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Homology decompositions in shape theory

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Abstract. In this paper, for each inverse sequence **G** of abelian groups and for each $n > 1$, we introduce the notion of approximate Moore space of type $(\mathbf{G}, n)$ and obtain shape theoretical homology decompositions for metric continua in the sense of E. H. Brown and A. H. Copeland.

1. Introduction

The Postnikov system of a space is a decomposition of the space into a tower of fibre CW complexes, each of which has finitely many nonvanishing homotopy groups that are isomorphic to the corresponding homotopy groups of the given space. The construction involves the technique of adding homotopy groups to CW complexes. E. H. Brown and A. H. Copeland [5] obtained a homology analogue of Postnikov system, whose construction involves the technique of adding homology groups to CW complexes. Those two techniques used in the constructions are dual to each other. B. Eckmann and P. J. Hilton introduced a systematic approach to the duality.

In this paper homology decompositions of metric continua are obtained in the framework of shape theory. Shape theory is formed in such a natural way that it deals with homotopy properties of general spaces. In shape theory, spaces and maps are respectively expanded into systems of polyhedra (alternatively, CW complexes or ANR's) and morphisms between systems in the pro-category pro-HTop of the homotopy category HTop of spaces and maps. One then studies the systems and the morphisms to study the shape properties of the spaces and

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maps. Postnikov systems in the pro-category are obtained [2], [6], but in order to obtain Postnikov systems in the shape category, one has to consider the limits of noncompact polyhedra, which may be empty. Noncompact polyhedra appear when one adds infinitely many cells in the process of adding homotopy groups. From this point of view, the process used for homology decompositions is more convenient than that used for homotopy decompositions because fewer cells are added in homology decompositions. In this paper the homology decompositions of metric continua are studied in the framework of shape theory. The constructions are based on those introduced by Brown and Copeland.

This paper is organized as follows. In the next section, basic notations and definitions are recalled. In Section 3, the notion of approximate Moore space is introduced, and the existence and uniqueness are discussed. In Section 4, cofibration sequences of shape morphisms are obtained. In Section 5, shape theoretical homology decompositions are proved to exist.

2. Preliminaries

Throughout the paper, space means pointed topological space, and map means base point preserving continuous map, and homotopy means base point preserving homotopy.

For any space X, let CX denote the cone over X, i.e., $CX = X \wedge I$, where $I = [0, 1]$, and $0 \in I$ is the base point of I. For any map $f : X \to Y$, let C_f denote the mapping cone (cofibre) of f, i.e., $C_f = CX \vee Y/(x \wedge 1) \sim f(x)$, and let Z_f denote the mapping cylinder of f, i.e., $Z_f = IX \vee Y/[(x, 1)] \sim f(x)$, where $IX = X \times I/\{x_0\} \times I$. For any map $f : X \to Y$, define a map $Cf : CX \to CY$ by $Cf(x \wedge t) \mapsto f(x) \wedge t$. Let SX be the suspension $X \wedge S^1$ of X, which is homeomorphic to the quotient space $X \times I/X \times 0 \cup \{x_0\} \times I \cup X \times 1$. For any map $f : X \to Y$, define a map $Sf : SX \to SY$ by $Sf = f \wedge 1_{S^1}$.

Let Top be the category of spaces and maps, let CM be the full subcategory of Top whose objects are metric continua. Let HTop be the homotopy category of Top, let HCM and HPol be the full subcategories of HTop whose objects are metric continua and spaces having the homotopy type of a connected polyhedron, respectively. Let Gp be the category of groups and homomorphisms, and let Ab denote the full subcategory of Gp whose objects are abelian groups.

Let $\mathcal{C}$ be any category. Let $\mathbb{N}$ be the set of all natural numbers. For any inverse sequence $\mathbf{X} = (X_i, p_{i,i+1})$ in $\mathcal{C}$, let $p_{ii'} = p_{i,i+1} \circ p_{i+1,i+2} \circ \cdots \circ p_{i'-1,i'}$ for $i < i'$, and let $p_{ii} = 1_{X_i}$ for each $i \in \mathbb{N}$. A system map $(f_i, f) : \mathbf{X} \to \mathbf{Y}$

between inverse sequences in $\mathcal{C}$ consists of a function $f : \mathbb{N} \to \mathbb{N}$ and maps $f_i : X_{f(i)} \to X_i$ for $i \in \mathbb{N}$ such that whenever $i_1 < i_2$, there is $i > f(i_1), f(i_2)$ with $f_{i_1} \circ p_{f(i_1)i} = q_{i_1 i_2} \circ f_{i_2} \circ p_{f(i_2)i}$. A system map $(f_i, f) : \mathbf{X} \to \mathbf{Y}$ is said to be a *level map* if the function f is the identity, in which case we write (f_i) for (f_i, f). For any two system maps $(f_i, f), (g_i, g) : \mathbf{X} \to \mathbf{Y}$, (f_i, f) and (g_i, g) are said to be equivalent, in notation, $(f_i, f) \sim (g_i, g)$, provided for each i, there exists $i' > f(i), g(i)$ such that $f_i \circ p_{f(i)i'} = g_i \circ q_{g(i)i'}$. The relation $\sim$ is an equivalence relation. A morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ in pro-$\mathcal{C}$ is an equivalence class of system maps $(f_i, f) : \mathbf{X} \to \mathbf{Y}$.

For any map $f : X \to Y$, let $[f]$ denote the homotopy class of f. For any inverse sequence $\mathbf{X} = (X_i, p_{i,i+1})$ in Top, let $[\mathbf{X}]$ denote the induced inverse sequence $(X_i, [p_{i,i+1}])$ in HTop. Since $(f_i, f) \sim (g_i, g)$ implies $([f_i], f) \sim ([g_i], g)$, every morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ in pro-Top induces a morphism $[\mathbf{f}] : [\mathbf{X}] \to [\mathbf{Y}]$ in pro-HTop.

In this paper we consider the shape category $\mathsf{Sh(CM)}$ restricted to the objects of CM. For any two morphisms $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ and $\mathbf{f}' : \mathbf{X}' \to \mathbf{Y}'$ in pro-HPol where $\mathbf{p} : X \to \mathbf{X}$, $\mathbf{q} : Y \to \mathbf{Y}$, $\mathbf{p}' : X \to \mathbf{X}'$ and $\mathbf{q}' : Y \to \mathbf{Y}'$ are HPol-expansions in the sense of [7, p. 20], we say that $\mathbf{f}$ and $\mathbf{f}'$ are equivalent, in notation, $\mathbf{f} \equiv \mathbf{f}'$, provided the following diagram is commutative.

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{\ \mathbf{f}\ } & Y \\ {\scriptstyle \mathbf{i}}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{j}} \\ \mathbf{X}' & \xrightarrow{\ \mathbf{f}'\ } & \mathbf{Y}' \end{array}$$

Here $\mathbf{i}$ and $\mathbf{j}$ are natural isomorphisms. A morphism $F : X \to Y$ in $\mathsf{Sh(CM)}$ is an equivalence class of morphisms $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ in pro-HPol. For more details on shape theory, the reader is referred to [7].

Let H_* and $\tilde{\mathrm{H}}_*$ denote the unreduced and reduced singular homology theories with the integer coefficients, respectively.

3. Approximate Moore spaces

For any abelian group G and $n > 1$, a Moore space of type (G, n) is a connected, simply connected space M such that

$$\mathrm{H}_q(M) \approx \begin{cases} \mathbb{Z} & (q = 0), \\ G & (q = n), \\ 0 & (q \neq 0, n). \end{cases}$$

Any two CW complexes that are Moore spaces of type (G, n) have the same homotopy type. Let $\mathrm{M}(G, n)$ denote the CW complex which is defined as the mapping cone C_δ of a map $\delta : \bigvee_J S^n \to \bigvee_I S^n$ inducing the homomorphism d for a presentation $0 \to \bigoplus_J \mathbb{Z} \xrightarrow{d} \bigoplus_I \mathbb{Z} \xrightarrow{p} G \to 0$ of G.

Proposition 3.1 ([3, 1.4.3])**.** *If $\alpha : G \to H$ is a homomorphism between abelian groups, there exists a map $\mathrm{M}(\alpha, n) : \mathrm{M}(G, n) \to \mathrm{M}(H, n)$, which is defined up to homotopy such that the correspondence $\mathrm{M}(\cdot, n) : \mathsf{Ab} \to \mathsf{HTop}$ is functorial.*

We write $\mu_n(\alpha)$ for $\mathrm{M}(\alpha, n)$ in Proposition 3.1.

Let $\mathbf{G}$ be an inverse sequence in Ab, and let $n > 1$. An inverse sequence $\mathbf{X}$ in Top or HTop is said to be *of type* $(\mathbf{G}, n)$ provided $\mathbf{X}$ is 1-connected and

$$\mathrm{H}_q(\mathbf{X}) \approx \begin{cases} (\mathbb{Z}) & (q = 0), \\ \mathbf{G} & (q = n), \\ 0 & (q \neq 0, n). \end{cases}$$

An *approximate Moore space of type* $(\mathbf{G}, n)$ is a metric continuum X which admits an HPol-expansion $\mathbf{p} : X \to \mathbf{X}$ such that $\mathbf{X}$ is an inverse sequence of type $(\mathbf{G}, n)$.

The following shows the existence of approximate Moore spaces.

Proposition 3.2. *If $\mathbf{G} = (G_i, \alpha_{i,i+1})$ is an inverse sequence in Ab such that all G_i are finitely generated abelian groups, then for each $n > 1$, there exists an approximate Moore space $\mathrm{M}(\mathbf{G}, n)$ of type $(\mathbf{G}, n)$ with dimension at most $n + 1$.*

Proof. Consider the Moore spaces $\mathrm{M}(G_i, n)$ for $i \in \mathbb{N}$. Since each G_i is finitely generated, $\mathrm{M}(G_i, n)$ is a connected compact polyhedron. By Proposition 3.1, we obtain an inverse sequence $\mathbf{M}(\mathbf{G}, n) = (\mathrm{M}(G_i, n), \mu_n(\alpha_{i,i+1}))$. Let $\mathbf{p} = (p_i) : \mathrm{M}(\mathbf{G}, n) \to \mathbf{M}(\mathbf{G}, n)$ be the inverse limit of $\mathbf{M}(\mathbf{G}, n)$. Then $\mathrm{M}(\mathbf{G}, n)$ is a 1-shape connected continuum with the desired pro-homology groups. Since each $\mathrm{M}(G_i, n)$ has dimension at most $n + 1$, the dimension of $\mathrm{M}(\mathbf{G}, n)$ is at most $n + 1$. □

Let $\mathcal{F}$ be the full subcategory of pro-Ab whose objects are inverse sequences consisting of finitely generated abelian groups.

Proposition 3.3. *If $\boldsymbol{\rho} : \mathbf{G} \to \mathbf{H}$ is a morphism in $\mathcal{F}$, then there exists a shape morphism $\mathrm{M}(\boldsymbol{\rho}, n) : \mathrm{M}(\mathbf{G}, n) \to \mathrm{M}(\mathbf{H}, n)$ such that the correspondence $\mathrm{M}(\cdot, n) : \mathcal{F} \to \mathsf{Sh}(\mathsf{CM})$ is functorial.*

PROOF. Let $\boldsymbol{\rho}$ be represented by a system map $(\rho_i, \rho) : \mathbf{G} = (G_i, \alpha_{i,i+1}) \to \mathbf{H} = (H_i, \beta_{i,i+1})$. By Proposition 3.1, this induces a system map $([\mathrm{M}(\rho_i, n)], \rho) : [\mathbf{M}(\mathbf{G}, n)] \to [\mathbf{M}(\mathbf{H}, n)]$. This represents a shape morphism $\mathrm{M}(\boldsymbol{\rho}, n) : \mathrm{M}(\mathbf{G}, n) \to \mathrm{M}(\mathbf{H}, n)$. We must show that this definition does not depend on the choice of the system map (ρ_i, ρ). To see this, let (ρ'_i, ρ') be another choice of system map representing $\boldsymbol{\rho}$. Then, for each i, there exists $j > i$ such that $\rho_i \circ \alpha_{\rho(i),j} = \rho'_i \circ \alpha_{\rho'(i),j}$. Since $\mathrm{M}(\cdot, n)$ is functorial, $\mathrm{M}(\rho_i, n) \circ \mathrm{M}(\alpha_{\rho(i),j}, n) \simeq \mathrm{M}(\rho'_i, n) \circ \mathrm{M}(\alpha_{\rho'(i),j}, n)$. This means that $([\mathrm{M}(\rho_i, n)], \rho)$ and $([\mathrm{M}(\rho'_i, n)], \rho')$ represent the same morphism $\mathrm{M}(\mathbf{G}, n) \to \mathrm{M}(\mathbf{H}, n)$ in pro-HPol, and hence the same shape morphism. It is readily seen that $\mathrm{M}(1_{\mathbf{G}}, n) = 1_{\mathrm{M}(\mathbf{G},n)}$, where $1_{\mathbf{G}} : \mathbf{G} \to \mathbf{G}$ and $1_{\mathrm{M}(\mathbf{G},n)} : \mathrm{M}(\mathbf{G}, n) \to \mathrm{M}(\mathbf{G}, n)$ are the identity morphisms in $\mathcal{F}$ and $\mathsf{Sh(CM)}$, respectively, and that if $\boldsymbol{\rho} : \mathbf{G} \to \mathbf{H}$ and $\boldsymbol{\tau} : \mathbf{H} \to \mathbf{K}$ are morphisms in $\mathcal{F}$, then $\mathrm{M}(\boldsymbol{\tau} \circ \boldsymbol{\rho}, n) = \mathrm{M}(\boldsymbol{\tau}, n) \circ \mathrm{M}(\boldsymbol{\rho}, n)$ □

Proposition 3.4 ([4, 3a.5, p. 268])**.** *For any space X and any abelian group G, and for $n > 1$, there exists a short exact sequence*

$$0 \longrightarrow \mathrm{Ext}(G, \pi_{n+1}(X)) \longrightarrow [\mathrm{M}(G, n), X] \xrightarrow{p} \mathrm{Hom}(G, \pi_n(X)) \longrightarrow 0.$$

Here p is the n-th homotopy group functor.

Proposition 3.5. *Let $\mathbf{G} = (G_i, \alpha_{i,i+1})$ be an inverse sequence consisting of finitely generated abelian groups, and let X and Y be approximate Moore spaces of type $(\mathbf{G}, n)$ with finite shape dimension. Then X and Y have the same shape type if one of the following conditions holds.*

(1) *Each G_i is free.*

(2) *Each of the two progroups* pro-$\pi_{n+1}(X)$, pro-$\pi_{n+1}(Y)$ *is divisible.*

Here a progroup is said to be divisible provided each term is a divisible group.

PROOF. Let X be an approximate Moore space of type $(\mathbf{G}, n)$. We show that if (1) holds, or if pro-$\mathrm{H}_{n+1}(X)$ is divisible, X has the shape type of $\mathrm{M}(\mathbf{G}, n)$ (see Proposition 3.2). Let $\mathbf{p} = (p_i) : X \to \mathbf{X}$ be the limit of some inverse sequence $\mathbf{X} = (X_i, p_{i,i+1})$ consisting of compact connected polyhedra such that $\dim X_i \le N$ for $i \in \mathbb{N}$, where N is some fixed nonnegative integer. By the Hurewicz theorem [7, Theorem 2, p. 136], conditions that $\mathrm{H}_q(\mathbf{X}) = 0$ for $q < n$ and that $\pi_1(\mathbf{X}) = 0$ imply $\pi_q(\mathbf{X}) = 0$ for $q < n$ and $\pi_n(\mathbf{X}) \approx \mathrm{H}_n(\mathbf{X}) \approx \mathbf{G}$. Let $(\psi_i, \psi) : \mathbf{G} \to \pi_n(\mathbf{X})$ be a system map which induces an isomorphism in pro-Ab. By [7, Theorem 3, p. 12], we can assume that the system map (ψ_i, ψ) is a level map (ψ_i). Since $[\mathrm{M}(G_i, n), X_i] \to \mathrm{Hom}(G_i, \pi_n(X_i))$ is onto by Proposition 3.4, then, for each i, there exists a map $f_i : \mathrm{M}(G_i, n) \to X_i$ which induces ψ_i. If

G_{i+1} is free, or if $\pi_{n+1}(X_i)$ is divisible, then $\mathrm{Ext}(G_{i+1}, \pi_{n+1}(X_i)) = 0$, which implies $[\mathrm{M}(G_{i+1}, n), X_i] \cong \mathrm{Hom}(G_{i+1}, \pi_n(X_i))$ by Proposition 3.4. Hence the commutativity of the diagram

$$\begin{array}{ccc} G_i & \xleftarrow{\alpha_{i,i+1}} & G_{i+1} \\ {\scriptstyle \pi_n(f_i)}\downarrow & & \downarrow{\scriptstyle \pi_n(f_{i+1})} \\ \pi_n(X_i) & \xleftarrow{\pi_n(p_{i,i+1})} & \pi_n(X_{i+1}) \end{array}$$

implies the homotopy commutativity of the following diagram.

$$\begin{array}{ccc} \mathrm{M}(G_i, n) & \xleftarrow{\mu_n(\alpha_{i,i+1})} & \mathrm{M}(G_{i+1}, n) \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_{i+1}} \\ X_i & \xleftarrow{p_{i,i+1}} & X_{i+1} \end{array}$$

Thus we have a level map $([f_i]) : [\mathbf{M}(\mathbf{G}, n)] \to [\mathbf{X}]$. Let $\mathbf{f}$ be the morphism in pro-HPol which is represented by $([f_i])$. Then $\mathbf{f}$ induces an isomorphism $\pi_n(\mathbf{f}) : \mathbf{G} \to \pi_n(\mathbf{X})$. Consider the commutative diagram

$$\begin{array}{ccc} \pi_n(\mathbf{M}(\mathbf{G}, n)) & \xrightarrow{\pi_n(\mathbf{f})} & \pi_n(\mathbf{X}) \\ {\scriptstyle \varphi_{\mathbf{M}(\mathbf{G},n)}}\downarrow & & \downarrow{\scriptstyle \varphi_{\mathbf{X}}} \\ \mathrm{H}_n(\mathbf{M}(\mathbf{G}, n)) & \xrightarrow{\mathrm{H}_n(\mathbf{f})} & \mathrm{H}_n(\mathbf{X}) \end{array}$$

where the vertical arrows are the Hurewicz morphisms, which are isomorphisms by the Hurewicz theorem. So, $\mathrm{H}_n(\mathbf{f}) : \mathrm{H}_n(\mathbf{M}(\mathbf{G}, n)) \to \mathrm{H}_n(\mathbf{X})$ is an isomorphism, and hence $\mathrm{H}_q(\mathbf{f}) : \mathrm{H}_q(\mathbf{M}(\mathbf{G}, n)) \to \mathrm{H}_q(\mathbf{X})$ is an isomorphism for $q \geq 0$. By the homological version of the Whitehead theorem [7, Theorem 11, p. 153] and the assumptions that $\dim X_i \leq N$ and $\dim \mathrm{M}(G_i, n) \leq n+1$ and that $\pi_1(\mathbf{X}) = 0$ and $\pi_1(\mathbf{M}(\mathbf{G}, n)) = 0$, we conclude that $\mathbf{f}$ is an isomorphism. □

Example 3.6. The following example shows that the finiteness of the shape dimension in Proposition 3.5 with condition (1) is essential. Recall the Kan continuum in [7, Example 1, p. 153]. J. F. Adams [1, Theorem 1.7] constructed a compact polyhedron Y and a map $a : S^r Y \to Y$ for some integer $r \geq 1$ such that for each m, the composite

$$a(S^r a)(S^{2r} a) \cdots (S^{(m-1)r} a) : S^{mr} Y \to Y$$

is essential. Consider the inverse sequence of compact polyhedra

$$Y \xleftarrow{a} S^rY \xleftarrow{S^r a} S^{2r}Y \longleftarrow \cdots .$$

Let A be its inverse limit. Then A is a continuum with infinite shape dimension. It is shown in [7, Example 1, p. 153] that A is not shape equivalent to a point, but pro-$\pi_q(A) = 0$ for $q \geq 0$. However, the latter implies that pro-$\mathrm{H}_q(A) = 0$ for $q \geq 0$ by the Hurewicz theorem. This means that A is an approximate Moore space of type $(0, n)$ for any $n \geq 2$.

Proposition 3.7. *Let $n > 1$, and let $\mathbf{X} = (X_i, p_{i,i+1})$ be an $(n-1)$-connected inverse sequence consisting of spaces such that $\mathrm{H}_n(X_i)$ is a finitely generated free abelian group for each i. Then there exist an inverse sequence $\mathbf{M}$ of type $(\mathrm{H}_n(\mathbf{X}), n)$ consisting of connected compact polyhedra, a cofinal subsequence $\mathbf{X}'$ of $\mathbf{X}$, and a level map $([f_i]) : [\mathbf{M}] \to [\mathbf{X}']$ which induces an isomorphism $\mathrm{H}_n(\mathbf{M}) \to \mathrm{H}_n(\mathbf{X}')$ in pro-Ab.*

Proof. Since $\mathbf{X}$ is $(n-1)$-connected, by the Hurewicz theorem [7, Theorem 2, p. 136] (see also [7, Remark 3, p. 138]), the Hurewicz morphism $\boldsymbol{\varphi} : \pi_n(\mathbf{X}) \to \mathrm{H}_n(\mathbf{X})$ is an isomorphism in pro-Ab. Let $\boldsymbol{\psi} : \mathrm{H}_n(\mathbf{X}) \to \pi_n(\mathbf{X})$ be the morphism which is the inverse of $\boldsymbol{\varphi}$. By [7, Theorem 3, p. 12], replacing $\mathbf{X}$ by a cofinal subsequence if necessary, we can assume that $\boldsymbol{\psi}$ is represented by a level map (ψ_i). Since $[\mathrm{M}(\mathrm{H}_n(X_i), n), X_i] \to \mathrm{Hom}(\mathrm{H}_n(X_i), \pi_n(X_i))$ is onto by Proposition 3.4, then, for each i, there exists a map $f_i : \mathrm{M}(\mathrm{H}_n(X_i), n) \to X_i$ which induces ψ_i. Since $\mathrm{H}_n(X_{i+1})$ is free, $\mathrm{Ext}(\mathrm{H}_n(X_{i+1}), \pi_n(X_i)) = 0$, and by Proposition 3.4, $[\mathrm{M}(\mathrm{H}_n(X_{i+1}), n), X_i] \cong \mathrm{Hom}(\mathrm{H}_n(X_{i+1}), \pi_n(X_i))$. This implies that the following diagram is homotopy commutative.

$$\begin{array}{ccc} \mathrm{M}(\mathrm{H}_n(X_i), n) & \xleftarrow{\mu_n(H_n(p_{i,i+1}))} & \mathrm{M}(\mathrm{H}_n(X_{i+1}), n) \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_{i+1}} \\ X_i & \xleftarrow{p_{i,i+1}} & X_{i+1} \end{array}$$

Since $\mathrm{H}_n(X_i)$ is finitely generated, $\mathrm{M}(\mathrm{H}_n(X_i), n)$ is a connected compact polyhedron. Thus, we have an inverse sequence $\mathbf{M} = (\mathrm{M}(\mathrm{H}_n(X_i), n), \mu_n(\mathrm{H}_n(p_{i,i+1})))$ of connected compact polyhedra and a level map $([f_i]) : [\mathbf{M}] \to [\mathbf{X}]$ with the required property. □

Proposition 3.8. *Let $n > 1$, and let X be an $(n-1)$-shape connected metric continuum such that pro-$\mathrm{H}_n(X)$ is isomorphic to an inverse sequence consisting of finitely generated free abelian groups. Then there exists an approximate Moore* *space Z of type* $(\text{pro-}\mathrm{H}_n(X), n)$ *and a shape morphism* $F : Z \to X$ *which induces an isomorphism* $\text{pro-}\mathrm{H}_n(F) : \text{pro-}\mathrm{H}_n(Z) \to \text{pro-}\mathrm{H}_n(X)$.

Proof. Let $\mathbf{p} = (p_i) : X \to \mathbf{X}$ be the limit of some inverse sequence $\mathbf{X} = (X_i, p_{i,i+1})$ consisting of connected compact polyhedra. Without loss of generality, we can assume that each $\mathrm{H}_n(X_i)$ is a finitely generated free abelian group. We can proceed with the proof as for Proposition 3.7 to obtain an inverse sequence $\mathbf{M}$ of type $(\mathrm{H}_n(\mathbf{X}), n)$ consisting of connected compact polyhedra, a cofinal subsequence $\mathbf{X}'$ of $\mathbf{X}$, and a level map $([f_i]) : [\mathbf{M}] \to [\mathbf{X}']$ which induces an isomorphism $\mathrm{H}_n(\mathbf{M}) \to \mathrm{H}_n(\mathbf{X}')$ in pro-Ab. Here note that compactness of each coordinate space of $\mathbf{M}$ is guaranteed by the fact that each $\mathrm{H}_n(X_i)$ is finitely generated. Let $\mathbf{q} = (q_i) : Z \to \mathbf{M}$ be the inverse limit of $\mathbf{M}$. Then Z is an approximate Moore space of type $(\text{pro-}\mathrm{H}_n(X), n)$, and the shape morphism $F : Z \to X$ represented by $([f_i])$ has the desired property. □

4. Cofibration sequences of shape morphisms

In this section we discuss the cofibration sequences, mapping cones, and mapping cylinders for shape morphisms.

Recall that each map $f : X \to Y$ admits a long sequence, which is called a cofibration sequence, or alternatively, a Puppe sequence,

$$X \xrightarrow{f} Y \xrightarrow{j} C_f \xrightarrow{k} SX \xrightarrow{Sf} SY \longrightarrow \cdots .$$

Here the map $j : Y \to C_f$ is the composition $Y \hookrightarrow Y \coprod CX \to C_f$ of the inclusion map and the identification map. The map $k : C_f \to SX$ is the composition $C_f \overset{k'}{\hookrightarrow} C_j \xrightarrow{\varphi} SX$ of the inclusion map k' and the map φ defined by

$$\begin{cases} \varphi(x \wedge t) = x \wedge t \ (\in SX) & (x \in X,\ 0 \le t \le 1), \\ \varphi(y \wedge t) = \quad * & (y \in Y,\ 0 \le t \le 1). \end{cases} \tag{4.1}$$

Lemma 4.1. *For any homotopy commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ X' & \xrightarrow{f'} & Y' \end{array}$$

there exists a map $\gamma : C_f \to C_{f'}$ such that the following diagram is homotopy commutative.

$$\begin{array}{ccccccccccc} X & \xrightarrow{f} & Y & \xrightarrow{j} & C_f & \xrightarrow{k} & SX & \xrightarrow{Sf} & SY & \longrightarrow & \cdots \\ \alpha\downarrow & & \downarrow\beta & & \downarrow\gamma & & \downarrow S\alpha & & \downarrow S\beta & & \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{j'} & C_{f'} & \xrightarrow{k'} & SX' & \xrightarrow{Sf'} & SY' & \longrightarrow & \cdots \end{array}$$

PROOF. Let $G : X \times I \to Y'$ be a homotopy such that $G_0 = f' \circ \alpha$ and $G_1 = \beta \circ f$. The desired map $\gamma : C_f \to C_{f'}$ is given by the following formula (see [4, p. 260] and also [8, p. 252]).

$$\begin{cases} \gamma(y) = \beta(y) \qquad (y \in Y), \\ \gamma(x \wedge t) = \begin{cases} \alpha(x) \wedge 2t & \left(x \in X,\ 0 \le t \le \dfrac{1}{2}\right), \\ G(x, 2t-1) & \left(x \in X,\ \dfrac{1}{2} \le t \le 1\right). \end{cases} \end{cases}$$

□

Proposition 4.2. *For any shape morphism $F : X \to Y$, there exists a metric continuum C and an exact sequence in* pro-Ab

$$\cdots \longrightarrow \text{pro-}\tilde{\mathrm{H}}_n(X) \xrightarrow{\text{pro-}\tilde{\mathrm{H}}_n(F)} \text{pro-}\tilde{\mathrm{H}}_n(Y) \xrightarrow{\text{pro-}\tilde{\mathrm{H}}_n(j)} \text{pro-}\tilde{\mathrm{H}}_n(C) \xrightarrow{\boldsymbol{\partial}} \text{pro-}\tilde{\mathrm{H}}_{n-1}(X) \longrightarrow \cdots . \tag{4.2}$$

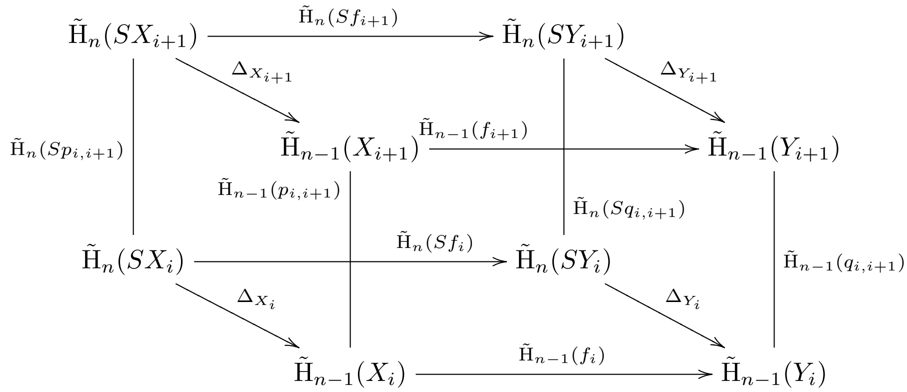
PROOF. By the simplicial approximation theorem [8, Theorem 2.5.16], we can find simplicial maps $f_i : X_i \to Y_i$ such that $\mathbf{X} = (X_i, p_{i,i+1})$ and $\mathbf{Y} = (Y_i, q_{i,i+1})$ are inverse sequences of connected compact polyhedra, and f_i form a level map $([f_i]) : [\mathbf{X}] \to [\mathbf{Y}]$ which represents F. By Lemma 4.1, for each i, there exists a map $\gamma_{i,i+1} : C_{f_{i+1}} \to C_{f_i}$ which makes the following diagram homotopy commutative.

$$\begin{array}{ccccccccccc} X_{i+1} & \xrightarrow{f_{i+1}} & Y_{i+1} & \xrightarrow{j_{i+1}} & C_{f_{i+1}} & \xrightarrow{k_{i+1}} & SX_{i+1} & \xrightarrow{Sf_{i+1}} & SY_{i+1} & \longrightarrow & \cdots \\ p_{i,i+1}\downarrow & & \downarrow q_{i,i+1} & & \downarrow \gamma_{i,i+1} & & \downarrow Sp_{i,i+1} & & \downarrow Sq_{i,i+1} & & \\ X_i & \xrightarrow{f_i} & Y_i & \xrightarrow{j_i} & C_{f_i} & \xrightarrow{k_i} & SX_i & \xrightarrow{Sf_i} & SY_i & \longrightarrow & \cdots \end{array}$$

Thus we obtain an inverse sequence $\mathbf{C}_{([f_i])} = (C_{f_i}, \gamma_{i,i+1})$ consisting of connected compact polyhedra. Let $\mathbf{t} = (t_i) : C \to \mathbf{C}_{([f_i])}$ be the limit of $\mathbf{C}_{([f_i])}$. For each i, consider the following commutative diagram (see [10, 2.39, 7.33]).

$$\begin{array}{ccccccccc} X_i & \xrightarrow{f_i} & Y_i & \xrightarrow{j_i} & C_{f_i} & \xrightarrow{k'_i} & C_{j_i} & \xrightarrow{l'_i} & C_{k'_i} \\ \| & & \| & & \| & & \downarrow{\scriptstyle v_i\circ\varphi_i} & & \downarrow{\scriptstyle \psi_i} \\ X_i & \xrightarrow{f_i} & Y_i & \xrightarrow{j_i} & C_{f_i} & \xrightarrow{v_i\circ k_i} & SX_i & \xrightarrow{Sf_i} & SY_i \end{array}$$

Here k'_i and l'_i are the inclusion maps, $v_i : SX_i \to SX_i$ is the homotopy inverse of the H-cogroup SX_i, $\varphi_i : C_{j_i} \to SX_i$ and $\psi_i : C_{k'_i} \to SY_i$ are the maps analogous to φ defined by (4.1). Since $v_i \circ \varphi_i$ and ψ_i are homotopy equivalences and since the top sequence induces an exact sequence in the reduced homology groups by the exact axiom, so does the bottom sequence. The following diagram is also commutative.



Here $\Delta_{X_{i+1}}$, Δ_{X_i}, $\Delta_{Y_{i+1}}$, and Δ_{Y_i} are the natural isomorphisms. Thus we have the following commutative diagram.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \tilde{H}_n(X_{i+1}) & \xrightarrow{\tilde{H}_n(f_{i+1})} & \tilde{H}_n(Y_{i+1}) & \xrightarrow{\tilde{H}_n(j_{i+1})} & \tilde{H}_n(C_{f_{i+1}}) \\ & & {\scriptstyle \tilde{H}_n(p_{i,i+1})}\downarrow & & \downarrow{\scriptstyle \tilde{H}_n(q_{i,i+1})} & & \downarrow{\scriptstyle \tilde{H}_n(\gamma_{i,i+1})} \\ \cdots & \longrightarrow & \tilde{H}_n(X_i) & \xrightarrow{\tilde{H}_n(f_i)} & \tilde{H}_n(Y_i) & \xrightarrow{\tilde{H}_n(j_i)} & \tilde{H}_n(C_{f_i}) \end{array}$$

$$\begin{array}{ccccccc} \xrightarrow{\Delta_{X_{i+1}}\circ\tilde{H}_n(v_{i+1}\circ k_{i+1})} & \tilde{H}_{n-1}(X_{i+1}) & \xrightarrow{\tilde{H}_{n-1}(f_{i+1})} & \tilde{H}_{n-1}(Y_{i+1}) & \longrightarrow & \cdots \\ & \downarrow{\scriptstyle \tilde{H}_{n-1}(p_{i,i+1})} & & \downarrow{\scriptstyle \tilde{H}_{n-1}(q_{i,i+1})} & & \\ \xrightarrow{\Delta_{X_i}\circ\tilde{H}_n(v_i\circ k_i)} & \tilde{H}_{n-1}(X_i) & \xrightarrow{\tilde{H}_{n-1}(f_i)} & \tilde{H}_{n-1}(Y_i) & \longrightarrow & \cdots \end{array}$$

Since the rows in this diagram are exact, we have the following exact sequence of progroups.

$$\cdots \longrightarrow \tilde{\mathrm{H}}_n(\mathbf{X}) \xrightarrow{(\tilde{\mathrm{H}}_n(f_i))} \tilde{\mathrm{H}}_n(\mathbf{Y}) \xrightarrow{(\tilde{\mathrm{H}}_n(j_i))} \tilde{\mathrm{H}}_n(\mathbf{C}_{([f_i])}) \xrightarrow{(\Delta_{X_i}\circ\tilde{\mathrm{H}}_n(v_i\circ k_i))}$$
$$\tilde{\mathrm{H}}_{n-1}(\mathbf{X}) \longrightarrow \tilde{\mathrm{H}}_{n-1}(\mathbf{Y}) \longrightarrow > \cdots$$

This represents the exact sequence (4.2). □

The inverse sequence $[\mathbf{C}_{([f_i])}]$ is called the *cofibre* of $([f_i])$. The metric continuum C in (4.2) is called a *cofibre* of F, but it may depend on the choice of a morphism which represents F.

Remark 4.3. We raise the following question: given a commutative diagram in $\mathsf{Sh}(\mathsf{CM})$

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ \alpha\downarrow & & \downarrow\beta \\ X' & \xrightarrow{F'} & Y' \end{array}$$

does there exist a shape morphism $\gamma : C \to C'$ which makes the following diagram commute in $\mathsf{Sh}(\mathsf{CM})$?

$$\begin{array}{cccccccccccc} X & \xrightarrow{F} & Y & \xrightarrow{j} & C & \xrightarrow{k} & SX & \xrightarrow{SF} & SY & \longrightarrow & \cdots \\ \downarrow\alpha & & \downarrow\beta & & \downarrow\gamma & & S\alpha\downarrow & & S\beta\downarrow & & \\ X' & \xrightarrow{F'} & Y' & \xrightarrow{j'} & C' & \xrightarrow{k'} & SX' & \xrightarrow{SF'} & SY' & \longrightarrow & \cdots \end{array}$$

We do not know the answer to this question at this moment, but if it is affirmative, then for any shape morphism $F : X \to Y$ between 1-shape connected metric continua X and Y with finite shape dimensions, the metric continuum C in Proposition 4.2 is unique up to shape. Indeed, suppose that there is another metric continuum C'. Then there would be a shape morphism $\gamma : C \to C'$ and the following commutative diagram.

$$\begin{array}{ccccccccc} \text{pro-}\tilde{\mathrm{H}}_n(X) & \xrightarrow{\text{pro-}\tilde{\mathrm{H}}_n(F)} & \text{pro-}\tilde{\mathrm{H}}_n(Y) & \xrightarrow{\text{pro-}\tilde{\mathrm{H}}_n(j)} & \text{pro-}\tilde{\mathrm{H}}_n(C) & \xrightarrow{\partial} & \text{pro-}\tilde{\mathrm{H}}_{n-1}(X) & \xrightarrow{\text{pro-}\tilde{\mathrm{H}}_n(F)} & \text{pro-}\tilde{\mathrm{H}}_n(Y) \\ & & & \searrow{\scriptstyle \text{pro-}\tilde{\mathrm{H}}_n(j')} & \Big|{\scriptstyle \text{pro-}\tilde{\mathrm{H}}_n(\gamma)} & \nearrow{\scriptstyle \partial'} & & & \\ & & & & \text{pro-}\tilde{\mathrm{H}}_n(C') & & & & \end{array}$$

By the five lemma (see [2, Appendix] or [7, Theorem 11, p. 153]), pro-$\tilde{\mathrm{H}}_n(\gamma)$ is an isomorphism for $n \geq 0$. Since C and C' are 1-shape connected, by the homological version of the Whitehead theorem [7, Theorem 11, p. 153], γ is an isomorphism.

The following is an immediate consequence of Proposition 4.2.

Proposition 4.4. *Let* $\mathbf{X}$ *be an inverse sequence of connected compact polyhedra such that* $\mathrm{H}_q(\mathbf{X}) = 0$ *for* $q > n$*, and let* $\mathbf{G}$ *be an inverse sequence consisting of finitely generated abelian groups. Suppose that there exists a level map* $([g_i]) : [\mathbf{M}(\mathbf{G}, n)] \to [\mathbf{X}]$ *which induces a trivial morphism* $\mathrm{H}_n(\mathbf{M}(\mathbf{G}, n)) \to \mathrm{H}_n(\mathbf{X})$ *in* pro-Ab*. Let* $(l_i) : \mathbf{X} \to \mathbf{C}_{([g_i])}$ *be the level map consisting of the inclusion maps* $X_i \hookrightarrow C_{g_i}$*. Then* (l_i) *induces an isomorphism* $\mathrm{H}_q(\mathbf{X}) \to \mathrm{H}_q(\mathbf{C}_{([g_i])})$ *in* pro-Ab *for* $q \neq n+1$*, and* $\mathrm{H}_{n+1}(\mathbf{C}_{([g_i])}) \approx \mathbf{G}$.

Recall that any map $f : X \to Y$ is the composite of a cofibration and a homotopy equivalence. Indeed, $f = hg$, where $g : X \to Z_f$ is the inclusion of X into $X \times I$ as $X \times \{0\}$, followed by the identification map, and $h : Z_f \to Y$ is the map induced by the identity map of Y and the map $IX \to Y$ defined by $[(x,t)] \mapsto f(x)$.

Lemma 4.5. *Let the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\beta} \\ X' & \xrightarrow{\ f'\ } & Y' \end{array}$$

be homotopy commutative, and let $X \xrightarrow{g} Z_f \xrightarrow{h} Y$ *and* $X' \xrightarrow{g'} Z_{f'} \xrightarrow{h'} Y'$ *be the factorizations of* f *and* f'*, respectively, such that* g *and* g' *are cofibrations and* h *and* h' *are homotopy equivalences. Then there exists a map* $\eta : Z_f \to Z_{f'}$ *such that* $\eta \circ g = g' \circ \alpha$ *and* $\beta \circ h \simeq h' \circ \eta$.

Proof. First, we define a map $\eta' : Z_f \to Z_{f'}$ such that

$$\eta' \circ g \simeq g' \circ \alpha,$$

and

$$\beta \circ h \simeq h' \circ \eta'. \tag{4.3}$$

Indeed, if $G : X \times I \to Y'$ is a homotopy such that $G_0 = f' \circ \alpha$ and $G_1 = \beta \circ f$, then the map $\eta' : Z_f \to Z_{f'}$ can be defined as

$$\begin{cases} \eta'([(x,t)]) = \begin{cases} [(\alpha(x), 2t)] & \left(x \in X,\ 0 \le t \le \dfrac{1}{2}\right), \\ G(x, 2t-1) & \left(x \in X,\ \dfrac{1}{2} \le t \le 1\right), \end{cases} \\ \eta'(y) = \beta(y) \qquad (y \in Y). \end{cases}$$

Since g is a cofibration, there exists a map $\eta : Z_f \to Z_{f'}$ such that

$$\eta \simeq \eta', \tag{4.4}$$

and

$$\eta \circ g = g' \circ \alpha.$$

By (4.3) and (4.4), we have $\beta \circ h \simeq h' \circ \eta$ and conclude that η is the desired map. □

Proposition 4.6. *For any inverse sequences* $\mathbf{X} = (X_i, p_{i,i+1})$ *and* $\mathbf{Y} = (Y_i, q_{i,i+q})$, *every level map* $([f_i]) : [\mathbf{X}] \to [\mathbf{Y}]$ *admits an inverse sequence* $\mathbf{Z} = (Z_{f_i}, \eta_{i,i+1})$ *of mapping cylinders and level maps* $(g_i) : \mathbf{X} \to \mathbf{Z}$ *and* $([h_i]) : [\mathbf{Z}] \to [\mathbf{Y}]$ *such that*

(1) *each* g_i *is the inclusion map of* X_i *into* Z_{f_i} *as* $X_i \times \{0\}$,

(2) *each* h_i *is a homotopy equivalence, and*

(3) $f_i = h_i \circ g_i$.

5. Homology decompositions

In this section we prove a homology decomposition theorem in shape theory. Before proving the main theorem, we establish the following lemma.

Lemma 5.1. *Let* $\mathbf{X} = (X_i, p_{ii'}, \mathbb{N})$ *and* $\mathbf{A} = (A_i, q_{ii'}, \mathbb{N})$ *be 1-shape connected inverse sequences of connected compact polyhedra such that each* $\mathrm{H}_n(X_i)$ *is a finitely generated free abelian group, each* A_i *is a subpolyhedron of* X_i, *and* $q_{ii'}$ *is a restriction of* $p_{ii'}$. *Let* $(f_i) : \mathbf{A} \to \mathbf{X}$ *be the level map consisting of the inclusion maps. Suppose that* (f_i) *induces an isomorphism* $\mathrm{H}_q(\mathbf{A}) \to \mathrm{H}_q(\mathbf{X})$ *in* pro-Ab *for* $q < n$, *and* $\mathrm{H}_q(\mathbf{A}) = 0$ *for* $q \geq n$. *Then there exist a cofinal subset* Λ *of* $\mathbb{N}$, *an inverse sequence* $\mathbf{M} = (M_i, \alpha_{ii'}, \Lambda)$ *of type* $(\mathrm{H}_n(\mathbf{X}), n-1)$, *level maps* $([g_i]) : [\mathbf{M}] \to [\mathbf{A}']$ *and* $([f_i']) : [\mathbf{C}_{([g_i])}] \to [\mathbf{X}']$ *for the cofinal subsequences* $\mathbf{X}' = (X_i, p_{ii'}, \Lambda)$ *and* $\mathbf{A}' = (A_i, q_{ii'}, \Lambda)$ *of* $\mathbf{X}$ *and* $\mathbf{A}$, *respectively, with the following properties:*

(1) *if* $j_i : A_i \hookrightarrow C_{g_i}$, $i \in \Lambda$, *are the inclusion maps into the mapping cones, then the following diagram is homotopy commutative for* $i < i'$,

$$\begin{array}{ccc} A_{i'} & \xrightarrow{j_{i'}} & C_{g_{i'}} \\ {\scriptstyle q_{ii'}}\downarrow & & \downarrow{\scriptstyle \gamma_{ii'}} \\ A_i & \xrightarrow{j_i} & C_{g_i} \end{array} \tag{5.1}$$

(2) *the level map* $([g_i]) : [\mathbf{M}] \to [\mathbf{A}']$ *induces a trivial morphism* $\mathrm{H}_{n-1}(\mathbf{M}) \to \mathrm{H}_{n-1}(\mathbf{A}')$ *in* pro- Ab,

(3) $f'_i \circ j_i = f_i$ *for each* $i \in \Lambda$,

(4) $([f'_i])$ *induces an isomorphism* $\mathrm{H}_q(\mathbf{C}_{([g_i])}) \to \mathrm{H}_q(\mathbf{X}')$ *in* pro- Ab *for* $q \le n$, *and*

(5) $\mathrm{H}_q(\mathbf{C}_{([g_i])}) = 0$ *for* $q > n$.

Proof. For each $i \in \mathbb{N}$, let $E_i = \{\omega \in X_i^I : \omega(0) = x_{0i}\}$ and $F_i = \{\omega \in X_i^I : \omega(0) = x_{0i}, \omega(1) \in A_i\}$, where $x_{0i} \in X_i$ is a base point of X_i and A_i. Define maps $u_{i,i+1} : E_{i+1} \to E_i : \omega \mapsto p_{i,i+1} \circ \omega$ and $v_{i,i+1} : F_{i+1} \to F_i : \omega \mapsto q_{i,i+1} \circ \omega$. Then we have inverse sequences $\mathbf{E} = (E_i, u_{i,i+1})$ and $\mathbf{F} = (F_i, v_{i,i+1})$.

By the exactness of the homology sequence of the pair $(\mathbf{X}, \mathbf{A})$ [7, Theorem 3, p. 125], $\mathrm{H}_q(\mathbf{X}, \mathbf{A}) = 0$ for $q < n$, and $\mathrm{H}_q(\mathbf{X}, \mathbf{A}) \approx \mathrm{H}_q(\mathbf{X})$ for $q \ge n$. Since $\pi_1(\mathbf{X}) = 0$ and $\pi_0(\mathbf{A}) = 0$, by the exactness of the homotopy sequence of the pair $(\mathbf{X}, \mathbf{A})$ [7, Theorem 8, p. 131], we have $\pi_1(\mathbf{X}, \mathbf{A}) = 0$. This together with $\pi_1(\mathbf{A}) = 0$ and the relative Hurewicz theorem [7, Theorem 6, p. 140] implies that $\pi_2(\mathbf{X}, \mathbf{A}) \approx \mathrm{H}_2(\mathbf{X}, \mathbf{A}) = 0$. Since $\mathrm{H}_q(\mathbf{X}, \mathbf{A}) = 0$ for $q < n$, repeatedly using the relative Hurewicz theorem, we have $\pi_q(\mathbf{X}, \mathbf{A}) = 0$ for $q < n$ and $\pi_n(\mathbf{X}, \mathbf{A}) \approx \mathrm{H}_n(\mathbf{X}, \mathbf{A})$. Since $\pi_q(\mathbf{F}) = \pi_{q+1}(\mathbf{X}, \mathbf{A})$ for $q \ge 0$ (see [8, 7.2.9]), $\mathbf{F}$ is $(n-2)$-connected. By the Hurewicz theorem [7, Theorem 2, p. 136], $\mathrm{H}_{n-1}(\mathbf{F}) \approx \pi_{n-1}(\mathbf{F})$, which is isomorphic to the inverse sequence $\mathrm{H}_n(\mathbf{X})$ consisting of finitely generated free abelian groups. Thus by Proposition 3.7, there exist a cofinal subsequence $\mathbf{F}' = (F_i, v_{ii'}, \Lambda)$ of $\mathbf{F}$, an inverse sequence $\mathbf{M} = (M_i, \alpha_{ii'}, \Lambda)$ of type $(\mathrm{H}_{n-1}(\mathbf{F}), n-1)$, and a level map $([g'_i]) : [\mathbf{M}] \to [\mathbf{F}']$, which induces an isomorphism in the $(n-1)$-th pro-homology. Define maps $\theta_i : E_i \to X_i : \omega \mapsto \omega(1)$ and $\psi_i : F_i \to A_i : \omega \mapsto \omega(1)$. Then the following diagrams are commutative for $i < i'$.

$$
\begin{array}{ccc}
E_{i'} & \xrightarrow{\theta_{i'}} & X_{i'} \\
{\scriptstyle u_{ii'}}\downarrow & & \downarrow{\scriptstyle p_{ii'}} \\
E_i & \xrightarrow{\theta_i} & X_i
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
F_{i'} & \xrightarrow{\psi_{i'}} & A_{i'} \\
{\scriptstyle v_{ii'}}\downarrow & & \downarrow{\scriptstyle q_{ii'}} \\
F_i & \xrightarrow{\psi_i} & A_i
\end{array}
$$

So, we have level maps $(\theta_i) : \mathbf{E}' = (E_i, u_{ii'}, \Lambda) \to \mathbf{X}'$ and $(\psi_i) : \mathbf{F}' \to \mathbf{A}'$. For each $i \in \Lambda$, let $g_i = \psi_i \circ g'_i : M_i \to A_i$. Then we have a level map $([g_i]) : [\mathbf{M}] \to [\mathbf{A}']$. For each $i \in \Lambda$, define a map $f'_i : C_{g_i} \to X_i$ by

$$
\begin{cases}
f'_i(a) = f_i(a) & (a \in A_i), \\
f'_i(x \wedge t) = g'_i(x)(t) & (x \in M_i,\ t \in I).
\end{cases}
$$

By Lemma 4.1, for $i < i'$, there exists a map $\gamma_{ii'} : C_{g_{i'}} \to C_{g_i}$ between mapping cones such that diagram (5.1) is homotopy commutative. This verifies property (1).

We claim that the following diagram is homotopy commutative for $i < i'$.

$$\begin{array}{ccc} C_{g_{i'}} & \xrightarrow{f'_{i'}} & X_{i'} \\ {\scriptstyle \gamma_{ii'}}\downarrow & & \downarrow{\scriptstyle p_{ii'}} \\ C_{g_i} & \xrightarrow{f'_i} & X_i \end{array}$$

Indeed, let $K : M_{i'} \times I \to F_i$ be a homotopy such that $K_0 = g'_i \circ \alpha_{ii'}$ and $K_1 = v_{ii'} \circ g'_{i'}$. For each $a \in A_{i'}$,

$$(p_{ii'} \circ f'_{i'})(a) = (p_{ii'} \circ f_{i'})(a) = (f_i \circ q_{ii'})(a) = (f'_i \circ \gamma_{ii'})(a).$$

For each $x \in M_i$ and $s \in I$,

$$(p_{ii'} \circ f'_{i'})(x \wedge s) = p_{ii'}(g'_{i'}(x)(s)),$$

$$(f'_i \circ \gamma_{ii'})(x \wedge s) = \begin{cases} g'_i(\alpha_{ii'}(x))(2s) & \left(0 \le s \le \dfrac{1}{2}\right), \\ f_i(K(x, 2s-1)(1)) & \left(\dfrac{1}{2} \le s \le 1\right). \end{cases}$$

The map $L : C_{g_{i'}} \times I \to X_i$ defined by

$$L(a,t) = (p_{ii'} \circ f'_{i'})(a) \quad \text{for } a \in A_{i'} \text{ and } 0 \le t \le 1,$$

$$L(x \wedge s, t) = \left\{ \begin{array}{ll} f_i\left(K(x,t)\left(\dfrac{2s}{t+1}\right)\right) & \left(0 \le s \le \dfrac{t+1}{2}\right) \\ f_i\left(K(x, 2s-1)(1)\right) & \left(\dfrac{t+1}{2} \le s \le 1\right) \end{array} \right\}$$

$$\text{for } x \wedge s \in C_{g_{i'}} \text{ and } 0 \le t \le 1,$$

gives a homotopy such that $L_0 = f'_i \circ \gamma_{ii'}$ and $L_1 = p_{ii'} \circ f'_{i'}$. Thus we have a level map $([f'_i]) : [\mathbf{C}_{([g_i])}] = (C_{g_i}, [\gamma_{ii'}], \Lambda) \to [\mathbf{X}']$. By the definition of f'_i, $f_i = f'_i \circ j_i$ for each $i \in \Lambda$, and this verifies property (3).

Since each $f_i \circ \psi_i$ is inessential, $\mathrm{H}_{n-1}(f_i) \circ \mathrm{H}_{n-1}(g_i) = \mathrm{H}_{n-1}(f_i) \circ \mathrm{H}_{n-1}(\psi_i) \circ \mathrm{H}_{n-1}(g'_i)$ is trivial. This together with the assumption that (f_i) induces an isomorphism $\mathrm{H}_{n-1}(\mathbf{A}') \to \mathrm{H}_{n-1}(\mathbf{X}')$ implies that $([g_i])$ induces a trivial morphism $\mathrm{H}_{n-1}(\mathbf{M}) \to \mathrm{H}_{n-1}(\mathbf{A}')$, verifying property (2).

By Proposition 4.2, $([j_i])$ induces an isomorphism $\mathrm{H}_q(\mathbf{A}') \to \mathrm{H}_q(\mathbf{C}_{([g_i])})$ for $q \ne n$. In particular, $\mathrm{H}_q(\mathbf{C}_{([g_i])}) \approx \mathrm{H}_q(\mathbf{A}') = 0$ for $q > n$, verifying property (5).

Moreover, since $f_i = f'_i \circ j_i$ for each $i \in \Lambda$ and (f_i) induces an isomorphism $\mathrm{H}_q(\mathbf{A}') \to \mathrm{H}_q(\mathbf{X}')$ for $q < n$, $([f'_i])$ induces an isomorphism $\mathrm{H}_q(\mathbf{C}_{([g_i])}) \to \mathrm{H}_q(\mathbf{X}')$ for $q < n$. To verify property (4), it remains to show that $([f'_i])$ induces an isomorphism $\mathrm{H}_n(\mathbf{C}_{([g_i])}) \to \mathrm{H}_n(\mathbf{X}')$. For each $i \in \Lambda$, define a map $g''_i : (CM_i, M_i) \to (E_i, F_i)$ by $g''_i(x \wedge s)(t) = g'_i(x)(st)$ for $x \wedge s \in CM_i$ and $t \in I$. Then the homotopy commutativity of the diagram

$$\begin{array}{ccc}
(CM_{i'}, M_{i'}) & \xrightarrow{g''_{i'}} & (E_{i'}, F_{i'}) \\
{\scriptstyle C\alpha_{ii'}}\downarrow & & \downarrow{\scriptstyle u_{ii'}} \\
(CM_i, M_i) & \xrightarrow{g''_i} & (E_i, F_i)
\end{array}$$

follows from the fact that for each $x \wedge s \in CM_{i'}$ and for $t \in I$, $(u_{ii'} \circ g''_{i'})(x \wedge s)(t) = v_{ii'}(g'_{i'}(x))(st)$, $(g''_i \circ C\alpha_{ii'})(x \wedge s)(t) = g'_i(\alpha_{ii'}(x))(st)$, and the homotopy-commutativity of the following diagram.

$$\begin{array}{ccc}
M_{i'} & \xrightarrow{g'_{i'}} & F_{i'} \\
{\scriptstyle \alpha_{ii'}}\downarrow & & \downarrow{\scriptstyle v_{ii'}} \\
M_i & \xrightarrow{g'_i} & F_i
\end{array}$$

This implies that we have an induced level map $([g''_i]) : ([C\mathbf{M}], [\mathbf{M}]) \to ([\mathbf{E}'], [\mathbf{F}'])$, which makes the following diagram commute.

$$\begin{array}{ccc}
([\mathbf{C}_{([g_i])}], [\mathbf{A}']) & \xleftarrow{([m_i])} & ([C\mathbf{M}], [\mathbf{M}]) \\
{\scriptstyle ([f'_i])}\downarrow & & \downarrow{\scriptstyle ([g''_i])} \\
([\mathbf{X}'], [\mathbf{A}']) & \xleftarrow{([\theta_i])} & ([\mathbf{E}], [\mathbf{F}])
\end{array}$$

Here $C\mathbf{M} = (CM_i, C\alpha_{ii'}, \Lambda)$, and $([m_i]) : ([C\mathbf{M}], [\mathbf{M}]) \to ([\mathbf{C}_{([g_i])}], [\mathbf{A}'])$ is a system map consisting of the inclusion maps $m_i : (CM_i, M_i) \hookrightarrow (C_{g_i}, A_i)$, $i \in \Lambda$. The map $f'_i : C_{g_i} \to X_i$ defines a map of pairs $f''_i : (C_{g_i}, A_i) \to (X_i, A_i)$. Let $\mathbf{f}'' : (\mathbf{C}_{([g_i])}, \mathbf{A}') \to (\mathbf{X}', \mathbf{A}')$ be the morphism represented by the level map $([f''_i])$. Consider the commutative diagram in pro-Ab

$$\begin{array}{ccccccc}
\mathrm{H}_n(\mathbf{C}_{([g_i])}) & \xrightarrow{\boldsymbol{\alpha}} & \mathrm{H}_n(\mathbf{C}_{([g_i])}, \mathbf{A}') & \xleftarrow{\mathrm{H}_n(\mathbf{m})} & \mathrm{H}_n(C\mathbf{M}, \mathbf{M}) & \xrightarrow{\boldsymbol{\partial}} & \mathrm{H}_{n-1}(\mathbf{M}) \\
{\scriptstyle \mathrm{H}_n(\mathbf{f}')}\downarrow & & {\scriptstyle \mathrm{H}_n(\mathbf{f}'')}\downarrow & & \downarrow{\scriptstyle \mathrm{H}_n(\mathbf{g}'')} & & \downarrow{\scriptstyle \mathrm{H}_{n-1}(\mathbf{g}')} \\
\mathrm{H}_n(\mathbf{X}') & \xrightarrow{\boldsymbol{\beta}} & \mathrm{H}_n(\mathbf{X}', \mathbf{A}') & \xleftarrow{\mathrm{H}_n(\boldsymbol{\theta})} & \mathrm{H}_n(\mathbf{E}', \mathbf{F}') & \xrightarrow{\boldsymbol{\partial}'} & \mathrm{H}_{n-1}(\mathbf{F}')
\end{array} \tag{5.2}$$

where $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\partial}$, and $\boldsymbol{\partial}'$ are morphisms from the pro-homology sequences of the pairs $(\mathbf{C}_{([g_i])}, \mathbf{A}')$, $(\mathbf{X}', \mathbf{A}')$, $(C\mathbf{M}, \mathbf{M})$, and $(\mathbf{E}', \mathbf{F}')$, respectively, and $\mathbf{g}'$, $\mathbf{g}''$, $\mathbf{m}$, and $\boldsymbol{\theta}$ are morphisms in pro-HPol represented by $([g_i'])$, $([g_i''])$, $([m_i])$, and $([\theta_i])$, respectively. Then $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are isomorphisms by the exactness of the homology sequences of the pairs $(\mathbf{C}_{([g_i])}, \mathbf{A}')$ and $(\mathbf{X}', \mathbf{A}')$, respectively, since $\mathrm{H}_n(\mathbf{A}') = 0$, and $([j_i])$ and $([f_i])$ induce isomorphisms $\mathrm{H}_{n-1}(\mathbf{A}') \to \mathrm{H}_{n-1}(\mathbf{C}_{([g_i])})$ and $\mathrm{H}_{n-1}(\mathbf{A}') \to \mathrm{H}_{n-1}(\mathbf{X}')$, respectively. Moreover, $\mathrm{H}_n(\mathbf{m})$ is an isomorphism since each $\mathrm{H}_n(m_i)$ is an isomorphism by the excision theorem. That $\boldsymbol{\partial}$ and $\boldsymbol{\partial}'$ are isomorphisms follows from the fact that $C\mathbf{M}$ and $\mathbf{E}'$ are trivial in pro-homotopy. Also, $\mathrm{H}_{n-1}(\mathbf{g}')$ is an isomorphism as seen above. Moreover, $\mathrm{H}_n(\boldsymbol{\theta})$ is an isomorphism. Indeed, consider the following commutative diagram.

$$\begin{array}{ccc} \pi_n(\mathbf{E}', \mathbf{F}') & \xrightarrow{\pi_n(\boldsymbol{\theta})} & \pi_n(\mathbf{X}', \mathbf{A}') \\ {\scriptstyle \boldsymbol{\varphi}_{(\mathbf{E}',\mathbf{F}')}}\big\downarrow & & \big\downarrow{\scriptstyle \boldsymbol{\varphi}_{(\mathbf{X}',\mathbf{A}')}} \\ \mathrm{H}_n(\mathbf{E}', \mathbf{F}') & \xrightarrow{\mathrm{H}_n(\boldsymbol{\theta})} & \mathrm{H}_n(\mathbf{X}', \mathbf{A}') \end{array} \tag{5.3}$$

Note that $\pi_q(\boldsymbol{\theta}) : \pi_q(\mathbf{E}', \mathbf{F}') \to \pi_q(\mathbf{X}', \mathbf{A}') = \pi_{q-1}(\mathbf{F}')$ is an isomorphism for $q \geq 0$ by the exactness of the pro-homology sequence since $\mathbf{E}'$ is trivial in pro-homotopy. In particular, $\pi_n(\boldsymbol{\theta})$ is an isomorphism. That $(\mathbf{X}, \mathbf{A})$ is $(n-1)$-connected implies that $(\mathbf{X}', \mathbf{A}')$ and $(\mathbf{E}', \mathbf{F}')$ are $(n-1)$-connected. By the relative Hurewicz theorem, the two Hurewicz morphisms $\boldsymbol{\varphi}_{(\mathbf{X}',\mathbf{A}')}$ and $\boldsymbol{\varphi}_{(\mathbf{E}',\mathbf{F}')}$ are isomorphisms. By the commutativity of diagram (5.3), we have that $\mathrm{H}_n(\boldsymbol{\theta})$ is an isomorphism. Thus, by the commutativity of diagram (5.2), we conclude that $\mathrm{H}(\mathbf{f}') : \mathrm{H}_n(\mathbf{C}_{([g_i])}) \to \mathrm{H}_n(\mathbf{X}')$ is an isomorphism. This verifies property (4) and completes the proof of the lemma. □

Theorem 5.2. *Let* $\mathbf{X} = (X_i, p_{i,i+1})$ *be a* 1*-connected inverse sequence of connected compact polyhedra such that each* $\mathrm{H}_n(X_i)$ *is a finitely generated free abelian group for* $n \geq 2$*. Then there exist inverse sequences* $\mathbf{X}_{(n)}$ *and* $\mathbf{M}_{(n+1)}$ *of compact connected polyhedra and morphisms* $\mathbf{f}_{(n)} : [\mathbf{X}_{(n)}] \to [\mathbf{X}]$ *and* $\mathbf{g}_{(n+1)} : [\mathbf{M}_{(n+1)}] \to [\mathbf{X}_{(n)}]$ *in* pro-HPol *for* $n \geq 2$ *with the following properties:*

(1) $\mathbf{X}_{(2)}$ *is of type* $(\mathrm{H}_2(\mathbf{X}), 2)$*,*

(2) $\mathbf{M}_{(n+1)}$ *is of type* $(\mathrm{H}_{n+1}(\mathbf{X}), n)$*,*

(3) $[\mathbf{X}_{(n+1)}]$ *is the cofibre of a level map representing* $\mathbf{g}_{(n+1)}$*,*

(4) $\mathrm{H}_q(\mathbf{f}_{(n)}) : \mathrm{H}_q(\mathbf{X}_{(n)}) \to \mathrm{H}_q(\mathbf{X})$ *is an isomorphism for* $q \leq n$*,*

(5) $\mathrm{H}_q(\mathbf{X}_{(n)}) = 0$ *for* $q > n$*, and*

(6) *the following diagram in* pro-HPol *commutes.*

$$\begin{array}{ccccc}
[\mathbf{M}_{(3)}] & \xrightarrow{\mathbf{g}_{(3)}} & [\mathbf{X}_{(2)}] & \xrightarrow{\mathbf{f}_{(2)}} & [\mathbf{X}] \\
 & & \Big| & \nearrow \mathbf{f}_{(3)} & \\
[\mathbf{M}_{(4)}] & \xrightarrow{\mathbf{g}_{(4)}} & [\mathbf{X}_{(3)}] & \mathbf{f}_{(4)} & \\
 & & \Big| & & \\
[\mathbf{M}_{(5)}] & \xrightarrow{\mathbf{g}_{(5)}} & [\mathbf{X}_{(4)}] & & \\
 & & \Big| & & \\
 & & \vdots & &
\end{array}$$

Proof. By Proposition 3.7, there exist a cofinal subsequence $\mathbf{X}'$ of $\mathbf{X}$, an inverse sequence $\mathbf{X}_{(2)}$ of type $(\mathrm{H}_2(\mathbf{X}), 2)$ and a level map $([(f_{(2)})_i]) : [\mathbf{X}_{(2)}] \to [\mathbf{X}']$ which induces an isomorphism $\mathrm{H}_q(\mathbf{X}_{(2)}) \to \mathrm{H}_q(\mathbf{X}')$ for $q \leq 2$. There exists a factorization $[\mathbf{X}_{(2)}] \xrightarrow{([(k_{(2)})_i])} [\mathbf{Z}_{(2)}] \xrightarrow{([(h_{(2)})_i])} [\mathbf{X}']$ of $([(f_{(2)})_i])$ with the properties in Lemma 4.6. Apply Lemma 5.1 to the level map $((k_{(2)})_i) : \mathbf{X}_{(2)} \to \mathbf{Z}_{(2)}$. Then there exist cofinal subsequences $[\mathbf{Z}'_{(2)}]$ and $[\mathbf{X}'_{(2)}]$ of $[\mathbf{Z}_{(2)}]$ and $[\mathbf{X}_{(2)}]$, respectively, an inverse sequence $[\mathbf{M}_{(3)}]$ of type $(\mathrm{H}_3(\mathbf{X}), 2)$, and a level map $([(g_{(3)})_i]) : [\mathbf{M}_{(3)}] \to [\mathbf{X}_{(2)}]$ representing a morphism $\mathbf{g}_{(3)}$ in pro-HPol, a level map $([(f'_{(3)})_i]) : [\mathbf{X}_{(3)}] \to [\mathbf{Z}'_{(2)}]$ such that $[\mathbf{X}_{(3)}]$ is the cofibre of $([(g_{(3)})_i])$, and $([(f'_{(3)})_i])$ induces an isomorphism $\mathrm{H}_q(\mathbf{X}_{(3)}) \to \mathrm{H}_q(\mathbf{Z}'_{(2)})$ for $q \leq 3$. Then the morphism $\mathbf{f}_{(3)}$ which is represented by the composite of system maps

$$[\mathbf{X}_{(3)}] \xrightarrow{([(f'_{(3)})_i])} [\mathbf{Z}'_{(2)}] \xrightarrow{\approx} [\mathbf{Z}_{(2)}] \xrightarrow{([(h_{(2)})_i])} [\mathbf{X}'] \xrightarrow{\approx} [\mathbf{X}]$$

induce an isomorphism $\mathrm{H}_q(\mathbf{f}_{(3)}) : \mathrm{H}_q(\mathbf{X}_{(3)}) \to \mathrm{H}_q(\mathbf{X})$ for $q \leq 3$. Repeat this process, using Lemma 5.1, and we inductively obtain inverse sequences $\mathbf{X}_{(n)}$ and $\mathbf{M}_{(n+1)}$ and morphisms $\mathbf{f}_{(n)} : [\mathbf{X}_{(n)}] \to [\mathbf{X}]$ and $\mathbf{g}_{(n+1)} : [\mathbf{M}_{(n+1)}] \to [\mathbf{X}_{(n)}]$ in pro-HPol for $n \geq 2$ with properties (2)–(4) and (6) as required.

It remains to prove (5). By (1), (5) holds for $n = 2$. Assume that (5) holds for a given $n \geq 2$. To prove (5) for $n + 1$, let $q > n + 1$, and consider the level map $([(g_{(n+1)})_i]) : \mathbf{M}_{(n+1)} \to \mathbf{X}_{(n)}$. Since $[\mathbf{X}_{(n+1)}]$ is the cofibre of $([(g_{(n+1)})_i])$, by the proof of Proposition 4.2, there exists an exact sequence

$$\cdots \to \mathrm{H}_q(\mathbf{X}_{(n)}) \to \mathrm{H}_q(\mathbf{X}_{(n+1)}) \to \mathrm{H}_{q-1}(\mathbf{M}_{(n+1)}) \to \cdots.$$

By the induction hypothesis, $\mathrm{H}_q(\mathbf{X}_{(n)}) = 0$ for $q > n+1 > n$. Since $\mathbf{M}_{(n+1)}$ is of type $(\mathrm{H}_{n+1}(\mathbf{X}), n)$ and $q-1 > n$, then $\mathrm{H}_{q-1}(\mathbf{M}_{(n+1)}) = 0$. Consequently, $\mathrm{H}_q(\mathbf{X}_{(n+1)}) = 0$ for $q > n+1$. This proves (5) and completes the proof of Theorem 5.2. □

Theorem 5.2 immediately implies

Corollary 5.3. *Every* 1*-shape connected metric continuum* X *with* pro-$\mathrm{H}_n(X)$ *being isomorphic to an inverse sequence of finitely generated free abelian groups for* $n \geq 2$ *admits metric continua* $X_{(n)}$ *and* $\mathrm{M}_{(n+1)}$ *and shape morphisms* $F_{(n)} : X_{(n)} \to X$ *and* $G_{(n+1)} : \mathrm{M}_{(n+1)} \to X_{(n)}$ *for* $n \geq 2$ *with the following properties:*

(1) $X_{(2)}$ *is an approximate Moore space of type* (pro-$\mathrm{H}_2(X), 2)$,

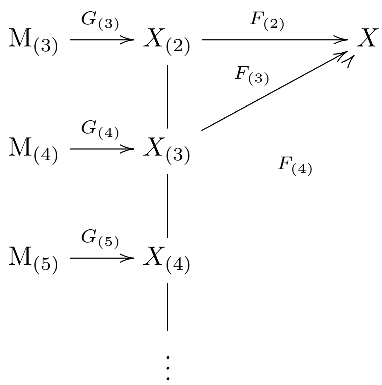
(2) $\mathrm{M}_{(n+1)}$ *is an approximate Moore space of type* (pro-$\mathrm{H}_{n+1}(X), n)$,

(3) $X_{(n+1)}$ *is a shape cofibre of* $G_{(n+1)}$,

(4) pro-$\mathrm{H}_q(F_{(n)})$: pro-$\mathrm{H}_q(X_{(n)}) \to$ pro-$\mathrm{H}_q(X)$ *is an isomorphism for* $q \leq n$,

(5) pro-$\mathrm{H}_q(\mathbf{X}_{(n)}) = 0$ *for* $q > n$, *and*

(6) *the following diagram in* Sh(CM) *commutes.*



Proof. Choose an HPol-expansion $\mathbf{p} = ([p_i]) : X \to [\mathbf{X}] = (X_i, [p_{i,i+1}])$ of X such that X_i are connected compact polyhedra. By Theorem 5.2, we obtain inverse sequences $\mathbf{X}_{(n)}$ and $\mathbf{M}_{(n+1)}$ of connected compact polyhedra and morphisms $\mathbf{f}_{(n)} : [\mathbf{X}_{(n)}] \to [\mathbf{X}]$ and $\mathbf{g}_{(n+1)} : [\mathbf{M}_{(n+1)}] \to [\mathbf{X}_{(n)}]$ in pro-HPol for $n \geq 2$ with the properties in Theorem 5.2. Let $X_{(n)}$ and $\mathrm{M}_{(n+1)}$ be the limits of $\mathbf{X}_{(n)}$ and $\mathbf{M}_{(n+1)}$, and let $F_{(n)} : X_{(n)} \to X$ and $G_{(n+1)} : \mathrm{M}_{(n+1)} \to X_{(n)}$ be the shape morphisms represented by $\mathbf{f}_{(n)}$ and $\mathbf{g}_{(n+1)}$, respectively. Then we have the theorem. □

Remark 5.4. The decomposition $X_{(2)} \to X_{(3)} \to \cdots$ in Corollary 5.3 depends on the choice of HPol-expansion of X. Indeed, it is known that the decomposition of X is not unique even for polyhedra X [5].

Theorem 5.5. *Let X be a 1-shape connected metric continuum such that* pro-$\mathrm{H}_n(X)$ *being isomorphic to an inverse sequence of finitely generated free abelian groups for $n \geq 2$. If X has a finite shape dimension, then there exists a decomposition $X_{(2)} \to X_{(3)} \to \cdots$ with the properties in Theorem 5.3 such that* dirlim $X_{(n)}$ *is shape equivalent to X.*

Proof. By [9] (or [7, Theorem 5, p. 251]), the shape dimension of X equals $c(X) = \max\{n \geq 0 : \check{\mathrm{H}}^n(X) \neq 0\}$, where $\check{\mathrm{H}}^*$ denotes the Čech cohomology theory with the integer coefficients. Then the finite shape dimensionality of X implies that there exists $N \geq 0$ such that $\check{\mathrm{H}}^n(X) = 0$ for $n \geq N$. This together with [7, Lemma 6, p. 156] implies that pro-$\mathrm{H}_n(X) = 0$ for $n \geq N$. Choose an HPol-expansion $\mathbf{p} = ([p_i]) : X \to [\mathbf{X}] = (X_i, [p_{i,i+1}])$ such that X_i are connected compact polyhedra with $\dim X_i \leq N$. In view of Proposition 3.5 and the proof of Lemma 5.1, for each $n \geq N$, we can choose the trivial system $(*)$ consisting of base points for $\mathbf{M}_{(n+1)}$ in the construction of the decomposition in Theorem 5.2. Then we can take metric continua $X_{(n+1)}$ so that $X_{(n+1)} = X_{(N)}$ for $n \geq N$. So, dirlim $X_{(n)}$ is shape equivalent to $X_{(N)}$. The shape morphism $F_{(N)} : X \to X_{(N)}$ induces an isomorphism pro-$\mathrm{H}_q(F_{(N)})$: pro-$\mathrm{H}_q(X_{(N)}) \to$ pro-$\mathrm{H}_q(X)$ for $q \geq 0$. By the homological version of the Whitehead theorem [7, Theorem 12, p. 155] and the fact that the shape dimensions of X and $X_{(N)}$ are finite, $F_{(N)}$ is a shape equivalence. This proves the theorem. □

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