints (i.e., $K_i = \mathbb{R}^n$, $i = 1, \dots, N$). In [11] it is considered the more general problem (1.1)-(1.3) and necessary optimality conditions are obtained in terms of local tents of constraints.

In [18], [19] MIRICĂ introduced the concept of generalized derived cone and it is proved that it is more general than the regular tangent cone introduced by POLOVINKIN and SMIRNOV ([23]) while the local generalized derived cone coincides with a local variant of regular tangent cone and contains as a particular case BOLTYANSKII's local tent ([5], [25]).

In general, the efficiency of a given type of local approximation in the theory of necessary optimality conditions seems to rely on certain "intersection properties" from which the best known is that introduced by DUBOVITSKII and MILLYUTIN ([10]) and developed by GIRSANOV ([12]) and BOLTYANSKII ([4], [5]). In [19] it is proved that the local generalized derived cones satisfy another type of intersection property called "the quasitangent intersection property" and which allows to obtain more powerful generalized multiplier rules for general Mathematical Programming problems.

The aim of the present paper is to obtain necessary optimality conditions in the form of the Maximum Principle for problem (1.1)–(1.3) in terms of local generalized derived cones of constraints and in terms of convex linearizations of the discrete inclusions.

On one hand, our result provides a significant example for which the method developed in [19] can be applied and on the other hand, since Boltyanskii's local tent is a particular case of local generalized derived cone, our result extends the necessary optimality conditions obtained in [11].

The paper is organized as follows: in Section 2 we recall some notations and some preliminary results to be used in the sequel and in Section 3 we present the main result of this paper.

2. Preliminaries

Denote by $\mathcal{P}(\mathbb{R}^n)$ the family of all subsets of $\mathbb{R}^n$ and by $B \subset \mathbb{R}^n$ the closed unit ball in $\mathbb{R}^n$. If $A \subset \mathbb{R}^n$ we denote by $cl(A)$ the closure of A and by $\overline{co}(A)$ the closed convex hull of A. In what follows, when the product $Z = Z_1 \times \cdots \times Z_N$ of metric spaces (Z_i, d_{Z_i}), $i = \overline{1, N}$, is considered, it is assumed that Z is equipped with the distance $d_Z((z_1, \dots, z_N), (z'_1, \dots, z'_N)) = \sum_{i=1}^{N} d_{Z_i}(z_i, z'_i)$.

For a set that is, in general, neither a differentiable manifold, nor a convex set, its infinitesimal properties may be characterized only by tangent cones in a generalized sense, extending the classical concepts of tangent cones in Differential Geometry and Convex Analysis, respectively.

We recall first the definitions of the main types of *intrinsic tangent cones* to an arbitrary subset $X \subseteq \mathbb{R}^n$ at a point $x \in X$, that have been most often used in the study of optimization, optimal control and many other problems involving nonsmooth sets and mappings (see, e.g. [1]). The contingent cone, the quasitangent (intermediate tangent) cone and Clarke's tangent cone are defined, respectively, by

$$K_xX = \left\{ v \in \mathbb{R}^n;\ \exists\, s_m \to 0+,\ x_m \in X : \frac{x_m - x}{s_m} \to v \right\}$$

$$Q_xX = \left\{ v \in \mathbb{R}^n;\ \forall\, s_m \to 0+,\ \exists x_m \in X : \frac{x_m - x}{s_m} \to v \right\}$$

$$C_xX = \left\{ v \in \mathbb{R}^n;\ \forall\, (x_m, s_m) \to (x, 0+),\ x_m \in X,\ \exists\, y_m \in X : \frac{y_m - x_m}{s_m} \to v \right\}$$

All the above sets are cones. K_xX, Q_xX, C_xX are closed and C_xX is convex. These cones are related as follows $C_xX \subset Q_xX \subset K_xX$. The rather large gap between Clarke's tangent cone and the quasitangent one may be diminished by several other types of tangent cones, from which we mention only the "asymptotic" variant of the quasitangent cone defined as follows $AQ_xX = \{v \in R^n;\ v + Q_xX \subset Q_xX\}$. AQ_xX is a convex cone and $AQ_xX \subset Q_xX$.

The efforts of some researchers to clarify and extend Pontryagin's Maximum Principle in optimal control theory resulted in the introduction of several "local convex approximations" that may not have the intrinsic character of the cones above but still served their purposes in the theory of necessary optimality conditions. We mention the *derived cone* by Hestenes ([13]), the *first order convex approximation* by Neustadt ([21]), the *cone of tangent vectors* by Lee and Markus ([16]) and the (local) *tent* by Boltyanskii ([4], [5]). Further on, Polovinkin and Smirnov ([23]) extended Boltyanskii's concept of tent to *regular tangent cone*. In the same way in which Boltyanskii relaxed the concept of tents to those of local tents, Bianchini ([3]) introduced the slightly more general concept of *locally regular tangent cone*.

In [18], [19] Mirică introduced the following concept which is a generalization of Hestenes' derived cone in [13].

Definition 2.1. A subset $D \subset \mathbb{R}^n$ is said to be a *generalized derived set* to $X \subset \mathbb{R}^n$ at $x \in X$ if for any $\varepsilon > 0$ and for any finite subset $\{v_1, \dots, v_m\} \subset D$, there exist $r_\varepsilon > 0$ and the continuous mappings $\rho(\varepsilon, r, .) : [0,1]^m \to \mathbb{R}^n$, $r \in [0, r_\varepsilon)$ such that

$$a(\varepsilon, r, s) := x + r\left[\sum_{i=1}^{m} s_i v_i + \rho(\varepsilon, r, s)\right] \in X,$$

$$\|\rho(\varepsilon, r, s)\| \leq \varepsilon, \quad \forall \varepsilon > 0,\ r \in [0, r_\varepsilon),\ s \in [0,1]^m$$

and is said to be *generalized derived cone* if, in addition, it is convex cone; a convex cone $C \subseteq \mathbb{R}^n$ is said to be *local generalized derived cone* if for any $v \in ri(C)$ there exist a generalized derived cone $K \subseteq C$ such that

$$v \in ri(K), \quad \mathrm{span}(C) = \mathrm{span}(K) = \left\{\sum_{i=1}^{k} s_i v_i,\ s_i \in \mathbb{R},\ v_i \in K\right\},$$

where $ri(C)$ denotes the relative interior and $\mathrm{span}(C)$ denotes the vector space generated by the vectors in C.

For the properties of generalized derived cones we refer to [19]. We recall that if D is a generalized derived set then the convex cone generated by D, defined by

$$cco(D) = \left\{\sum_{i=1}^{k} \lambda_j v_j;\ \lambda_j \geq 0,\ k \in \mathbb{N},\ v_j \in D,\ j = 1, \ldots, k\right\}$$

is a generalized derived cone.

As it is proved in [19], the regular tangent cones are particular cases of generalized derived cones but the local regular tangent cones coincide with the local generalized derived cones. At the same time Boltyanskii's local tent is a particular case of local generalized derived cone.

We recall that two cones $C_1, C_2 \subset \mathbb{R}^n$ are said to be *separable* if there exists $q \in \mathbb{R}^n \backslash \{0\}$ such that:

$$\langle q, v\rangle \leq 0 \leq \langle q, w\rangle \quad \forall v \in C_1,\ w \in C_2.$$

We denote by C^+ the positive dual cone of $C \subset \mathbb{R}^n$

$$C^+ = \{q \in \mathbb{R}^n;\ \langle q, v\rangle \geq 0,\ \forall\, v \in C\}$$

The negative dual cone of $C \subset \mathbb{R}^n$ is $C^- = -C^+$.

The following "quasitangent intersection property" of local generalized derived cones, obtained in [19], is a key tool in the proof of our necessary optimality conditions.

Lemma 2.1 ([19]). *Let $X_1, X_2 \subset \mathbb{R}^n$ be given sets, $x \in X_1 \cap X_2$, and let C_1, C_2 be local generalized derived cones to X_1, resp. to X_2 at x. If C_1 and C_2 are not separable, then*

$$Cl(C_1 \cap C_2) = (Cl(C_1)) \cap (Cl(C_2)) \subset Q_x(X_1 \cap X_2).$$

We recall also the following easy corollary of the theorem on separation of two convex sets.

Lemma 2.2 ([17]). *Let $C \subset \mathbb{R}^n$ be a convex cone, and let $h(.) : \mathbb{R}^n \to \mathbb{R}$ be sublinear (i.e., positively homogeneous and subadditive). If h(.) satisfies the condition $h(v) \geq 0\ \forall\, v \in C$, then there exists $q \in C^+$ such that $\langle q, v\rangle \leq h(v)$ $\forall\, v \in \mathbb{R}^n$.*

For a mapping $g(.) : X \subset \mathbb{R}^n \to \mathbb{R}$ which is not differentiable, the classical (Fréchet) derivative is replaced by some generalized directional derivatives. We recall only the upper right-contingent derivative, defined by

$$D_K g(x;v) = \limsup_{(\theta,w)\to(0+,v)} \frac{g(x+\theta w) - g(x)}{\theta}, \quad v \in K_x X$$

and in the case when $g(.)$ is locally-Lipschitz at $x \in int(X)$ by Clarke's generalized directional derivative, defined by:

$$D_C g(x;v) = \limsup_{(y,\theta)\to(x,0+)} \frac{g(y+\theta v) - g(y)}{\theta}, \quad v \in \mathbb{R}^n.$$

The result in the next section will be expressed in terms of the Clarke generalized gradient, defined by

$$\partial_C g(x) = \{q \in \mathbb{R}^n;\ \langle q, v\rangle \leq D_C g(x;v)\ \forall\, v \in \mathbb{R}^n\}.$$

Corresponding to each type of tangent cone, say $\tau_x X$, one may introduce a *set-valued directional derivative* of a multifunction $G(.) : X \subset \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ (in particular of a single-valued mapping) at a point $(x, y) \in \mathrm{Graph}(G)$ as follows

$$\tau_y G(x,v) = \{w \in \mathbb{R}^n;\ (v,w) \in \tau_{(x,y)} \mathrm{Graph}(G)\}, \quad v \in \tau_x X.$$

Let $A : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ be a set-valued map. A is called *closed (respectively, convex) process* if $\mathrm{Graph}(A(.))$ is a closed (respectively, convex) cone. The adjoint process $A^* : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ of the closed convex process A is defined by

$$A^*(p) = \{q \in \mathbb{R}^n;\ \langle q, v\rangle \leq \langle p, v'\rangle\ \forall\,(v,v') \in \mathrm{Graph}\, A(.)\}.$$

For other properties of closed convex processes we refer to [1].

In what follows we are concerned with the discrete inclusion

$$x_i \in F_i(x_{i-1}), \quad i = 1, 2, \dots, N,\ x_0 \in K_0, \tag{2.1}$$

where $F_i : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$, $i = 1, 2, \dots, N$ and $K_0 \subseteq \mathbb{R}^n$.

Denote by S_F the solution set of inclusion (2.1), i.e.

$$S_F := \{x = (x_0, x_1, \dots, x_N);\ x \text{ is a solution of } (2.1)\}.$$

and by $R_F^N := \{x_N;\ x \in S_F\}$ the reachable set of inclusion (2.1).

In the sequel we assume the following hypotheses.

Hypothesis 2.1. i) *$K_0 \subset \mathbb{R}^n$ is a compact convex set and $K_i \subset \mathbb{R}^n$, $i = 1, \dots, N$ are closed sets.*

ii) *The values of $F_i(.)$ are compact convex, $\forall i \in \{1, \dots, N\}$ and there exists $L > 0$ such that $F_i(.)$ is Lipschitz with the Lipschitz constant L, $\forall i \in \{1, \dots, N\}$.*

Consider $\overline{x} = (\overline{x}_0, \overline{x}_1, \dots, \overline{x}_N) \in \mathbb{R}^{(N+1)n}$ a solution of (2.1). We wish to linearize $F_i(.)$ and K_i along $\overline{x}$.s

Consider $A_i : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$, $i = 1, 2, \dots, N$ a family of closed convex processes such that

$$A_i(v) \subset Q_{\overline{x}_i} F_i(\overline{x}_{i-1}; v), \quad \forall v \in \mathbb{R}^n,\ \forall i \in \{1, \dots, N\}. \tag{2.2}$$

Note that if F_i are expressed in the parametrized form

$$F_i(x) = \bigcup_{u_i \in U_i} f_i(x, u_i) \quad \forall x \in \mathbb{R}^n,\ i = 1, \dots, N$$

and $f_i(., u)$ is differentiable $\forall u \in U_i,\ i = 1, \dots, N$ then one may take $A_i(v) = \frac{\partial f_i}{\partial x}(\overline{x}_{i-1}, \overline{u}_i)v$, $i = 1, \dots, N$, where $\overline{u} = (\overline{u}_1, \overline{u}_2, \dots, \overline{u}_N)$ is a control corresponding to solution $\overline{x}$.

Let C_0 be a local generalized derived cone to K_0 at $\overline{x}_0$. To problem (2.1) we associate the linearized problem

$$w_i \in A_i(w_{i-1}), \quad i = 1, 2, \dots, N,\ w_0 \in C_0. \tag{2.3}$$

If K_0 is convex as in Hypotheses 2.1 then as (local) generalized derived cone to K_0 at $\overline{x}_0$ we take

$$C_0 = Cl\{t(y - \overline{x}_0); \quad y \in K_0,\ t \geq 0\}, \tag{2.4}$$

the tangent cone in the sense of Convex Analysis to K_0 at $\overline{x}_0$.

Denote by S_A the solution set of inclusion (2.3).

A characterization of the positive dual of the solution set of problem (2.3) is obtained in [27].

Lemma 2.3 ([27]). *Assume that Hypotheses 2.1 is satisfied. Then, one has*

$$S_A^+ = \{w = (w_0, w_1, \dots, w_N);\ \exists p = (p_0, p_1, \dots, p_N) \in \mathbb{R}^{(N+1)n} \text{ with } p_0 \in C_0^+,\\ p_0 \in A_1^*(p_1) + w_0, p_1 \in A_2^*(p_2) + w_1, \dots, p_{N-1} \in A_N^*(p_N)\\ + w_{N-1},\ p_N = w_N\}.$$

3. The main results

We prove first that the solution set S_A of the variational inclusion (2.3) is a generalized derived cone to S_F at $\overline{x} = (\overline{x}_0, \overline{x}_1, \dots, \overline{x}_N)$.

Theorem 3.1. *Assume that Hypothesis 2.1 is satisfied and let C_0 be a generalized derived cone to K_0 at $\overline{x}_0$.*

Then S_A is a generalized derived cone to S_F at $\overline{x}$.

Proof. In view of Definition 2.1, let $\varepsilon > 0$ and $\{v_1, \dots, v_m\} \subset S_A$; $v_i = (v_0^i, v_1^i, \dots, v_N^i)$, $i = 1, \dots, m$.

Since $\{v_0^1, v_0^2, \dots, v_0^m\} \subset C_0$ and C_0 is a generalized derived cone to K_0 at $\overline{x}_0$, there exist $r_\varepsilon^0 > 0$ and the continuous mappings $\rho_0(\varepsilon, r, .) : [0,1]^m \to \mathbb{R}^n$, $r \in [0, r_\varepsilon^0)$ such that

$$a_0(\varepsilon, r, s) := \overline{x}_0 + r\left[\sum_{i=1}^m s_i v_0^i + \rho_0(\varepsilon, r, s)\right] \in K_0,$$

$$\|\rho_0(\varepsilon, r, s)\| \le \varepsilon, \quad \forall \varepsilon > 0,\ r \in [0, r_\varepsilon^0),\ s \in [0,1]^m$$

Further on, for any $s = (s_1, \dots, s_m) \in [0,1]^m$ we denote

$$v(\varepsilon, r, s) := r\sum_{j=1}^m s_j v_j, \quad y(\varepsilon, r, s) = \overline{x} + v(\varepsilon, r, s).$$

For $x = (x_0, x_1, \dots, x_N) \in \mathbb{R}^n \times \mathbb{R}^n \times \dots \times \mathbb{R}^n$ we define

$$F(x) = (K_0, F_1(x_0), F_2(x_1), \dots, F_N(x_{N-1})).$$

From the compactness of K_0 and the fact that the values of $F_i(.)$, $i = 1, \dots, N$ are compact there exists a mapping $\gamma(.) = (\gamma_0(.), \dots, \gamma_N(.)) : \mathbb{R}^{(N+1)n} \to S_F$ satisfying

$$\|x_0 - \gamma_0(x)\| = d(x_0, K_0),$$

$$\|x_i - \gamma_i(x)\| = d(x_i, F_i(\gamma_{i-1}(x))), \quad i = 1, \dots, N.$$

Moreover, by convexity of K_0 and of the values of $F_i(.)$, the mapping $\gamma(.)$ is continuous.

On the other hand, from the lipschitzianity of $F_i(.)$ we have for any $i=1,\dots,N$

$$\|x_i-\gamma_i(x)\|=d(x_i,F_i(\gamma_{i-1}(x)))\leq d(x_i,F_i(x_{i-1}))$$
$$+L\|x_{i-1}-\gamma_{i-1}(x)\|\leq d(x,F(x))+L\|x_{i-1}-\gamma_{i-1}(x)\|.$$

Thus, there exists $M>0$ depending only on L and N such that

$$\|x-\gamma(x)\|\leq Md(x,F(x)),\quad \forall x\in\mathbb{R}^{(N+1)n}.$$

We define $a(\varepsilon,r,s)=\gamma(y(\varepsilon,r,s))$ and

$$\rho(\varepsilon,r,s):=\frac{a(\varepsilon,r,s)-\overline{x}}{r}-\sum_{j=1}^m s_jv_j,\quad ,r\in[0,r_\varepsilon^0),\ s\in[0,1]^m.$$

From the continuity of $\gamma(.)$ we deduce the continuity of $\rho(\varepsilon,r,.)$.

Since $A_i(.)$, $i=1,\dots,N$ are convex process and

$$v_i^k\in A_i(v_{i-1}^k),\quad i=1,\dots,N,\ k=1,\dots,m$$

it follows that

$$v(\varepsilon,1,s)_i\in A_i(v(\varepsilon,1,s)_{i-1})\subset Q_{\overline{x}_i}F_i(\overline{x}_{i-1};v(\varepsilon,1,s)_{i-1}),\quad i=1,\dots,N$$

and taking into account the characterization of the quasitangent derivative of lipschitzian set-valued maps (e.g., [1]) we obtain that

$$\lim_{h\to 0+}\frac{1}{h}d(\overline{x}_i+hv(\varepsilon,1,s)_i,F_i(\overline{x}_{i-1}+hv(\varepsilon,1,s)_{i-1}))=0,\quad i=1,\dots,N.$$

Therefore, for $\varepsilon>0$ there exist $r_\varepsilon^i>0$, $i=1,\dots,N$ such that if $r\in[0,r_\varepsilon^i)$

$$\frac{1}{r}d(\overline{x}_i+v(\varepsilon,r,s)_i,F_i(\overline{x}_{i-1}+v(\varepsilon,r,s)_{i-1}))<\varepsilon.$$

It remains to take $r_\varepsilon:=\min\{r_\varepsilon^i,\ i=0,1,\dots,N\}$ and note that

$$\|\rho(\varepsilon,r,s)\|=\frac{1}{r}\|a(\varepsilon,r,s)-y(\varepsilon,r,s)\|=\frac{1}{r}\|\gamma(y(\varepsilon,r,s))-y(\varepsilon,r,s)\|$$
$$\leq\frac{M}{r}d(y(\varepsilon,r,s),F(y(\varepsilon,r,s)))\leq M\|\rho_0(\varepsilon,r,s)\|$$
$$+M\sum_{j=1}^N\frac{1}{r}d(y(\varepsilon,r,s)_j,F_j(y(\varepsilon,r,s)_{j-1}))\leq M(N+1)\varepsilon.\qquad\square$$

We are able to prove our main result in which we obtain necessary optimality condition for a solution $\overline{x} = (\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_N)$ to problem (1.1)–(1.3) in the form of maximum principle.

Theorem 3.2. *Let $\overline{x} = (\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_N) \in \mathbb{R}^{(N+1)n}$ be an optimal solution to problem (1.1)–(1.3), assume that Hypothesis 2.1 is satisfied, let C_0 be defined as in (2.4) and let $g(.) : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function.*

Then, for any local generalized derived cone C_i to K_i at $\overline{x}_i$, $i = 1, \ldots, N$ there exist $\lambda \in \{0, 1\}$, $(p_0, p_1, \ldots, p_N) \in \mathbb{R}^{(N+1)n}$ and $(\eta_1, \ldots, \eta_N) \in \mathbb{R}^{Nn}$ such that

$$\begin{aligned} &p_0 \in C_0^+, \quad p_0 \in A_1^*(p_1), \quad p_1 \in A_2^*(p_2) + \eta_1, \quad \ldots, \\ &p_{N-2} \in A_{N-1}^*(p_{N-1}) + \eta_{N-2}, \quad p_{N-1} \in A_N^*(p_N) + \eta_{N-1}, \end{aligned} \tag{3.1}$$

$$\eta_i \in C_i^-, \quad i = 0, 1, \ldots, N-1, \tag{3.2}$$

$$p_N \in \lambda \partial_C g(\overline{x}_N) + C_N^-, \tag{3.3}$$

$$\langle -p_i, \overline{x}_i \rangle = \max\{\langle -p_i, v \rangle;\ v \in F_i(\overline{x}_{i-1})\}, \quad i = 1, \ldots, N, \tag{3.4}$$

$$\lambda + \sum_{i=0}^{N} \|p_i\| + \sum_{i=1}^{N} \|\eta_i\| > 0. \tag{3.5}$$

Proof. Consider the set-valued map $B_i(.) : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ $i = 1, 2, \ldots, N$ defined by

$$B_i(y) = cl(A_i(y) + \bigcup_{t>0} \frac{1}{t}(F_i(\overline{x}_{i-1}) - \overline{x}_i)).$$

Then, by Proposition 3.5 in [14], $\{B_i\}_{i=1,\ldots,N}$ is a family of closed convex processes satisfying (2.2), $A_i \subset B_i$ and, moreover

$$B_i^*(y) = \{\begin{cases} A_i^*(y) & \text{if } \langle -y, \overline{x}_i \rangle = \max\{\langle -y, v \rangle;\ v \in F_i(\overline{x}_{i-1})\}, \\ \emptyset & \text{otherwise.} \end{cases} \tag{3.6}$$

Therefore, it what follows A_i will be replaced by B_i.

Set

$$\mathcal{C} = \mathbb{R}^n \times C_1 \times \cdots \times C_N$$

and

$$\gamma(x) = x_N \quad \forall\, x = (x_0, x_1, \ldots, x_N) \in \mathbb{R}^{(N+1)n}.$$

We have two cases.

Case 1. $\mathcal{C}$ and S_B are separable. Then there exists $q \in \mathbb{R}^{(N+1)n}\backslash\{0\}$ such that

$$\langle q, c\rangle \leq 0 \leq \langle q, a\rangle \quad \forall\, c \in \mathcal{C},\ a \in S_B,$$

It means that $q \in \mathcal{C}^- \cap S_B^+$. On one hand, from $q \in \mathcal{C}^-$ it follows that $q = (0, \eta_1, \dots, \eta_N)$ with $\eta_i \in C_i^-, \quad i = 0, 1, \dots, N$.

On the other hand, $q \in S_B^+$ and taking into account Lemma 2.3 there exists $(p_0, p_1, \dots, p_N) \in \mathbb{R}^{(N+1)n}$ such that

$$\begin{aligned} &p_0 \in C_0^+, \quad p_0 \in B_1^*(p_1), \quad p_1 \in B_2^*(p_2) + \eta_1, \quad \dots, \\ &p_{N-2} \in B_{N-1}^*(p_{N-1}) + \eta_{N-2}, \quad p_{N-1} \in B_N^*(p_N) + \eta_{N-1}, \quad p_N = \eta_N \in C_N^- \end{aligned} \tag{3.7}$$

From (3.6) and (3.7) it follows the adjoint inclusions (3.1) and the maximum conditions (3.4). It is enough to take $\lambda = 0$ and to note that the non triviality condition (3.5) holds true in this case. Indeed, if $\sum_{i=1}^N \|\eta_i\| = 0$ then it follows that $q = 0$, which is a contradiction.

Case 2. $\mathcal{C}$ and S_B are not separable. In this case (e.g., [2]) $(S_B \cap \mathcal{C})^+ = S_B^+ + \mathcal{C}^+$.

Since, by Theorem 3.1, S_B is a generalized derived cone to S_F at $\overline{x}$ (and thus a local generalized derived cone) and, obviously, $\mathcal{C}$ is a local generalized derived cone to $\mathbb{R}^n \times K_1 \times \dots \times K_N$ at $\overline{x}$ we apply Lemma 2.1 and we deduce that

$$S_B \cap \mathcal{C} \subset Q_{\overline{x}}(S_F \cap K).$$

From the definition of quasitangent cone we have that if $v = (v_0, v_1, \dots, v_N) \in Q_{\overline{x}}(S_F \cap K)$ then $v_N \in Q_{\overline{x}_N}(R_F^N \cap K_N)$. We have $g(\overline{x}_N) = \min\{g(x) : x \in K_N \cap R_F^N\}$ and from definitions it follows (e.g., Proposition 4.1 in [19])

$$D_C g(\overline{x}_N; v) \geq D_K g(\overline{x}_N; v) \geq 0 \quad \forall v \in Q_{\overline{x}_N}(K_N \cap R_F^N).$$

Therefore,

$$D_C g(\overline{x}_N; v) \geq 0, \quad \forall v \in \gamma(S_B \cap \mathcal{C}).$$

We apply Lemma 2.2 with $h(.) = D_C g(\overline{x}_N, .)$ and we find that there exists $\alpha \in \partial_C g(\overline{x}_N)) \cap [\gamma(S_B \cap \mathcal{C})]^+ = \partial_C g(\overline{x}_N) \cap \gamma^{*-1}([S_B \cap \mathcal{C}]^+) = \partial_C g(\overline{x}_N)) \cap \gamma^{*-1}(S_B^+ + \mathcal{C}^+)$.

Hence for some $\eta \in \mathcal{C}^-$, $\gamma^*(\alpha) + \eta \in S_B^+$. As in Case 1, $\eta = (0, \eta_1, \dots, \eta_N)$ with $\eta_i \in C_i^-, \quad i = 0, 1, \dots, N$. We apply again Lemma 2.3 to deduce the existence of $(p_0, p_1, \dots, p_N) \in \mathbb{R}^{(N+1)n}$ such that, if $\gamma^*(\alpha) + \eta = (w_0, w_1, \dots, w_N)$, one has

$$\begin{aligned} &p_0 \in C_0^+, \quad p_0 \in B_1^*(p_1) + w_0, \quad p_1 \in B_2^*(p_2) + w_1, \quad \dots, \\ &p_{N-2} \in B_{N-1}^*(p_{N-1}) + w_{N-2}, \quad p_{N-1} \in B_N^*(p_N) + w_{N-1}, \quad p_N = w_N. \end{aligned}$$

We have

$$\langle \gamma^*(\alpha) + \eta, x \rangle = \langle w, x \rangle \quad \forall \quad x = (x_0, x_1, \ldots, x_N) \in \mathbb{R}^{(N+1)n}.$$

If $x \in \mathbb{R}^{(N+1)n}$ with $x_0 = x_1 = \cdots = x_{N-1} = 0$ it follows that $\alpha + \eta_N = w_N$. Therefore $p_N = w_N = \alpha + \eta_N \in \partial_C g(\overline{x}_N) + C_N^-$. On the other hand

$$\langle w - \eta, x \rangle = \langle \gamma^*(\alpha), x \rangle = \langle \alpha, \gamma(x) \rangle = \langle \alpha, x_N \rangle$$

$\forall \ x = (x_0, x_1, \ldots, x_N) \in \mathbb{R}^{(N+1)n}$.

Since $\alpha = -\eta_N + w_N$ it follows

$$\langle w - \eta, x \rangle = \langle w_N - \eta_N, x_N \rangle \quad \forall \ x = (x_0, x_1, \ldots, x_N) \in \mathbb{R}^{(N+1)n}.$$

In particular, $w_i = \eta_i$, $i = 1, \ldots, N-1$, $w_0 = 0$.

The adjoint inclusion (3.1) and the maximum condition (3.4) follows as in Case 1. It remains to take $\lambda = 1$ and the proof is complete. □

Remark 3.1. Several remarks are in order.

i) If in the above theorem we assume that $g(.)$ is differentiable at $\overline{x}_N$, then by a very slight modification of the proof, inclusion (3.3) can be replaced by

$$p_N \in \lambda \nabla g(\overline{x}_N) + C_N^-.$$

ii) If in Theorem 3.2, C_i are local tents at K_i at $\overline{x}_i$ then Theorem 3.2 yields the main result in [11], namely Theorem 3.

iii) Theorem 3.2 extends the main results in [9], obtained for an optimization problem with only end point constraints (i.e., $K_i = \mathbb{R}^n, i = 1, \ldots, N-1$); result obtained in terms of Hestenes' derived cone to K_N at $\overline{x}_N$.

iv) A result similar to that of Theorem 3.2 can be obtained without any convexity assumptions in terms of the so-called limiting normal cones introduced by B. Mordukhovich and subdifferentials by applying the generalized Lagrange Multipler Rule as in [20].

In particular, when F_i are expressed in the parametrized form

$$F_i(x_{i-1}) = \bigcup_{u_i \in U_i} f_i(x_{i-1}, u_i) \quad \forall x_{i-1} \in \mathbb{R}^n, \ i = 1, \ldots, N,$$

i.e., inclusion (1.2) became the nonlinear discrete system

$$x_i = f_i(x_{i-1}, u_i), \quad u_i \in U_i, \quad i = 1, \ldots, N, \quad x_0 \in K_0 \tag{3.8}$$

taking $A_i(v) = \frac{\partial f_i}{\partial x}(\overline{x}_{i-1}, \overline{u}_i)v$, $i = 1, \ldots, N$ we obtain the following consequence of Theorem 3.2.

Corollary 3.1. *Let $U_i \subset \mathbb{R}^n$ $i = 1, \dots, N$ be compact set, let $f_i(.,.) : \mathbb{R}^n \times U_i \to \mathbb{R}^n$ be such that $f_i(., u_i)$ is differentiable and the multifunction F_i satisfy Hypothesis 2.1, $i = 1, \dots, N$, let $\overline{x} = (\overline{x}_0, \overline{x}_1, \dots, \overline{x}_N) \in \mathbb{R}^{(N+1)n}$ be an optimal solution for problem* (1.1), (3.8), (1.3) *and $\overline{u} = (\overline{u}_1, \overline{u}_2, \dots, \overline{u}_N)$ be a control corresponding to solution $\overline{x}$. Consider C_0 defined in (2.4) and $g(.) : \mathbb{R}^n \to \mathbb{R}$ a locally Lipschitz function.*

Then, for any local generalized derived cone C_i to K_i at $\overline{x}_i$, $i = 1, 2, \dots, N$ there exist $\lambda \in \{0, 1\}$, $(p_0, p_1, \dots, p_N) \in \mathbb{R}^{(N+1)n}$ and $(\eta_1, \dots, \eta_N) \in \mathbb{R}^{Nn}$ such that

$$p_0 \in C_0^+, \quad p_0 = \left(\frac{\partial f_1}{\partial x}(\overline{x}_0, \overline{u}_1)\right)^* (p_1),$$

$$p_1 = \left(\frac{\partial f_N}{\partial x}(\overline{x}_{N-1}, \overline{u}_N)\right)^* (p_2) + \eta_1, \dots p_{N-1} = \left(\frac{\partial f_N}{\partial x}(\overline{x}_{N-1}, \overline{u}_N)\right)^* (p_N) + \eta_{N-1},$$

$$\eta_i \in C_i^-, \quad \forall i = 1, \dots, N-1,$$

$$p_N \in \lambda \partial_C g(\overline{x}_N) + C_N^-,$$

$$\langle -p_i, \overline{x}_i \rangle = \max\{\langle -p_i, f_i(\overline{x}_{i-1}, u_i)\rangle, \quad u_i \in U_i\},\ i = 1, \dots, N,$$

$$\lambda + \sum_{i=0}^{N} \|p_i\| + \sum_{i=1}^{N} \|\eta_i\| > 0.$$

References

[1] J. P. Aubin and H. Frankowska, Set-Valued Analysis, *Birkhäuser*, *Basel*, 1990.

[2] M. S. Bazaraa and C. M. Shetty, Foundations of Optimization, *Springer*, *New York*, 1976.

[3] R. M. Bianchini, Variational Cones and Higher Order Maximum Principles, Preprint 12, *Dipartimento di Matematica, Universita di Firenze*, *Firenze*, 1994.

[4] V. G. Boltjanskii, Optimal Control of Discrete Systems, *Nauka*, *Moskow*, 1972 (in *Russian*).

[5] V. G. Boltjanskii, The method of tents in the theory of extremal problems, *Russian Math. Surveys* **30** (1975), 1–54.

[6] A. Cernea, On the maximum principle for discrete inclusions with end point constraints, *Math. Reports* **7**(57) (2005), 13–20.

[7] A. Cernea, Second-order necessary conditions for discrete inclusions with end point constraints, *Discuss. Math. Diff. Incl. Control Optim.* **25** (2005), 47–58.

[8] A. Cernea, Minimum principle and controllability for multiparameter discrete inclusions via derived cones, *Discrete Dyn. Nat. Soc.* **ID 96505** (2006), 1–12.

[9] A. Cernea, Derived cones to reachable sets of discrete inclusions, *Nonlinear Studies* **14** (2007), 177–187.

[10] A. Ya. Dubovitskii and A. A. Milyutin, Extremal problems with constraints, *Dokl. Akad. Nauk SSSR, Mat.* **149** (1963), 759–762.

[11] E. Duda, A. Korzun and O. Minchenko, On local controllability of discrete systems, *Diff. Uravn.* **33** (1997), 462–469.

[12] I. V. Girsanov, Lectures on Mathematical Theory of Extremum Problems, *Springer*, *Berlin*, 1972.

[13] M. R. Hestenes, Calculus of Variations and Optimal Control Theory, *Wiley*, *New York*, 1966.

[14] B. Kaskosz, Maximum principle in relaxed controls, *Nonlinear Anal.* **14** (1990), 357–367.

[15] A. G. Kusraev, Discrete maximum principle, *Math. Notes* **34** (1984), 617–619.

[16] E. B. Lee and L. Markus, Foundations of Optimal Control Theory, *Wiley*, *New York*, 1966.

[17] Ş. Mirică, New proof and some generalizations of the minimum principle in optimal control, *J. Optim. Theory Appl.* **74** (1992), 487–508.

[18] Ş. Mirică, Generalized derived cones and other convex approximations of sets, *An. Univ. Bucureşti, Mat.* **47** (1998), 155–168.

[19] Ş. Mirică, Intersection properties of tangent cones and generalized multiplier rules, *Optimization* **46** (1999), 135–163.

[20] B. Mordukhovich, Discrete approximations and refined Euler-Lagrange conditions for nonconvex differential inclusions, *SIAM J. Control Optimi.* **33** (1995), 882–915.

[21] L. W. Neustadt, An abstract variational theory with applications to a broad class of optimization problems,, I. General theory, *SIAM J. Control* **4** (1966), 505–528.

[22] V. N. Phat, Controllability of nonlinear discrete systems without differentiability assumption, *Optimization* **19** (1988), 133–142.

[23] E. S. Polovinkin and G. V. Smirnov, An approach to differentiation of many-valued mapping and necessary condition for optimization of solution of differential inclusions, *Diff. Uravn.* **22** (1986), 944–954.

[24] A. I. Propoi, The maximum principle for discrete control systems, *Automat. Remote Control* **26** (1965), 451–461.

[25] B. N. Pshenihnyi, Convex analysis and extremal problems, *Nauka*, *Moskow*, 1980 (in *Russian*).

[26] R. Pytlak, A variational approach to discrete maximum principle, *IMA J. Math. Control and Information* **9** (1992), 197–220.

[27] H. D. Tuan and Y. Ishizuka, On controllability and maximum principle for discrete inclusions, *Optimization* **34** (1995), 293–316.

[28] N. D. Yen and T. C. Dien, On local controllability of nondifferentiable discrete time systems with nonconvex constraints on control, *Optimization* **20** (1989), 889–899.

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