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Cohomogeneity one Minkowski space $\mathbb{R}_1^n$

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Abstract. In this paper we study cohomogeneity one Minkowski space $\mathbb{R}_1^n$. Among other results, we prove that the orbit space is homeomorphic to $\mathbb{R}$ or $[0,\infty)$. We show that if there is a spacelike principal orbit, then each of the orbits is spacelike and principal. If $n=3$ and there is a singular orbit, we characterize the orbits up to isometry, and the acting group up to conjugacy.

1. Introduction

Cohomogeneity one Rimannian manifolds have been studied by many mathematicians, see [1], [2], [3], [6], [21], [22], [23]. When the metric is indefinite there are not so much papers in the literature. With this paper we want to begin the study of cohomogeneity one pseudo-Riemannian manifolds. Here we take $M=\mathbb{R}_1^n$, i.e. a Minkowski space, and suppose that a connected closed Lie subgroup $G\subset \mathrm{Iso}(\mathbb{R}_1^n)$ acts properly on $\mathbb{R}_1^n$ with an orbit of codimension one.

The main result of this paper (found in § 3) is that if there is a spacelike principal orbit, then there is no singular orbit and each orbit is isometric to $\mathbb{R}^{n-1}$. When $n=3$ and there is a singular orbit B, we prove that B is a timelike affine subspace of $\mathbb{R}_1^3$, each principal orbit is isometric to $\mathbb{R}_1^1\times S^1(r)$, $r>0$, and G is conjugate to $\mathbb{R}\times SO(2)$.

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2. Preliminaries

Let (M, g) be a complete pseudo-Riemannian manifold of dimension n and G a connected closed Lie subgroup of $\mathrm{Iso}_g(M)$ which acts properly on M. We say that M *is of cohomogeneity one* under the action of G, if G has an orbit of codimension one. For a general theory of (Riemannian) cohomogeneity one manifolds we refer to [2], [3], [4], [6], [21], [22]. Here we remind some of the indispensable backgrounds.

Definition 2.1 ([8, p. 53]). An action of a group G on a manifold M is said to be proper if the mapping $\varphi : G \times M \to M \times M$, $(g, x) \mapsto (g.x, x)$ is proper.

Here is some of the main properties of proper actions which we use.

The orbit space M/G of a proper action of G on M is Hausdorff and the orbits are closed submanifolds in M, and the stabilizer subgroups are compact [8, p. 149].

Lemma 2.2 ([8, p. 150]). *If there is a proper action of a Lie group G on a connected manifold M, then M possesses a G-invariant Riemannian structure which can be assumed complete.*

Throughout the paper we assume that the investigated group action is effective and proper, so we can pass to a Riemannian manifold (M, g'), in which the Riemannian metric g' on M is G-invariant. A result by Mostert (see [17]) for the compact case, and Berard Bergery [4] for the general case, says that the orbit space M/G, equipped with the quotient topology, is homeomorphic to $\mathbb{R}$, S^1, $[0, +\infty)$ or $[0, 1]$. When M is homotopy equivalent to $\mathbb{R}^n$, we prove in Lemma 3.3 that M/G is homeomorphic to $\mathbb{R}$ or $[0, +\infty)$.

Consider the projection map $M \to M/G$ to the orbit space. Given a point $x \in M$, we say that the orbit $G(x)$ is *principal* (resp. *singular*) if the corresponding image in the orbit space M/G is an internal (resp. boundary) point. A point x whose orbit is principal (resp. singular) will be called *regular* (resp. *singular*). All principal orbits are diffeomorphic to each other, each singular orbit is of dimension less than or equal to $n-1$, where $n = \dim M$. A singular orbit of dimension $n-1$ is called an *exceptional* orbit. Note that an exceptional orbit is never simply connected, and if M is simply connected then exceptional orbits do not exist. If M/G is homeomorphic to $\mathbb{R}$, then each orbit is principal and the orbits form a foliation on M. If M/G is homeomorphic to $[0, +\infty)$ and B is the singular orbit with $\dim B = n-m$, then M is homeomorphic to $G \times_H V$, where H is the isotropy subgroup of a singular point $y \in M$ and the manifold V is H-homeomorphic to an m-dimensional Euclidean space in which the subgroup H acts linearly and transitively on the unit sphere $S^{m-1} \subset V$ (see [2]). Since $G \times_H V$ is a V-bundle over G/H, M is a fibre bundle with typical fibre $\mathbb{R}^m$ over B. If $p : M \to B$ is the fibre bundle, then p restricted to a principal orbit is a fibre bundle with typical fibre S^{m-1} (see [6, p. 181] and [5]).

Proposition 2.3 ([8, p. 152])**.** *Suppose that a Lie group G acts properly on a manifold M such that the orbit space M/G is connected. Then the union M_0 of all regular points is open and dense in M.*

Lemma 2.4 ([8, p. 137])**.** *Let G be a connected Lie group and H a Lie subgroup of G. If $\pi_n(G/H) = 0$ for each $n \geqslant 0$, where π_n is the n-th homotopy group, then the manifold G/H is diffeomorphic to $\mathbb{R}^m$, where $m = \dim G/H$.*

Throughout in the following $\mathbb{R}^n_p$ denotes the n-dimensional real vector space $\mathbb{R}^n$ with a scalar product of signature $(p, n-p)$ given by

$$\langle x, y\rangle = -\sum_{i=1}^{p} x_i y_i + \sum_{j=p+1}^{n} x_j y_j .$$

The set of all linear isometries $\mathbb{R}^n_p \to \mathbb{R}^n_p$ is a Lie group, which may be identified with the group $O(p, n-p)$ of all matrices $A \in GL(n, \mathbb{R})$ that preserve the scalar product defined above. The identity component of $O(p, n-p)$ is denoted by $SO_o(p, n-p)$. It is known that each maximal compact subgroup of $SO_o(p, n-p)$ is conjugate to $SO(p)\times SO(n-p)$ (see [11] or [16, p. 25]) where $SO(p)\times SO(n-p)$ is considered with respect to the standard decomposition $\mathbb{R}^n_p = \mathbb{R}^p_p \oplus \mathbb{R}^{n-p}$. We write $S^n_1(r)$ for the hypersurface $\{x \in \mathbb{R}^{n+1}_1 \mid \langle x, x\rangle = r^2\}$ and $H^n(r)$ for a connected component of the hypersurface $H^n_o(r) = \{x \in \mathbb{R}^{n+1}_1 \mid \langle x, x\rangle = -r^2\}$.

Definition 2.5 ([18]). Let $L^{n+1} = \mathbb{R}^{n+1}_1$ and let N be a connected spacelike hypersurface. N is said to be isoparametric if its shape operator S has constant eigenvalues (principal curvatures).

In [18] Nomizu showed that a complete connected spacelike isoparametric hypersurface in $\mathbb{R}^{n+1}_1$ has at most two distinct constant principal curvatures. If it has two distinct constant principal curvatures, then it is isometric to

$$H^k(r) \times E^{n-k} = \{(x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1}_1 \mid -x_1^2 + x_2^2 + \cdots + x_{k+1}^2 = -r^2\} .$$

If it is a complete, connected, totally umbilic spacelike hypersurface (has exactly one principal curvature) then it is isometric to either $\mathbb{R}^{n-1}$ or $H^n(r)$ (see [19, p. 117]).

Definition 2.6. Let $N \subset (M, g)$ be a pseudo-Riemannian submanifold of M. N is called extrinsically homogeneous, if there is a Lie group $G \subset \mathrm{Iso}_g(M)$ such that G acts on N transitively.

Suppose that $M = \mathbb{R}^n_1$ is of cohomogeneity one under the proper action of a connected, closed Lie subgroup $G \subset \mathrm{Iso}(M)$. Then each orbit $G(x)$ is extrinsically homogeneous, so isoparametric (see [20]). Furthermore, if $G(x)$ is a spacelike principal orbit, then it is a complete submanifold (see [10]).

Lemma 2.7 ([14])**.** *If a Lie group G is compact, or connected and semisimple, then any smooth representation of G by affine transformations of $\mathbb{R}^n$ admits a fixed point.*

3. Main results

We begin with the formulation of our main Theorem.

Theorem 3.1. *Let $\mathbb{R}^n_1$ be of cohomogeneity one under the proper action of a connected, closed Lie subgroup $G \subset \mathrm{Iso}(\mathbb{R}^n_1)$. If there is a spacelike principal orbit, then each orbit is spacelike and isometric to $\mathbb{R}^{n-1}$. In particular there is no singular orbit.*

We prove the Theorem via proving some lemmas.

Lemma 3.2. *If $\mathbb{R}^n_p$, $1 \leqslant p \leqslant n-1$, is of cohomogeneity one under the action of a connected, closed Lie subgroup $G \subset \mathrm{Iso}(M)$, then G is not compact.*

Proof. If G is compact then each (principal) orbit is compact, but there is no compact pseudo-Riemannian hypersurface in $\mathbb{R}^n_p$ (see [19, p. 125]), so G is not compact. □

Lemma 3.3. *If $M = \mathbb{R}^n_p$, $1 \leqslant p \leqslant n-1$, is of cohomogeneity one under the proper action of a connected Lie group G, then the orbit space M/G is a one dimensional Hausdorff space homeomorphic to $\mathbb{R}$ or $[0, +\infty)$. In particular, there is at most one singular orbit.*

Proof. Since the action of G on M is proper, by Lemma 2.2 M possesses a G-invariant Riemannian structure, hence we may assume that (M, g') is a Riemannian manifold and $G \subset \mathrm{Iso}_{g'}(M)$ acts on M by cohomogeneity one, therefore M/G is homeomorphic to one of the spaces (i) $\mathbb{R}$, (ii) S^1, (iii) $[0, +\infty)$, (iv) $[0, 1]$, (see [4]), and by Proposition 2.3 the set of regular points is dense in M. So using the same argument as in the proof of Proposition 3.3 of [22] we obtain that the case (iv) is impossible. We claim that case (ii) is also not possible. If $M/G \cong S^1$ then by [2] the projection $\pi : M \to S^1$ is a fibration with fibre G/K, where K is the stabilizer of a regular point. By Theorem 4.41 of [9, p. 379] there is a long exact sequence

$$\to \pi_m(G/K, x_0) \to \pi_m(M, x_0) \to \pi_m(S^1, b_0) \to \pi_{m-1}(G/K, x_0) \to \dots \\ \to \pi_0(M, x_0) \to 0$$

of homotopy groups, where $x_0 \in \pi^{-1}(b_0)$ and $b_0 \in S^1$.

Hence $\pi_0(G/K, x_0) \cong \mathbb{Z}$ and this contradicts the connectedness of G/K. □

Lemma 3.4. *Let $M = \mathbb{R}^n_p$, $1 \leqslant p \leqslant n-1$, be of cohomogeneity one under the proper action of a connected Lie group G. If there is a singular orbit $G(y)$, then it is diffeomorphic to $\mathbb{R}^k$ for some $1 \leqslant k \leqslant n-2$.*

Proof. Suppose that $G(y)$ is a singular orbit. If $\dim G(y)=0$, then $G_y=G$, so by the properness of the action G must be compact, which contradicts Lemma 3.2. Thus $1 \leqslant \dim G(y) \leqslant n-2$. As there is at most one singular orbit by Lemma 3.3, and the action is proper, by Lemma 2.2 and [2] M is homeomorphic to $G \times_{G_y} V$ where V is an $(n-k)$-dimensional vector space. Hence M is a fibre bundle with base G/G_y. Thus M and $G(y)$ are of the same homotopy type, therefore $G(y)$ is diffeomorphic to $\mathbb{R}^k$ by Lemma 2.4. □

Lemma 3.5. *Let $\mathbb{R}^n_1$ be of cohomogeneity one under the proper action of a connected, closed Lie subgroup $G \subset \mathrm{Iso}(\mathbb{R}^n_1)$. Then*

(a) *If there is a spacelike principal orbit, there is no singular orbit.*

(b) *Each spacelike principal orbit is isometric to $\mathbb{R}^{n-1}$.*

Proof. (a) Suppose that $G(x_\circ)$ is a spacelike principal orbit, for some $x_\circ \in \mathbb{R}^n_1$. Since each orbit is extrinsically homogeneous, it is a complete isoparametric hypersurface (see [10] and [20]), so by [18] and [19, p. 117] it is isometric to one of the following spaces

(i) $\mathbb{R}^{n-1}$;

(ii) $H^{n-1}(r)$, a connected component of $H^{n-1}_\circ(r)$;

(iii) $H^k(r) \times \mathbb{R}^{n-k-1}$ where $0 < k < n$ and $r > 0$

where each of them, and so $G(x_\circ)$, is diffeomorphic to $\mathbb{R}^{n-1}$. We claim that there is no singular orbit. If $G(y)$ is a (unique) singular orbit, it is diffeomorphic to $\mathbb{R}^k$ by Lemma 3.4. Hence $G(x_\circ)$ must be a spherical fibre bundle over $G(y)$, (see [6, p. 181]), which is not obviously possible.

(b) Based on the proof of the case (a) it is enough to show that the cases $H^{n-1}(r)$ and $H^k(r) \times \mathbb{R}^{n-k-1}$ do not occur.

Case $b-1$: If $G(x_\circ)$ is isometric to $H^{n-1}(r)$, for some $x_\circ \in \mathbb{R}^n_1$ and $r>0$, then without loss of generality we may assume that $G(x_\circ) = H^{n-1}(r)$ with $\langle x_\circ, x_\circ\rangle = -r^2$. We show that $G \subset SO_o(1, n-1) \times \{0\}$. Fix an arbitrary element $(A, a) \in G \subset SO_\circ(1, n-1) \ltimes \mathbb{R}^n$ and $y \in G(x_\circ)$. Then

$$Ay + a = (A, a).y \in G(x_\circ) = H^{n-1}(r)$$

Since a linear isometry $A : \mathbb{R}^n_1 \longrightarrow \mathbb{R}^n_1$ carries $H^{n-1}_\circ(r)$ on to itself and $H^{n-1}_\circ(r)$ is a pseudo-Riemannian submanifold, $A \mid H^{n-1}_\circ(r) \in \mathrm{Iso}(H^{n-1}_\circ(r))$. So

$$\langle Ay, Ay\rangle = -r^2, \quad \langle Ay + a, Ay + a\rangle = -r^2,$$

hence

$$\langle Ay, a\rangle = -\frac{1}{2}\langle a, a\rangle$$

so

$$\langle x, a\rangle = -\frac{1}{2}\langle a, a\rangle \quad \text{for all } x \in A(H^{n-1}(r)).$$

Thus for each $x \in A(H^{n-1}(r))$ and each curve γ in $A(H^{n-1}_\circ(r))$ with $\gamma(0) = x$ we have $\langle \gamma'(0), a\rangle = 0$, so $a \perp T_x A(H^{n-1}(r))$ for all $x \in A(H^{n-1}(r))$. But $A(H^{n-1}(r))$ is one of the connected components of $H^{n-1}_\circ(r)$, so $a = 0$. Therefore $G \subset SO_\circ(1, n-1)$ and this implies that G stabilizes the origin. In particular G must be compact, hence $G(x) = H^{n-1}(r)$ is compact, which is not true.

Case $b-2$: $G(x) = H^k(r) \times \mathbb{R}^{n-k-1} \subset \mathbb{R}^{k+1}_1 \oplus \mathbb{R}^{n-k-1}$. Fixing an arbitrary element $(A, a) \in G \subset SO_o(1, n-1) \ltimes \mathbb{R}^n$, let $A = (A_1, \ldots, A_n)$, where A_i is i-th row of A and $a = (a_1, \ldots, a_n)^T$. Let $p_1 : \mathbb{R}^{k+1}_1 \oplus \mathbb{R}^{n-k-1} \longrightarrow \mathbb{R}^{k+1}_1$ be the canonical projection, and denote by $\langle \mid \rangle$ the usual inner product in $\mathbb{R}^n$. Since

$$(A, a).x = Ax + a \in H^k(r) \times \mathbb{R}^{n-k-1}, \quad \forall\, x \in H^k(r) \times \mathbb{R}^{n-k-1},$$

it follows that

$$p_1(Ax + a) = \begin{bmatrix} \langle A_1 \mid x\rangle \\ \vdots \\ \langle A_{k+1} \mid x\rangle \end{bmatrix} + \begin{bmatrix} a_1 \\ \vdots \\ a_{k+1} \end{bmatrix} \in H^k(r).$$

If

$$A_i = (A_{i_1}, A_{i_2}) \in \mathbb{R}^{k+1} \times \mathbb{R}^{n-k-1} \quad \text{and} \quad x = (x_1, x_2) \in H^k(r) \times \mathbb{R}^{n-k-1}\ ,$$

then
$$\begin{bmatrix} \langle A_{1_1} \mid x_1\rangle \\ \vdots \\ \langle A_{k+1_1} \mid x_1\rangle \end{bmatrix} + \begin{bmatrix} \langle A_{1_2} \mid x_2\rangle \\ \vdots \\ \langle A_{k+1_2} \mid x_2\rangle \end{bmatrix} + \begin{bmatrix} a_1 \\ \vdots \\ a_{k+1} \end{bmatrix} \in H^k(r).$$

If we fix $x_1 \in H^k(r)$ and choose $x_2 \in \mathbb{R}^{n-k-1}$ arbitrarily, then we get

$$A_{1_2} = \cdots = A_{k+1_2} = 0.$$

Hence
$$A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} \in SO_\circ(1, n),$$

where $B \in O(1, k)$ and $C \in O(n - k - 1)$. Thus $A(H^k_\circ(r) \times \{0\}) = H^k_\circ(r) \times \{0\}$, i.e., $BH^k_\circ(r) = H^k_\circ(r)$ and

$$Bx_1 + \begin{bmatrix} a_1 \\ \vdots \\ a_{k+1} \end{bmatrix} \in H^k(r), \quad \forall\, x_1 \in H^k(r);$$

hence by *subcase* $b - 1$ one gets that $a_1 = \cdots = a_{k+1} = 0$.

Thus

$$G = \left\{ \left(\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}, \begin{bmatrix} 0 \\ b \end{bmatrix} \right) \mid B \in O(1, k),\ C \in O(n - k - 1),\ b \in \mathbb{R}^{n-k-1} \right\}$$

This shows that $G(0) = \mathbb{R}^{n-k-1}$ is a singular orbit which contradicts to (a). $\square$

Now we are in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Suppose that there is a spacelike principal orbit $G(x_\circ)$. We show that $G(x)$ is spacelike and principal for all $x \in \mathbb{R}^n_1$. Without loss of generality we may suppose that $x_\circ = 0$. Then $G(0)$ is isometric to $\mathbb{R}^{n-1}$ and there is no singular orbit by Lemma 3.5. Consider the foliation defined by

$$\{f_x = x + G(0) \mid x \in \mathbb{R}^n_1\}.$$

Here each leaf is a translation of $G(0)$. Denote the space of leaves by Δ, and consider the canonical projection map $\pi : \mathbb{R}^n_1 \to \Delta$. Then Δ with the quotient topology is a manifold diffeomorphic to $\mathbb{R}$. Each vector field X tangent to Δ has a unique lift $\bar{X}$ normal to the fibres on $M = \mathbb{R}^n_1$, hence one may define a scalar product $\langle\,|\,\rangle$ by $\langle X \mid X\rangle \circ \pi = \langle \bar{X}, \bar{X}\rangle$. Thus $(\Delta, \langle\,|\,\rangle)$ is isometric to $\mathbb{R}^1_1$, and π is a pseudo-Riemannian submersion. Now define the (isometric) action of G on Δ as follows

$$G \times \Delta \to \Delta, \quad (g, \pi(x)) \mapsto \pi(gx)$$

Since $G(\pi(0)) = \pi(0)$, G fixes Δ pointwise, i.e. G acts on Δ trivially. Thus $G(x)$ is contained in f_x for each $x \in \mathbb{R}^n_1$, which implies that $G(x)$ is spacelike, and hence isometric to $\mathbb{R}^{n-1}$ by Lemma 3.5. □

By Theorem 3.1, if there is a spacelike principal orbit, then each orbit is spacelike. One would like to get that this result holds, when there is a Lorentzian principal orbit. However the following example shows that this is expectation false. In fact, in the next example there is an open dense subset of M consisting of Lorentzian principal orbits, but there is another degenerate principal orbit!

Example 3.6. Let $M = \mathbb{R}^3_1$ and

$$G = \left\{ (A_t, b_{t,s}) \in SO_o(1,2) \ltimes \mathbb{R}^3 \mid A_t = \begin{bmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{bmatrix},\ b_{t,s} = \begin{bmatrix} s \\ s \\ t \end{bmatrix} s, t \in \mathbb{R} \right\}.$$

Then G is a subgroup of $\mathrm{Iso}(\mathbb{R}^3_1)$, the action of G on $\mathbb{R}^3_1$ is proper, and the orbit $G(0) = \{(s,s,t) \mid s,t \in \mathbb{R}\}$ is a two dimensional subspace of $\mathbb{R}^3_1$. Since $\{(1,1,0),(0,0,1)\}$ is a basis of this subspace, $v = (1,1,0)$ is null and normal to $u = (0,0,1)$, $G(0)$ is a degenerate subspace. If $a = (x_1, x_2, x_3)$ is an arbitrary point in $\mathbb{R}^3_1$ such that $x_1 \neq x_2$ then the shape operator of $G(a)$ at $A_t a + b_{t,s}$ is

$$S = -e^t \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

so the minimal polynomial of the shape operator is x^2 (the shape operator is not diagonalizable), hence $G(a)$ is a generalized cylinder of type 1 (see [13]). Thus for such an $a \in \mathbb{R}^3_1$ the orbit $G(a)$ is a Lorentzian orbit, but $G(0)$ is not Lorentzian.

As a final result, we characterize the orbits of $\mathbb{R}^3_1$ up to isometry, and the acting group up to conjugacy, when there is a singular orbit.

Theorem 3.7. *Let $\mathbb{R}^3_1$ be of cohomogeneity one under the proper action of a connected and closed Lie subgroup $G \subset \mathrm{Iso}(\mathbb{R}^3_1)$. If there is a singular orbit B then:*

(a) *B is a one dimensional timelike affine subspace of $\mathbb{R}^3_1$.*

(b) *G is conjugate to*

$$S = \left\{ \left(\begin{bmatrix} 1 & 0 \\ 0 & SO(2) \end{bmatrix}, \begin{bmatrix} t \\ 0 \end{bmatrix} \right) \mid t \in \mathbb{R} \right\}$$

(c) *Each principal orbit D is isometric to $\mathbb{R}^1_1 \times S^1(r)$ for some $r > 0$.*

For the proof we need the following lemma.

Lemma 3.8. *Let $\mathbb{R}^n_1$ be of cohomogeneity one under the proper action of a connected, closed Lie subgroup $G \subset \mathrm{Iso}(\mathbb{R}^n_1)$, let B be a singular orbit, and $H = G_b$ the isotropy subgroup at a point $b \in B$. Then H is maximal compact subgroup in G.*

Proof. Suppose that H is not a maximal compact subgroup, and that $H \subsetneqq H'$, where H' is a compact Lie subgroup of G. There is a point $x_0 \in \mathbb{R}^n_1$ which is fixed under the action of H', so under H, by Lemma 2.7. We note that $G(x_0)$ is necessarily a singular orbit and x_0 does not belong to the orbit B, since otherwise H and H' would be conjugate, and hence equal. The unique geodesic γ through b and x_0 is fixed pointwise under the action of H, since b and x_0 are fixed. By the properness of the action M_0 is open and dense in M, so $\gamma(t_0)$ is a regular point for some $t_0 \in \mathbb{R}$ and is fixed under the action of H, hence $H \subset G_{\gamma(t_0)}$, which is a contradiction. $\square$

Proof of Theorem 3.7. Since there is no exceptional orbit, by Lemma 3.4 we conclude that $\dim B = 1$. By Lemma 3.3, B is the only singular orbit, hence B is homotopic to M (see [22]), so B is diffeomorphic to $\mathbb{R}$ by Lemma 2.4.

We prove that B is a one dimensional affine subspace of $\mathbb{R}^3_1$. Fixing $y \in B$, as G_y is compact by the properness of the action and connected by Proposition 17 of [19, p. 309], $G_y(x)$ is a compact connected submanifold of $B \cong \mathbb{R}$ for each $x \in B$, so $G_y(x) = \{x\}$. Therefore B is left invariant by G_y pointwise and the geodesic γ through y and $x \in B$ is left invariant by G_y pointwise as well. By the uniqueness of the singular orbit B we get that $B = \gamma(\mathbb{R})$.

Suppose that

$$SO(1) \times SO(2) = \left\{ \left(\begin{bmatrix} 1 & 0 \\ 0 & SO(2) \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) \right\} \subset SO_o(1,2) \ltimes \mathbb{R}^3.$$

We claim that G_y is conjugate to $SO(1) \times SO(2)$. We know that G_y is a maximal compact subgroup in G by Lemma 3.8, and by [11, p. 275] any maximal compact subgroup of $SO_o(1,2) \ltimes \mathbb{R}^3$ is conjugate to $SO(1) \times SO(2)$, so G_y is conjugate to some subgroup H of $SO(1) \times SO(2)$. If $H \subsetneqq SO(1) \times SO(2)$, then $\dim H = 0$, since by Proposition 17 of [19, p. 309] H is connected, so $H = \{I\}$ and this is impossible since $G_x \subsetneqq H = \{I\}$ for each regular point x, a contradiction. Hence G_y is conjugate to $SO(1) \times SO(2)$. Since G_y leaves B pointwise invariant, it is a normal subgroup in G, and G is isomorphic to $G_y \times \mathbb{R}$, so G is Abelian and $\dim G = 2$, hence $G = Z_G(G_y)$ is conjugate to a Lie subgroup of

$$Z_{SO_o(1,2)\ltimes\mathbb{R}^3}(SO(1) \times SO(2)) = \left\{ \left(\begin{bmatrix} 1 & 0 \\ 0 & SO(2) \end{bmatrix}, \begin{bmatrix} t \\ 0 \end{bmatrix} \right) \mid t \in \mathbb{R} \right\} \quad ;$$

thus G is conjugate to S.

Now we prove that B is timelike. Without loss of generality we may suppose that

$$G = \left\{ \left(\begin{pmatrix} 1 & 0 \\ 0 & SO(2) \end{pmatrix}, \begin{pmatrix} t \\ 0 \end{pmatrix} \right) \mid t \in \mathbb{R} \right\}.$$

Hence $B = G(0) = \mathbb{R}^1_1 \times \{0\}$ is timelike. Each element of $SO_o(1,2)$ preserves time- and space-orientation, thus the singular orbit is a timelike subspace.

The statement (c) is a consequence of (a) and (b). □

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