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On categories of ordered sets with a closure operator

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Abstract. We define and study two categories of partially ordered sets endowed with a closure operator. The first category has order-preserving and continuous maps as morphisms and it is shown to be concretely isomorphic to a category of ordered sets endowed with a compatible preorder. The second category has closed maps as morphisms and it is proved to be cartesian closed. Consequences of these results for categories of closure spaces and, in particular, of topological spaces are discussed.

1. Introduction and preliminaries

Closure operators that are not additive in general occur in many branches of mathematics, in particular in algebra (algebraic closure operators) and in geometry (convex hulls). They are closely related to (complete) lattices as has already been known for quite a long time - see the pioneering book [3] by G. Birkhoff (and, for more details, see [8]). For this reason, such closure operators were studied by many authors including W. Sierpiński, whose book [13] was among the first to deal with the topic. From the categorical point of view, they were investigated, for instance, in [4], [5] and [6]. Non-additive closure operators have numerous applications also in other disciplines such as informatics (data analysis and knowledge representation – see [9]), formal logic (see [11]), physics (quantum mechanics – see [2] and [12]), etc. In these applications, the closure operators are even non-grounded.

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Closure operators are usually understood to be maps from the Boolean algebra given by the power-set of an underlying set into itself. But in set-theory and algebra, more general closure operators are often used that are defined to be maps from a given partially ordered set into itself (for example, every Galois connection between partially ordered sets gives rise to such closure operators). In this note, we study two categories of certain partially ordered sets endowed with a closure operator. The first category is defined to have order-preserving and continuous maps as morphisms. We show that this category is isomorphic to a category of ordered sets endowed with a compatible preorder whose morphisms are maps which preserve both the order and the compatible preorder. The restrictions of this isomorphism to the full subcategory given by ordered sets with grounded closure operators and to that given by ordered sets with additive closure operators are discussed. As a consequence, we obtain a category of sets endowed with a preorder on the corresponding power set which is isomorphic to the category Top of topological spaces and continuous maps. The second category under investigation has also certain ordered sets with a closure operator as objects but its morphisms are closed maps. This category, into which the category of topological spaces and closed maps between the corresponding power-sets may be fully embedded, is proved to be cartesian closed.

Definition 1.1. Let $X = (X, \leq)$ be a partially ordered set. A *closure operator* on X is any map $u : X \to X$ which fulfills the following three axioms:

(i) for all $x \in X$, $x \leq u(x)$ (*extensiveness*),

(ii) for all $x, y \in X$, $x \leq y \Rightarrow u(x) \leq u(y)$ (*monotonicity*),

(iii) for all $x \in X$, $u(u(x)) = u(x)$ (*idempotency*).

If u is a closure operator on a partially ordered set X, then the pair (X, u) is called a *closure system*. An element $x \in X$ is said to be *closed* in (X, u) if $u(x) = x$. We distinguish between the concept of closure systems and that of closure spaces: By a *closure space* we understand, as usual, a pair (X, u) where X is a (generally non-ordered) set and u is a closure operator on the Boolean algebra $2^X = (2^X, \subseteq)$ where 2^X denotes the power-set of X.

A closure operator u on a partially ordered set X and the closure system (X, u) are called

(iv) *grounded* if X has a least element 0 and $u(0) = 0$,

(v) *additive* if X is a join-semilattice and $u(x \vee y) = u(x) \vee u(y)$ for all $x, y \in X$.

Throughout the paper, partial orders on generally different sets will usually be denoted by the same symbol $\leq$. This will cause no confusion as it will always be clear which partially ordered set is considered.

The well-known concepts of continuous and closed maps are transferred from the classical closure operators to our more general setting as follows:

Definition 1.2. Let (X, u) and (Y, v) be closure systems and $f : X \to Y$ be a map. Then f is said to be

(a) a *continuous map* from (X, u) into (Y, v) if $f(u(x)) \leq v(f(x))$ for every $x \in X$,

(b) a *closed map* from (X, u) into (Y, v) if $f(x)$ is closed in (Y, v) whenever x is closed in (X, u).

Remark 1.3. One can easily see that:

(a) A composition $g \circ f$ of two continuous maps f and g between closure systems is continuous whenever g is order-preserving.

(b) An order-preserving map $f : (X, u) \to (Y, v)$ is closed if and only if $f(u(x)) \geq v(f(x))$ for every $x \in X$. Thus, among the order-preserving maps, maps which are both continuous and closed coincide with the closure preserving maps, i.e., maps $f : (X, u) \to (Y, v)$ with $f(u(x)) = v(f(x))$ for every $x \in X$.

(c) Every order-preserving and continuous map $f : (X, u) \to (Y, v)$ has the property that, for every $x \in X$, $f(u(x)) = f(x)$ whenever $f(x)$ is closed.

We will use some basic topological, category-theoretical and lattice-theoretical concepts only - for their definitions see e.g. [7], [1] and [10], respectively. Let us just note here that, by a *complete join-semilattice* (*complete meet-semilattice*), we mean an ordered set in which each nonempty subset has a join (meet).

2. A category of closure systems with order-preserving continuous maps as morphisms

We denote by *Cont* the category whose objects are the closure systems (X, u) such that $X = (X, \leq)$ is a complete join-semilattice and whose morphisms are maps that are both order-preserving and continuous.

Definition 2.1. Let $X = (X, \leq)$ be a partially ordered set that is a complete join-semilattice. A preorder ρ on the set X is said to be *compatible* (with the partial order $\leq$) and the pair (X, ρ) is said to be a *preordered complete join-semilattice* if the following two axioms are satisfied:

(1) For all $x, y \in X$, $x \leq y \Rightarrow x\rho y$.

(2) If $x_i \in X$ for every $i \in I$ ($I \neq \emptyset$ a set) and $x \in X$, then $\bigvee\{x_i;\ i \in I\}\rho x$ whenever $x_i \rho x$ for every $i \in I$.

We denote by *Copo* the category whose objects are the preordered complete join-semilattices (X, ρ) and whose morphisms are the maps that preserve both the partial order and the compatible preorder.

We will need the following

Lemma 2.2. *Let (X, ρ) be a preordered complete join-semilattice, $X = (X, \leq)$. Then every equivalence class of $\rho \cap \rho^{-1}$ has a greatest element (with respect to $\leq$).*

Proof. Let A be an arbitrary equivalence class of $\rho \cap \rho^{-1}$ and let $x \in A$ be an element. By the axiom (2) of Definition 2.1, $\bigvee A \rho x$. Conversely, since $x \leq \bigvee A$, we have $x\rho \bigvee A$ by the axiom (1) of Definition 2.1. Thus, $\bigvee A \in A$, which proves the statement. □

Remark 2.3. It can easily be seen that the greatest element of an equivalence class from Lemma 2.2 equals the greatest element of the set $\{x \in X;\ x\rho y\}$ where y is an arbitrary element of the class.

Given a closure system (X, u), we denote by ρ_u the preorder on X defined by $\rho_u = \{(x, y) \in X^2;\ x \leq u(y)\}$. We put $F(X, u) = (X, \rho_u)$ for every object (X, u) of *Cont* and $Ff = f$ for every morphism f in *Cont*.

Theorem 2.4. *F is a concrete isomorphism of Cont onto Copo.*

Proof. If (X, u) is an object of *Cont*, then (X, ρ_u) is clearly a preordered complete join-semilattice, hence an object of *Copo*. Let $f : (X, u) \to (Y, v)$ be a morphism in *Cont* and let $x, y \in X$, $x\rho_u y$. Then $x \leq u(y)$, which implies $f(x) \leq f(u(y)) \leq v(f(y))$. Thus, $f(x)\rho_v f(y)$, which means that $f : (X, \rho_u) \to (Y, \rho_v)$ is a morphism in *Copo*. Therefore, $F : Cont \to Copo$ is a (faithful) functor.

Let $f : (X, \rho_u) \to (Y, \rho_v)$ be a morphism in *Copo* and let $x \in X$. Then $u(x) \leq u(x)$ implies $u(x)\rho_u x$ and, consequently, $f(u(x))\rho_v f(x)$. This yields $f(u(x)) \leq v(f(x))$ so that $f : (X, u) \to (Y, v)$ is a morphism in *Cont*. We have shown that the functor F is full.

Let (X, u), (Y, v) be objects of *Cont* such that $F(X, u) = F(Y, v)$, i.e., such that $\rho_u = \rho_v$. Then, for arbitrary $x, y \in X$, we have $x \leq u(y) \Leftrightarrow x \leq v(y)$. Consequently, $u(y) = v(y)$ for every $y \in X$. Therefore, F is injective on objects.

Let (X, ρ) be an object of *Copo*. By Lemma 2.2, for every $x \in X$, there exists a greatest element $u_\rho(x)$ of the equivalence class of $\rho \cap \rho^{-1}$ containing x. Then, clearly, $x \leq u_\rho(x)$ and $u_\rho(u_\rho(x)) = u_\rho(x)$. Let $x, y \in X$, $x \leq y$. Then $x \leq u_\rho(y)$ and we have $u_\rho(x)\rho x\rho u_\rho(y)$. Consequently, $u_\rho(x)\rho u_\rho(y)$. Since $u_\rho(y)\rho u_\rho(y)$ and $u_\rho(y) \leq u_\rho(x) \vee u_\rho(y)$, we get $u_\rho(y)\rho(u_\rho(x) \vee u_\rho(y))\rho u_\rho(y)$. Therefore, $u_\rho(y) = u_\rho(u_\rho(y)) = u_\rho(u_\rho(x) \vee u_\rho(y))$. Thus, $u_\rho(x) \leq u_\rho(x) \vee u_\rho(y) \leq u_\rho(u_\rho(x) \vee u_\rho(y))$ implies $u_\rho(x) \leq u_\rho(y)$. We have proved that u_ρ is a closure operator on X, i.e., that (X, u_ρ) is an object of *Cont*. Let $x, y \in X$ and suppose that $x\rho_{u_\rho}y$. Then $x \leq u_\rho(y)$. It follows that $u_\rho(x) \leq u_\rho(y)$, hence $u_\rho(x)\rho u_\rho(y)$. Since $x\rho u_\rho(x)$ and $u_\rho(y)\rho y$, we have $x\rho y$. Conversely, let $x\rho y$. Then $y\rho y$ and $y \leq x \vee y$ imply $y\rho(x \vee y)\rho y$. This yields $u_\rho(y) = u_\rho(x \vee y)$. Thus, $x \leq x \vee y \leq u_\rho(x \vee y)$ implies $x \leq u_\rho(y)$. Therefore, $x\rho_{u_\rho}y$. We have shown that F is surjective on objects. The proof is complete. □

Recall [3] that a lattice is said to be *atomistic* if each of its elements is a join of atoms.

Proposition 2.5. *Let (X, u) be a closure system where X has a least element 0. Then*

(a) *(X, u) is grounded if and only if, for every $x \in X$, $x\rho_u 0 \Rightarrow x = 0$;*

(b) *if X is a distributive atomistic lattice, then (X, u) is additive if and only if, for every $x, y \in X$ and every atom $a \in X$, $a\rho_u(x \vee y) \Leftrightarrow a\rho_u x$ or $a\rho_u y$.*

Proof. Statement (a) is obvious. Let $X = (X, \leq)$ be a distributive atomistic lattice and let $x, y \in X$. Then $u(x \vee y) = u(x) \vee u(y)$ is equivalent to $a \leq u(x \vee y) \Leftrightarrow a \leq (u(x) \vee u(y))$ for every atom $a \in X$, which is equivalent to $a\rho_u(x \vee y) \Leftrightarrow a \wedge (u(x) \vee u(y)) = a$ for every atom $a \in X$. Clearly, for every atom $a \in X$, we have $a \wedge (u(x) \vee u(y)) = a \Leftrightarrow (a \wedge u(x)) \vee (a \wedge u(y)) = a \Leftrightarrow (a \leq u(x)$ or $a \leq u(y)) \Leftrightarrow (a\rho_u x$ or $a\rho_u y)$. This proves statement (b). □

Example 2.6. a) Let *ContCl* denote the category whose objects are the closure spaces and whose morphisms are the usual continuous maps, i.e., the maps $f : (X, u) \to (Y, v)$ with $f(u(A)) \subseteq v(f(A))$ whenever $A \subseteq X$. Then there is an embedding $G : ContCl \to Cont$ given by $G(X, u) = (2^X, u)$ for every object (X, u) of *ContCl* and $Gf = \bar{f} : (2^X, u) \to (2^Y, v)$ for every morphism $f : (X, u) \to (Y, v)$ in *ContCl* where $\bar{f}$ denotes the lifting (extension) of f to the corresponding power-sets. Thus, by Theorem 2.4, $F \circ G : ContCl \to Copo$ is an embedding too. The objects of the category $F \circ G(ContCl)$ are precisely the preordered sets $(2^X, \rho)$, X a set, satisfying the following two conditions:

(1') For all $A, B \subseteq X$, $A \subseteq B \Rightarrow A\rho B$.

(2') If $A_i \subseteq X$ for every $i \in I$ ($I \neq \emptyset$ a set) and $A \subseteq X$, then $\bigcup\{A_i;\ i \in I\}\rho A$ whenever $A_i\rho A$ for every $i \in I$.

The morphisms in $F \circ G(ContCl)$ are the preorder-preserving maps $g : (2^X, \rho) \to (2^Y, \sigma)$ such that $g = \bar{f}$ where $f : X \to Y$ is a map (i.e., such that $g(A)$ is a singleton whenever A is a singleton and $g(\bigcup_{i\in I} A_i) = \bigcup_{I\in I} g(A_i)$ whenever $\{A_i;\ i \in I\} \subseteq 2^X$).

b) Let $ContCl^*$ denote the full subcategory of $ContCl$ given by the objects (X, u) of $ContCl$ such that u is a grounded closure operator on $2^X = (2^X, \subseteq)$. Then the embedding of $ContCl^*$ into $Copo$ obtained by restricting $F \circ G$ coincides with the embedding found in [15] by the help of results from [14].

c) Let Top denote the category of topological spaces and the usual continuous maps, i.e, the full subcategory of $ContCl$ given by the objects (X, u) of $ContCl$ such that u is a grounded and additive closure operator on $2^X = (2^X, \subseteq)$. By Proposition 2.5, the image of Top under the restriction of $F \circ G$ to Top is the full subcategory of $F \circ G(ContCl)$ given by just the objects $(2^X, \rho)$ of $F \circ G(ContCl)$ that fulfill the following two conditions:

(3') For every $A \subseteq X$, $A\rho\emptyset \Rightarrow A = \emptyset$.

(4') For every $x \in X$ and every $A, B \subseteq X$, $\{x\}\rho(A \cup B) \Leftrightarrow \{x\}\rho A$ or $\{x\}\rho B$.

3. A category of closure systems with closed maps as morphisms

Recall [1] that a category $\mathbf{C}$ is *cartesian closed* if it has finite products and, for any two objects $A, B \in \mathbf{C}$, there exists an object B^A in $\mathbf{C}$ and a morphism $ev : A \times B^A \to B$ with the property that, for each morphism $g : A \times C \to B$ in $\mathbf{C}$, there exists a unique morphism $g^* : C \to B^A$ such that $ev \circ (id_A \times g^*) = g$. Cartesian closed categories possess a well behaved operation of exponentiation of objects and, therefore, have many useful applications. For instance, in computer science they are used as models of important foundational programming languages, the so-called typed lambda-calculi. Therefore, it is always a useful result to find a new cartesian closed category. (Since Top is not cartesian closed, it is often replaced by some of its full subcategories or supercategories that are cartesian closed.)

We denote by Clo the category whose objects are the closure systems (X, u) such that, for every $x \in X$, the upper interval $[x)$ in X is a complete meet-semilattice and whose morphisms are the closed maps.

Remark 3.1. a) Let $(X, \leq)$ be a partially ordered set such that, for every $x \in X$, the upper interval $[x)$ in X is a complete meet-semilattice. Then there is a one-to-one correspondence between the closure operators on the partially ordered set X and the subsets $A \subseteq X$ satisfying the following condition:

$$(*) \quad \bigwedge(A \cap [x)) \in A \quad \text{for every } x \in X.$$

Indeed, such a correspondence is given by assigning to every closure operator u on X the set of all closed elements in (X, u). The converse correspondence is obtained by assigning to every subset $A \subseteq X$ satisfying $(*)$ the closure operator on X defined by $u(x) = \bigwedge(A \cap [x))$ for every $x \in X$.

b) Observe that morphisms in *Clo* are not required to be order-preserving. Therefore, objects (X, u) and (Y, v) may be isomorphic in *Clo* even if the partially ordered sets X and Y are not isomorphic.

Theorem 3.2. *Clo is a cartesian closed category.*

Proof. Clearly, given a family (X_i, u_i), $i \in I$ (I a set), of objects of *Clo*, the object $(\prod_{I \in I} X_i, u)$ with u the product of u_i (and $\prod_{I \in I} X_i$ the direct product of the partially ordered sets X_i, $i \in I$) is a product of the family in *Clo*. Closed elements in $(\prod_{I \in I} X_i, u)$ are precisely the elements $\varphi \in \prod_{i \in I} X_i$ with $\varphi(i)$ closed in (X_i, u_i) for every $i \in I$. For any two objects $A = (X, u)$ and $B = (Y, v)$ of *Clo*, put $B^A = (Y^X, w)$ where Y^X is the set of all maps of X into Y ordered point-wise and $w : Y^X \to Y^X$ is the map given by $w(f) = \bigwedge\{g \in Y^X;\ f \le g,\ g : (X, u) \to (Y, v) \text{ closed}\}$. For every $f \in Y^X$, the upper interval $[f)$ in Y^X is a complete meet-semilattice (because, for every subset $\mathcal{H} \subseteq [f)$, $(\bigwedge \mathcal{H})(x) = \bigwedge\{h(x);\ h \in \mathcal{H}\}$ where $\{h(x);\ h \in \mathcal{H}\} \subseteq [f(x))$ for every $x \in X$). Denoting by $\mathcal{G}$ the set of all closed maps of (X, u) into (Y, v), we get $(\bigwedge(\mathcal{G} \cap [f))(x)) = (\bigwedge\{g(x);\ g \in \mathcal{G} \cap [f)\}) \le \bigwedge\{(g(x));\ g \in \mathcal{G} \cap [f)\} = \bigwedge\{g(x);\ g \in \mathcal{G} \cap [f)\} = \bigwedge(\mathcal{G} \cap [f))(x)$. Consequently, $\bigwedge(\mathcal{G} \cap [f)) \in \mathcal{G}$ and, by Remark 3.1, w is a closure operator on Y^X such that the closed elements in (Y^X, w) are precisely the closed maps of (X, u) into (Y, v). We have shown that B^A is an object of *Clo*.

Obviously, the evaluation map $ev : A \times B^A \to B$ (given by $ev(x, f) = f(x)$ whenever $x \in X$ and $f \in Y^X$) is closed. If $C = (Z, p)$ is an object of *Clo* and $g : A \times C \to B$ is a closed map, then the map $g^* : C \to B^A$ given by $g^*(z)(x) = g(x, z)$ whenever $z \in Z$ and $x \in X$ is clearly closed too. Of course, $g^* : C \to B^A$ is a unique morphism in *Clo* such that $ev \circ (id_A \times g^*) = g$. This proves the statement. □

Remark 3.3. a) It is evident that the full subcategory Clo^* of *Clo* whose objects are the grounded objects of *Clo* is closed under both products and formation of power-objects in *Clo*. Therefore, Clo^* is cartesian closed too. But this is not true for the full subcategory of *Clo* given by its additive objects.

b) Of course, the category of closure spaces and the usual closed maps is not cartesian closed neither is the category of closure spaces with continuous maps nor is its full subcategory of grounded closure spaces. In [4], a cartesian closed topological hull of this subcategory is constructed.

Example 3.4. Let *CloSp* denote the category whose objects are the closure spaces and whose morphisms $(X,u) \to (Y,v)$ are the closed maps $(2^X,u) \to (2^Y,v)$ between the corresponding closure systems. Then, putting $H(X,u) = (2^X,u)$ for every object (X,u) of *CloSp* and $Hf = f$ for every morphism f in *CloSp*, we get a full embedding of *CloSp* into *Clo*. The full subcategory $H(CloSp)$ of *Clo* is closed under products in *Clo* (because $2^X \times 2^Y \cong 2^{X \sqcup Y}$) and also under the formation of power-objects (because $(2^Y)^{(2^X)} \cong 2^{2^X \times Y}$). Therefore, $H(CloSp)$ is cartesian closed and so is *CloSp*. Observe that products in *CloSp* are defined by $(X,u) \times (Y,v) = (X \sqcup Y, p)$ where $p : 2^{X \sqcup Y} \to 2^{X \sqcup Y}$ is given by $p(A) = u(A \cap X) \sqcup v(A \cap Y)$ whenever $A \subseteq X \sqcup Y$ (here, $\sqcup$ denotes coproduct, i.e., disjoint union, in *Set*). Power-objects in *CloSp* are given by $(Y,v)^{(X,u)} = (2^X \times Y, w)$ where $w(B) = \bigcap\{C \subseteq 2^X \times Y;\ B \subseteq C,\ A \subseteq X$ closed implies $\{y \in Y;\ (A,y) \in C\}$ closed$\}$ whenever $B \subseteq 2^X \times Y$.

Of course, among the full subcategories of the cartesian closed category *CloSp*, there is the one whose objects are exactly the topological spaces.

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