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On Randers metrics of isotropic S-curvature II

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Abstract. We have classified all Randers metrics of isotropic S-curvature when their Riemann metrics from navigation problem are locally conformally flat. In this paper, we study Randers metrics of isotropic S-curvature and constant S-curvature on product manifolds by navigation method, in which the corresponding Riemann product metrics from navigation problem are generally not locally conformally flat.

1. Introduction

In Finsler geometry, the S-curvature was originally introduced for the volume comparison theorem ([7]), and it is a non-Riemannian quantity, that is, $\mathbf{S} = \mathbf{0}$ for Riemannian metrics. Recent studies show that the S-curvature plays a very important role in Finsler geometry (cf. [2], [3], [5], [8], [10], [12]). For a Finsler manifold (M, F), the flag curvature $\mathbf{K} = \mathbf{K}(P, y)$ is an important Riemannian quantity, where $P \subset T_xM$ is a tangent plane and $y \in P$ is a non-zero vector. It is an analogue of the sectional curvature for a Riemann manifold. A Finsler metric F is of *scalar flag curvature* if $\mathbf{K} = \mathbf{K}(x, y)$ is independent of P for any non-zero vector $y \in P$. F is of *constant flag curvature* if $\mathbf{K} =$ constant. The flag curvature and the S-curvature are closely related ([3]).

A Randers metric is a Finsler metric in the form $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is Riemannian and $\beta = b_i(x)y^i$ is a 1-form with $||\beta||_\alpha < 1$. A Randers metric $F = \alpha + \beta$ naturally arises from the navigation problem on a Rieman manifold (M, h) under an external force field $W = W^i \frac{\partial}{\partial x^i}$. Then F can be

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expressed as ([9])

$$F = \frac{\sqrt{h(x,y)^2 - \left[h(x,W_x)^2 h(x,y)^2 - \langle y, W_x\rangle_h^2\right]}}{1 - h(x,W_x)^2} - \frac{\langle y, W_x\rangle_h}{1 - h(x,W_x)^2}. \tag{1}$$

According to [4] and [11], a Randers metric F is of isotropic S-curvature $\mathbf{S} = (n+1)c(x)F$ if and only if

$$W_{i,j} + W_{j,i} = -4c(x)h_{ij}, \tag{2}$$

where the covariant derivative $W_{i,j}$ is taken with respect to h, and $W_i := h_{ir}W^r$.

It is proved that a Randers metric of isotropic S-curvature is of scalar flag curvature if and only if the Riemann metric h is of isotropic sectional curvature ([5]). This leads to the classification of Randers metrics of isotropic S-curvature and scalar flag curvature for $n > 2$ ([5], [10]). On the other hand, it is known that a Randers metric F is of constant flag curvature if and only if F is of constant S-curvature and h is of constant sectional curvature, which gives the classification of Randers metrics of constant flag curvature ([2]).

Further the present author in [12] classifies all Randers metrics of isotropic S-curvature when their h's are locally conformally flat. In particular, all two-dimensional Randers metrics of isotropic S-curvature are classified. The class in [12] (the dimensions $n \geq 2$) includes all Randers metrics of scalar flag curvature and isotropic S-curvature.

Then a natural question arises: are there any Randers metrics of isotropic S-curvature if the Riemann metric h is not locally conformally flat? Our main results are the following Theorem 1.1 and Theorem 1.2

Before stating our main results, we make some conventions in the following discussions. Let M and $\widetilde{M}$ be two manifolds with the dimensions m and $\widetilde{m}$ respectively. Let $n = m + \widetilde{m}$ and $\{(x, \widetilde{x})\}$ be a local coordinate on the product manifold $M \times \widetilde{M}$. In a tensor, the indices i, j, k run from 1 to m, and the indices ξ, η, γ run from $m+1$ to n.

Theorem 1.1. *Let $F = \alpha + \beta$ be an n-dimensional Randers metric determined by the navigation data (h, W) on a product manifold $M \times \widetilde{M}$ with $h = \sqrt{H^2 + \widetilde{H}^2}$, where (M, H) and $(\widetilde{M}, \widetilde{H})$ are locally conformally flat Riemann manifolds, and $W = W(x, \widetilde{x})$ is a vector field on $M \times \widetilde{M}$. Locally put $H_{ij}(x) = e^{\sigma(x)}\delta_{ij}$ and $\widetilde{H}_{\xi\eta}(\widetilde{x}) = e^{\widetilde{\sigma}(\widetilde{x})}\delta_{\xi\eta}$. Then we have*

(i) *($m \geq 2$, $\widetilde{m} \geq 2$) F is of isotropic S-curvature if and only if the following hold (for arbitrarily fixed i, j and ξ, η):*

$$\frac{\partial W^i}{\partial x^j} + \frac{\partial W^j}{\partial x^i} = 0 \quad (\forall\, i \neq j), \qquad\qquad \frac{\partial W^i}{\partial x^i} = \frac{\partial W^j}{\partial x^j} \quad (\forall\, i, j), \tag{3}$$

$$\frac{\partial W^\xi}{\partial \widetilde{x}^\eta} + \frac{\partial W^\eta}{\partial \widetilde{x}^\xi} = 0 \quad (\forall\, \xi \neq \eta), \qquad \frac{\partial W^\xi}{\partial \widetilde{x}^\xi} = \frac{\partial W^\eta}{\partial \widetilde{x}^\eta} \quad (\forall\, \xi, \eta), \tag{4}$$

$$e^\sigma \frac{\partial W^i}{\partial \widetilde{x}^\xi} + e^{\widetilde{\sigma}} \frac{\partial W^\xi}{\partial x^i} = 0, \qquad 2\tau + W^k \sigma_k = 2\widetilde{\tau} + W^\xi \widetilde{\sigma}_\xi, \tag{5}$$

where $\sigma_k = \frac{\partial \sigma}{\partial x^k}, \widetilde{\sigma}_\xi = \frac{\partial \widetilde{\sigma}}{\partial \widetilde{x}^\xi}$, $\tau = \tau(x, \widetilde{x}) = \frac{\partial W^i}{\partial x^i}$ *(not summed) and* $\widetilde{\tau} = \widetilde{\tau}(x, \widetilde{x}) = \frac{\partial W^\xi}{\partial \widetilde{x}^\xi}$ *(not summed). In this case, the isotropic S-curvature* $\mathbf{S} = (n+1)c(x, \widetilde{x})F$ *is determined by*

$$c(x, \widetilde{x}) = -\frac{1}{4}\big[2\tau(x, \widetilde{x}) + W^k \sigma_k\big] = -\frac{1}{4}\big[2\widetilde{\tau}(\widetilde{x}, x) + W^\lambda \widetilde{\sigma}_\lambda\big], \tag{6}$$

(ii) $(m \geq 3,\ \widetilde{m} \geq 3)$ *F is of isotropic S-curvature if and only if*

$$W^i = -2\big(\lambda(\widetilde{x}) + \langle a(\widetilde{x}), x\rangle\big)x^i + |x|^2 a^i(\widetilde{x}) + q^i_k(\widetilde{x})x^k + b^i(\widetilde{x}), \tag{7}$$

$$W^\xi = -2\big(\widetilde{\lambda}(x) + \langle \widetilde{a}(x), \widetilde{x}\rangle\big)\widetilde{x}^\xi + |\widetilde{x}|^2 \widetilde{a}^\xi(x) + \widetilde{q}^\xi_\gamma(x)\widetilde{x}^\gamma + \widetilde{b}^\xi(x), \tag{8}$$

where $\lambda(\widetilde{x})$ *and* $\widetilde{\lambda}(x)$ *are scalar functions,* $a(\widetilde{x})$, $b(\widetilde{x})$, $\widetilde{a}(x)$ *and* $\widetilde{b}(x)$ *are vectors and the matrices* $(q^j_i(\widetilde{x}))$ *and* $(\widetilde{q}^\xi_\gamma(x))$ *are skew-symmetric, and these functions satisfy* (5). *In this case, the isotropic S-curvature* $\mathbf{S} = (n+1)c(x, \widetilde{x})F$ *is determined by* (6) *with* $c(x, \widetilde{x})$ *given by*

$$c(x, \widetilde{x}) = \lambda(\widetilde{x}) + \langle a(\widetilde{x}), x\rangle - \frac{1}{4}W^k \sigma_k = \widetilde{\lambda}(x) + \langle \widetilde{a}(x), \widetilde{x}\rangle - \frac{1}{4}W^\xi \widetilde{\sigma}_\xi. \tag{9}$$

Theorem 1.1 is similar to Theorem 1.1 in [12], and their proofs are also similar.

In general, the product of two locally conformally flat Riemann metrics are not locally conformally flat (Lemma 2.1). So by Theorem 1.1, it is possible to construct Randers metrics of isotropic S-curvature with h being not locally conformally flat. We will show such class of Randers metrics in the following Theorem 1.2 by assuming H and $\widetilde{H}$ are of constant sectional curvature.

In Theorem 1.1, if the vector field W satisfies $W^i = W^i(x)$, $W^\xi = W^\xi(\widetilde{x})$, then the S-curvature must be a constant, that is, $c(x, \widetilde{x}) =$ constant in (6). In Section 5, we will construct, in every dimension $n \geq 3$, examples of constant S-curvature with the corresponding Riemann metrics h's being not locally conformally flat.

Theorem 1.2. *Let $F = \alpha + \beta$ be an n-dimensional Randers metric determined by the navigation data (h, W) on a product manifold $M \times \widetilde{M}$ with*

$h=\sqrt{H^2+\widetilde{H}^2}$, *where* (M,H) *and* $(\widetilde{M},\widetilde{H})$ *are two Riemann manifolds of constant sectional curvatures* μ_1 *and* μ_2 *respectively, and* $W=W(x,\widetilde{x})$ *is a vector field on* $M\times\widetilde{M}$. *Locally put* $H_{ij}(x)=e^{\sigma(x)}\delta_{ij}$ *and* $\widetilde{H}_{\xi\eta}(\widetilde{x})=e^{\widetilde{\sigma}(\widetilde{x})}\delta_{\xi\eta}$. *with*

$$\sigma(x)=ln\frac{4}{(1+\mu_1|x|^2)^2},\qquad \widetilde{\sigma}(\widetilde{x})=ln\frac{4}{(1+\mu_2|\widetilde{x}|^2)^2}. \tag{10}$$

Suppose $m\ge 3,\widetilde{m}\ge 3$. *Then* F *is of isotropic S-curvature if and only if one of the following cases holds:*

(i) $\mu_1=\mu_2=0$, *and*

$$W^p=-2(\lambda+\langle a,u\rangle)u^p+|u|^2a^p+q^p_ru^r+b^p, \tag{11}$$

where $u:=(x,\widetilde{x})$, λ *is a constant number,* $a\in\mathbf{R}^n$ *and* $b\in\mathbf{R}^n$ *are constant vectors and* (q^p_r) *is a constant skew-symmetric matrix* $(1\le p,\ r\le n)$. *In this case, the isotropic S-curvature* $\mathbf{S}=(n+1)c(u)F$ *is given by*

$$c(u)=\lambda+\langle a,u\rangle. \tag{12}$$

(ii) $\mu:=\mu_1=-\mu_2\ne 0$, *and*

$$W^i=-2\Big\{\frac{\kappa(1+\mu|\widetilde{x}|^2)}{1-\mu|\widetilde{x}|^2}-\mu\langle\theta,x\rangle\Big\}x^i+(1-\mu|x|^2)\theta^i+q^i_kx^k, \tag{13}$$

$$W^\xi=-2\Big\{\frac{\kappa(1-\mu|x|^2)}{1+\mu|x|^2}+\mu\langle\widetilde{\theta},\widetilde{x}\rangle\Big\}\widetilde{x}^\xi+(1+\mu|\widetilde{x}|^2)\widetilde{\theta}^\xi+\widetilde{q}^\xi_\gamma\widetilde{x}^\gamma, \tag{14}$$

where κ *is a constant number,* $\theta\in\mathbf{R}^m$ *and* $\widetilde{\theta}\in\mathbf{R}^{\widetilde{m}}$ *are constant vectors, and* (q^i_k) *and* $(\widetilde{q}^\xi_\gamma)$ *are constant skew-symmetric matrices. In this case, the isotropic S-curvature* $\mathbf{S}=(n+1)c(x,\widetilde{x})F$ *is given by*

$$c(x,\widetilde{x})=\frac{\kappa(1-\mu|x|^2)(1+\mu|\widetilde{x}|^2)}{(1+\mu|x|^2)(1-\mu|\widetilde{x}|^2)}. \tag{15}$$

(iii) $\mu:=\mu_1=-\mu_2/2\ne 0$ *(or similarly* $\mu:=\mu_2=-\mu_1/2\ne 0$*), and*

$$W^i=-2\Big\{\frac{\langle T,\widetilde{x}\rangle}{(1-2\mu|\widetilde{x}|^2)}-\mu\langle\theta,x\rangle\Big\}x^i+(1-\mu|x|^2)\theta^i+q^i_kx^k, \tag{16}$$

$$\begin{aligned}W^\xi&=-2\langle 2\mu\widetilde{\theta}-T,\widetilde{x}\rangle\widetilde{x}^\xi-\frac{1+\mu(1+\mu|x|^2)|\widetilde{x}|^2}{\mu(1+\mu|x|^2)}T^\xi\\&\quad+(1+2\mu|\widetilde{x}|^2)\widetilde{\theta}^\xi+\widetilde{q}^\xi_\gamma\widetilde{x}^\gamma,\end{aligned} \tag{17}$$

where $T \in \mathbf{R}^{\widetilde{m}}$ *is a constant vector, and the meanings of* $\theta, \widetilde{\theta}, (q^i_k), (\widetilde{q}^{\xi}_{\gamma})$ *are referred to case* (ib). *In this case, the isotropic S-curvature* $\mathbf{S} = (n+1)c(x, \widetilde{x})F$ *is given by*

$$c(x, \widetilde{x}) = \frac{(1-\mu|x|^2)\langle T, \widetilde{x}\rangle}{(1+\mu|x|^2)(1-2\mu|\widetilde{x}|^2)}. \tag{18}$$

(iv) $\mu_1 \neq -\mu_2$, $\mu_1 \neq -\mu_2/2$, $\mu_1 \neq -2\mu_2$, $\mu_1 \neq 0$, $\mu_2 \neq 0$, *and*

$$W^i = 2\mu_1\langle\theta, x\rangle x^i + (1-\mu_1|x|^2)\theta^i + q^i_k x^k, \tag{19}$$

$$W^{\xi} = 2\mu_2\langle\widetilde{\theta}, \widetilde{x}\rangle\widetilde{x}^{\xi} + (1-\mu_2|\widetilde{x}|^2)\widetilde{\theta}^{\xi} + \widetilde{q}^{\xi}_{\gamma}\widetilde{x}^{\gamma}, \tag{20}$$

where the meanings of θ, $\widetilde{\theta}$, (q^i_k), $(\widetilde{q}^{\xi}_{\gamma})$ *are referred to case* (ib). *In this case, the isotropic S-curvature* $\mathbf{S} = 0$.

(v) $\mu_1 = 0$, $\mu_2 \neq 0$ *(or similarly* $\mu_2 = 0$, $\mu_1 \neq 0$*), and*

$$W^i = q^i_k x^k + \theta^i, \tag{21}$$

$$W^{\xi} = 2\mu_2\langle\widetilde{\omega}, \widetilde{x}\rangle\widetilde{x}^{\xi} + (1-\mu_2|\widetilde{x}|^2)\widetilde{\omega}^{\xi} + \widetilde{q}^{\xi}_{\gamma}\widetilde{x}^{\gamma}, \tag{22}$$

where the meanings of $\theta, (q^i_k), (\widetilde{q}^{\xi}_{\gamma})$ *are referred to case* (ib), *and* $\widetilde{\omega} \in \mathbf{R}^{\widetilde{m}}$ *are constant vectors. In this case, the isotropic S-curvature* $\mathbf{S} = 0$.

Theorem 1.2(i) was obtained in [10], since the Riemann metric h is flat when $\mu_1 = \mu_2 = 0$. By Lemma 2.1(iii), the Riemann metrics h's in cases (iii), (iv) and (v) are not locally conformally flat.

In Theorem 1.2, if $\mu_1 = \mu_2 \neq 0$, then h is not of constant sectional curvature, but it is an Einstein metric. So for $m \geq 2$ and $\widetilde{m} \geq 2$, if $\mu_1 = \mu_2 \neq 0$ in Theorem 1.2, then the Randers metrics are not of scalar flag curvature by [5], but they are Einstein metrics by [1] and [6].

2. Preliminaries

Let F be a Finsler metric on an n-dimensional manifold M with (x^i, y^i) the standard local coordinate system in TM, then for any $y \neq 0$, the matrix $g_{ij} := \frac{1}{2}[F^2]_{y^i y^j}$ is positive definite in our consideration. The Hausdorff–Busemann volume form $dV = \sigma_F(x)dx^1 \wedge \ldots \wedge dx^n$ is defined by

$$\sigma_F(x) := \frac{\mathrm{Vol}(B^n)}{\mathrm{Vol}\left\{(y^i) \in R^n | F(y^i \frac{\partial}{\partial x^i}|_x) < 1\right\}}.$$

The Finsler metric F induces a vector field $G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ on TM defined by

$$G^i = \frac{1}{4} g^{il} \big\{ [F^2]_{x^k y^l}(x, y) y^k - [F^2]_{x^l}(x, y) \big\}.$$

Then the S-curvature is defined by

$$\mathbf{S} := \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m}(ln\sigma_F).$$

S is said to be *isotropic* if there is a scalar function $c(x)$ on M such that

$$\mathbf{S} = (n+1)c(x)F.$$

If $c(x)$ is a constant, then we call F is of *constant S-curvature.*

For the discussions in this paper, we need the following result about the direct product of two Riemann metrics on a product manifold.

Lemma 2.1. *Let $(h, M \times \widetilde{M})$ be an n-dimensional Riemann manifold with $h := \sqrt{H^2 + \widetilde{H}^2}$, where (H, M) and $(\widetilde{H}, \widetilde{M})$ are two Riemann manifolds with the dimensions $m(\geq 2)$ and $\widetilde{m}(\geq 1)$ respectively. Suppose (H, M) is locally conformally flat.*

(i) *If h is locally conformally flat, then H is of isotropic sectional curvature $\mu(x)$, and*

$$\widetilde{\rho}(\widetilde{x}) = -\widetilde{m}(\widetilde{m} - 1)\mu(x), \tag{23}$$

where $\widetilde{\rho}(\widetilde{x})$ is the scalar curvature of $\widetilde{H}$. If further $\widetilde{m} \geq 2$, then $\mu(x)$ and $\widetilde{\rho}(\widetilde{x})$ are constant.

(ii) *If $\widetilde{m} = 1$, then h is locally conformally flat if and only if H is of constant sectional curvature.*

(iii) *If $\widetilde{m} \geq 2$ and $\widetilde{H}$ is locally conformally flat, then h is locally conformally flat if and only if H and $\widetilde{H}$ are of constant sectional curvatures μ and $\widetilde{\mu}$ respectively satisfying $\mu + \widetilde{\mu} = 0$.*

Proof. (i) Assume H and h are locally conformally flat. If $m \geq 3$, then for $1 \leq i, j, k, l \leq m$, we respectively have

$$C_{ijk}^{\ l} = R_i{}^l{}_{jk} + \delta_k^l L_{ij} + H_{ij} L_k^l - \delta_j^l L_{ik} - H_{ik} L_j^l = 0, \tag{24}$$

$$\widehat{C}_{ijk}^{\ l} = R_i{}^l{}_{jk} + \delta_k^l \widehat{L}_{ij} + H_{ij} \widehat{L}_k^l - \delta_j^l \widehat{L}_{ik} - H_{ik} \widehat{L}_j^l = 0, \tag{25}$$

where C and $\widehat{C}$ are the Weyl conformal curvature tensors of H and h respectively, $R_i{}^l{}_{jk}$ is the curvature tensor of h ($R_{ij} = R_i{}^r{}_{rj}$), and

$$L_{ij} = \frac{1}{m-2}R_{ij} - \frac{\rho}{2(m-1)(m-2)}H_{ij}, \quad L_j^k = H^{lk}L_{jl}, \quad \rho = H^{ij}R_{ij},$$

$$\widehat{L}_{ij} = \frac{1}{m+\widetilde{m}-2}R_{ij} - \frac{\rho+\widetilde{\rho}}{2(m+\widetilde{m}-1)(m+\widetilde{m}-2)}H_{ij}, \quad \widehat{L}_j^k = H^{lk}\widehat{L}_{jl}.$$

By (24) and (25) we have

$$\delta_k^l L_{ij} + H_{ij}L_k^l - \delta_j^l L_{ik} - H_{ik}L_j^l = \delta_k^l \widehat{L}_{ij} + H_{ij}\widehat{L}_k^l - \delta_j^l \widehat{L}_{ik} - H_{ik}\widehat{L}_j^l. \tag{26}$$

Contracting (26) in k and l gives

$$R_{ij} = \frac{\widetilde{m}\rho - (m-1)\widetilde{\rho}}{\widetilde{m}(m+\widetilde{m}-1)}H_{ij}. \tag{27}$$

By (27), H is an Einstein metric. Thus H is of constant sectional curvature μ since H is locally conformally flat.

We further prove that, in case of $m \geq 2$, if H is of sectional curvature $\mu(x)$, then (25) is equivalent to (23). Suppose H is of sectional curvature $\mu(x)$, then we have

$$R_i{}^l{}_{jk} = \mu(H_{ik}\delta_j^l - H_{ij}\delta_k^l), \quad R_{ij} = (m-1)\mu H_{ij}, \qquad \rho = m(m-1)\mu.$$

Therefore we have

$$\widehat{C}_{ijk}^{\ l} = \big[\widetilde{m}(\widetilde{m}-1)\mu + \widetilde{\rho}\big](H_{ik}\delta_j^l - H_{ij}\delta_k^l).$$

Thus (25) is equivalent to (23).

(iii) Since $\widetilde{m} \geq 2$ and H and $\widetilde{H}$ are locally conformally flat, by case (i), H and $\widetilde{H}$ are of constant sectional curvatures μ and $\widetilde{\mu}$ respectively, provided that h is locally conformally flat. Then (23) implies that $\mu + \widetilde{\mu} = 0$. The converse is easily verified by the last conclusion in the proof of case (i).

(ii) Suppose $\widetilde{m} = 1$ and h is locally conformally flat. If $m \geq 3$, then by case (i) H is of constant sectional curvature. If $m = 2$, then H is of sectional curvature $\mu(x)$. Since h is locally conformally flat, we have

$$\widehat{L}_{ij,k} = \widehat{L}_{ik,j}, \tag{28}$$

where the derivative "$_{,k}$" is taken with respect to h. Now it is easily seen that $\widehat{L}_{ij} = \frac{1}{2}\mu(x)H_{ij}$. Plug this into (28) and we get $\mu_{,k}H_{ij} = \mu_{,j}H_{ik}$, which implies that $\mu(x) = $ constant. □

3. Proof of Theorem 1.1

This proof is similar to that of a theorem in [12].

(i) First suppose that $F=\alpha+\beta$ is of isotropic S-curvature $\mathbf{S}=(n+1)c(x,\widetilde{x})F$. Let Γ^t_{rs} be the Levi–Civita connection coefficients of the Rieman metric h, where the indices r, s, t run from 1 to $n(=m+\widetilde{m})$. It is easily seen that $\Gamma^t_{rs}=0$ if among the indices r, s, t, one runs from 1 to m and another runs from $m+1$ to n. In other cases we have

$$\Gamma^k_{ij}(x)=\frac{1}{2}(\sigma_i\delta^k_j+\sigma_j\delta^k_i-\sigma_k\delta^j_i),\quad \left(\sigma_i=\frac{\partial\sigma(x)}{\partial x^i}\right),$$

$$\Gamma^\gamma_{\xi\eta}(\widetilde{x})=\frac{1}{2}(\widetilde{\sigma}_\xi\delta^\gamma_\eta+\widetilde{\sigma}_\eta\delta^\gamma_\xi-\widetilde{\sigma}_\gamma\delta^\eta_\xi),\quad \left(\widetilde{\sigma}_\xi=\frac{\partial\widetilde{\sigma}(\widetilde{x})}{\partial\widetilde{x}^\xi}\right).$$

We see that (2) is equivalent to

$$\frac{\partial W_i}{\partial x^j}+\frac{\partial W_j}{\partial x^i}-(\sigma_iW_j+\sigma_jW_i-W_k\sigma_k\delta_{ij})=-4c(x,\widetilde{x})e^\sigma\delta_{ij},\tag{29}$$

$$\frac{\partial W_\xi}{\partial \widetilde{x}^\eta}+\frac{\partial W_\eta}{\partial \widetilde{x}^\xi}-(\widetilde{\sigma}_\xi W_\eta+\widetilde{\sigma}_\eta W_\xi-W_\gamma\widetilde{\sigma}_\gamma\delta_{\xi\eta})=-4c(x,\widetilde{x})e^{\widetilde{\sigma}}\delta_{\xi\eta},\tag{30}$$

$$\frac{\partial W_i}{\partial \widetilde{x}^\xi}+\frac{\partial W_\xi}{\partial x^i}=0.\tag{31}$$

Note that $W^i=e^{-\sigma}W_i, W^\xi=e^{-\widetilde{\sigma}}W_\xi$. Then it is easy to verify that (29), (30) and (31) are equivalent respectively to the following

$$\frac{\partial W^i}{\partial x^j}+\frac{\partial W^j}{\partial x^i}=-(W^k\sigma_k+4c(x,\widetilde{x}))\delta_{ij},\tag{32}$$

$$\frac{\partial W^\xi}{\partial \widetilde{x}^\eta}+\frac{\partial W^\eta}{\partial \widetilde{x}^\xi}=-(W^\gamma\widetilde{\sigma}_\gamma+4c(x,\widetilde{x}))\delta_{\xi\eta},\tag{33}$$

$$e^\sigma\frac{\partial W^i}{\partial \widetilde{x}^\xi}+e^{\widetilde{\sigma}}\frac{\partial W^\xi}{\partial x^i}=0.\tag{34}$$

So by (32), (33) and (34) and for arbitrarily fixed i, j and ξ, η we have

$$\frac{\partial W^i}{\partial x^j}+\frac{\partial W^j}{\partial x^i}=0\quad(\forall\, i\neq j),\qquad \frac{\partial W^i}{\partial x^i}=\frac{\partial W^j}{\partial x^j}\quad(\forall\, i,j),$$

$$\frac{\partial W^\xi}{\partial \widetilde{x}^\eta}+\frac{\partial W^\eta}{\partial \widetilde{x}^\xi}=0\quad(\forall\, \xi\neq\eta),\qquad \frac{\partial W^\xi}{\partial \widetilde{x}^\xi}=\frac{\partial W^\eta}{\partial \widetilde{x}^\eta}\quad(\forall\, \xi,\eta),$$

$$e^\sigma\frac{\partial W^i}{\partial \widetilde{x}^\xi}+e^{\widetilde{\sigma}}\frac{\partial W^\xi}{\partial x^i}=0.$$

By (32) and (33) we get

$$2\tau + W^k\sigma_k = 2\widetilde{\tau} + W^\gamma\widetilde{\sigma}_\gamma.$$

Thus we obtain (3), (4) and (5).

Conversely, suppose that the vector field W satisfies (3), (4) and (5). Then by (3) and (4), there are two scalar functions $\tau(x,\widetilde{x})$ and $\widetilde{\tau}(x,\widetilde{x})$ defined by

$$\tau(x,\widetilde{x}) := \frac{\partial W^i}{\partial x^i}, \qquad \widetilde{\tau}(x,\widetilde{x}) := \frac{\partial W^\xi}{\partial \widetilde{x}^\xi},$$

where the indices i,ξ are arbitrarily fixed. Now by (5), we define a scalar function

$$c(x,\widetilde{x}) := -\frac{1}{4}\big[2\tau + W^k\sigma_k\big] = -\frac{1}{4}\big[2\widetilde{\tau} + W^\gamma\widetilde{\sigma}_\gamma\big].$$

It is easy to verify that for arbitrary indices i, j and ξ, η, the formulas (32), (33) and (34) hold. Thus (2) holds. This shows that the Randers metric F is of isotropic S-curvature.

By the proof above, we have seen that the isotropic S-curvature $\mathbf{S} = (n+1)c(x,\widetilde{x})F$ is given by (6).

(ii) For arbitrarily fixed $\widetilde{x}$, let $F_1 = F_1(\widetilde{x})$ be a Randers metric determined by the navigation data $(H, \{W^i\})$ and for arbitrarily fixed x, let $F_2 = F_2(x)$ be another Randers metric determined by the navigation data $(\widetilde{H}, \{W^\xi\})$. Then by (3), (4) and Theorem 1.1 in [12], we see that F_1 is of isotropic S-curvature $\mathbf{S} = (m+1)c(x,\widetilde{x})F_1$ and F_2 is also of isotropic S-curvature $\mathbf{S} = (\widetilde{m}+1)c(x,\widetilde{x})F_2$. Then by Theorem 1.1 in [12], we obtain the expressions of W^i, W^ξ given by (7), (8) in case of $m \geq 3$, $\widetilde{m} \geq 3$. Finally by the item (i), the functions in (7) and (8) satisfy (5), and this establishes (9). $\square$

4. Proof of Theorem 1.2

To prove Theorem 1.2, we first state the following lemma, whose proof is elementary.

Lemma 4.1. *Define $f : \mathbf{R}^m \mapsto \mathbf{R}^{\widetilde{m}}$ and $g : \mathbf{R}^{\widetilde{m}} \mapsto \mathbf{R}^m$ which satisfy*

$$\langle f(x), \widetilde{x}\rangle = \langle g(\widetilde{x}), x\rangle.$$

Then $f(x)$ and $g(\widetilde{x})$ can be expressed as

$$f(x) = Ax, \quad g(\widetilde{x}) = B\widetilde{x},$$

where A, B are two matrices with $B = transpose(A)$.

Now we begin to prove Theorem 1.2.

Assume that F is of isotropic S-curvature. Since $m \geq 3$, $\widetilde{m} \geq 3$, by Theorem 1.1(ii), the vector field W is given by (7) and (8), which satisfies (5). In the following we will use (5) to determine μ_1, μ_2, $\lambda(\widetilde{x})$, $\widetilde{\lambda}(x)$, $a(\widetilde{x})$, $\widetilde{a}(x)$, $q^i_k(\widetilde{x})$, $\widetilde{q}^\gamma_\xi(x)$, $b(\widetilde{x})$ and $\widetilde{b}(x)$ in (7) and (8).

Now plug (7), (8) and (10) into (9) or the second formula of (5), and we get

$$c(x,\widetilde{x}) = \frac{\lambda(\widetilde{x})(1-\mu_1|x|^2) + \langle \rho(\widetilde{x}), x\rangle}{1+\mu_1|x|^2} = \frac{\widetilde{\lambda}(x)(1-\mu_2|\widetilde{x}|^2) + \langle \widetilde{\rho}(x), \widetilde{x}\rangle}{1+\mu_2|\widetilde{x}|^2}, \qquad (35)$$

where

$$\rho(\widetilde{x}) := \mu_1 b(\widetilde{x}) + a(\widetilde{x}), \qquad \widetilde{\rho}(x) := \mu_2 \widetilde{b}(x) + \widetilde{a}(x).$$

Put $x = 0$, or $\widetilde{x} = 0$ in (35) and we respectively obtain

$$\lambda(\widetilde{x}) = \frac{\kappa(1-\mu_2|\widetilde{x}|^2) + \langle \widetilde{T}, \widetilde{x}\rangle}{1+\mu_2|\widetilde{x}|^2}, \qquad \widetilde{\lambda}(x) = \frac{\kappa(1-\mu_1|x|^2) + \langle T, x\rangle}{1+\mu_1|x|^2}, \qquad (36)$$

where $\kappa := \lambda(0) = \widetilde{\lambda}(0)$ is a constant number, and $\widetilde{T} := \widetilde{\rho}(0)$, $T := \rho(0)$ are constant vectors. Then plug (36) into (35) and we obtain

$$\langle \widetilde{f}(x), \widetilde{x}\rangle = \langle f(\widetilde{x}), x\rangle, \qquad (37)$$

where

$$\widetilde{f}(x) := (1+\mu_1|x|^2)\widetilde{\rho}(x) - (1-\mu_1|x|^2)\widetilde{T}, \qquad (38)$$

$$f(\widetilde{x}) := (1+\mu_2|\widetilde{x}|^2)\rho(\widetilde{x}) - (1-\mu_2|\widetilde{x}|^2)T. \qquad (39)$$

Then by (37) and Lemma 4.1, we have

$$f(\widetilde{x}) = \widetilde{A}\widetilde{x}, \qquad \widetilde{f}(x) = Ax,$$

where $A = (A^\gamma_i)$, $\widetilde{A} = (\widetilde{A}^i_\gamma)$ are two matrices with $\widetilde{A} = \text{transpose}(A)$. Again by (38) and (39) we get

$$\rho(\widetilde{x}) = \frac{(1-\mu_2|\widetilde{x}|^2)T + \widetilde{A}\widetilde{x}}{1+\mu_2|\widetilde{x}|^2}, \qquad \widetilde{\rho}(x) = \frac{(1-\mu_1|x|^2)\widetilde{T} + Ax}{1+\mu_1|x|^2}. \qquad (40)$$

By (35), (36) and (40), the isotropic S-curvature $\mathbf{S} = (n+1)c(x,\widetilde{x})F$ is determined by

$$c(x,\widetilde{x}) = \frac{Y_1}{Y_2}, \qquad (41)$$

where we define

$$Y_1 := \kappa(1-\mu_1|x|^2)(1-\mu_2|\widetilde{x}|^2)+(1-\mu_1|x|^2)\langle\widetilde{T},\widetilde{x}\rangle+(1-\mu_2|\widetilde{x}|^2)\langle T,x\rangle+\langle Ax,\widetilde{x}\rangle,$$

$$Y_2 := (1+\mu_1|x|^2)(1+\mu_2|\widetilde{x}|^2).$$

By (40), we have

$$\begin{aligned} a(\widetilde{x}) &= \frac{(1-\mu_2|\widetilde{x}|^2)T+\widetilde{A}\widetilde{x}}{1+\mu_2|\widetilde{x}|^2}-\mu_1 b(\widetilde{x}),\\ \widetilde{a}(x) &= \frac{(1-\mu_1|x|^2)\widetilde{T}+Ax}{1+\mu_1|x|^2}-\mu_2\widetilde{b}(x). \end{aligned} \tag{42}$$

By (36), we have

$$\frac{\partial\lambda}{\partial\widetilde{x}^\xi} = \frac{-2\mu_2(2\kappa+\langle\widetilde{T},\widetilde{x}\rangle)\widetilde{x}^\xi+(1+\mu_2|\widetilde{x}|^2)\widetilde{T}^a}{(1+\mu_2|\widetilde{x}|^2)^2}, \tag{43}$$

$$\frac{\partial\widetilde{\lambda}}{\partial x^i} = \frac{-2\mu_1(2\kappa+\langle T,x\rangle)x^i+(1+\mu_1|x|^2)T^i}{(1+\mu_1|x|^2)^2}. \tag{44}$$

By (42), we have

$$\frac{\partial a^i}{\partial\widetilde{x}^\xi} = \frac{-2\mu_2(2T^i+\widetilde{A}^i_\gamma\widetilde{x}^\gamma)\widetilde{x}^\xi+\widetilde{A}^i_\xi(1+\mu_2|\widetilde{x}|^2)}{(1+\mu_2|\widetilde{x}|^2)^2}-\mu_1\frac{b^i}{\widetilde{x}^\xi}, \tag{45}$$

$$\frac{\partial\widetilde{a}^\xi}{\partial x^i} = \frac{-2\mu_1(2\widetilde{T}^\xi+A^\xi_k x^k)x^i+A^\xi_i(1+\mu_1|x|^2)}{(1+\mu_1|x|^2)^2}-\mu_2\frac{\widetilde{b}^\xi}{x^i}. \tag{46}$$

By (7) and (8) we have

$$\frac{\partial W^i}{\partial\widetilde{x}^\xi} = -2\Big(\frac{\partial\lambda}{\partial\widetilde{x}^\xi}+\langle\frac{\partial a}{\partial\widetilde{x}^\xi},x\rangle\Big)x^i+|x|^2\frac{\partial a^i}{\partial\widetilde{x}^\xi}+\frac{\partial q^i_k}{\partial\widetilde{x}^\xi}x^k+\frac{\partial b^i}{\partial\widetilde{x}^\xi}, \tag{47}$$

$$\frac{\partial W^\xi}{\partial x^i} = -2\Big(\frac{\partial\widetilde{\lambda}}{\partial x^i}+\langle\frac{\partial\widetilde{a}}{\partial x^i},\widetilde{x}\rangle\Big)\widetilde{x}^\xi+|\widetilde{x}|^2\frac{\partial\widetilde{a}^\xi}{\partial x^i}+\frac{\partial\widetilde{q}^\xi_\gamma}{\partial x^i}\widetilde{x}^\gamma+\frac{\partial\widetilde{b}^\xi}{\partial x^i}. \tag{48}$$

Now plug (43)–(48) into the first formula of (5) and we can get

$$\begin{aligned} &-\Big\{(1+\mu_2|\widetilde{x}|^2)^2\frac{\partial q^i_k}{\partial\widetilde{x}^\xi}x^k+(1+\mu_1|x|^2)^2\frac{\partial\widetilde{q}^\xi_\gamma}{\partial x^i}\widetilde{x}^\gamma\Big\}\\ &-\Big\{(1-\mu_1|x|^2)(1+\mu_2|\widetilde{x}|^2)^2\frac{\partial b^i}{\partial\widetilde{x}^\xi}+(1-\mu_2|\widetilde{x}|^2)(1+\mu_1|x|^2)^2\frac{\partial\widetilde{b}^\xi}{\partial x^i}\Big\} \end{aligned}$$

$$
\begin{aligned}
&-\left\{2(1+\mu_2|\widetilde{x}|^2)^2\mu_1\frac{\partial b^k}{\partial\widetilde{x}^\xi}x^kx^i+2(1+\mu_1|x|^2)^2\mu_2\frac{\partial\widetilde{b}^\gamma}{\partial x^i}\widetilde{x}^\gamma\widetilde{x}^\xi\right\}\\
&=\big\{8(\mu_1+\mu_2)\kappa+4(2\mu_1+\mu_2)\langle\widetilde{T},\widetilde{x}\rangle\\
&\quad+4(\mu_1+2\mu_2)\langle T,x\rangle+4(\mu_1+\mu_2)\langle Ax,\widetilde{x}\rangle\big\}x^i\widetilde{x}^\xi\\
&\quad-2\big\{1+(2\mu_1+\mu_2)|\widetilde{x}|^2\big\}\widetilde{T}^\xi x^i-2\big\{1+(\mu_1+2\mu_2)|x|^2\big\}T^i\widetilde{x}^\xi\\
&\quad-2\big\{1+(\mu_1+\mu_2)|\widetilde{x}|^2\big\}A^\xi_kx^kx^i-2\big\{1+(\mu_1+\mu_2)|x|^2\big\}\widetilde{A}^i_\gamma\widetilde{x}^\gamma\widetilde{x}^\xi\\
&\quad+(1+\mu_2|\widetilde{x}|^2)|x|^2\widetilde{A}^i_\xi+(1+\mu_1|x|^2)|\widetilde{x}|^2A^\xi_i.
\end{aligned}\tag{49}
$$

Put $x=0$ and $\widetilde{x}=0$ in (49) respectively. We have

$$
\frac{\partial b^i}{\partial\widetilde{x}^\xi}=\frac{Y_3}{(1+\mu_2|\widetilde{x}|^2)^2},\qquad\frac{\partial\widetilde{b}^\xi}{\partial x^i}=\frac{Y_4}{(1+\mu_1|x|^2)^2},\tag{50}
$$

where we define

$$
\begin{aligned}
Y_3&:=-\widetilde{Q}^i_{\xi\gamma}\widetilde{x}^\gamma-(1-\mu_2|\widetilde{x}|^2)\widetilde{B}^i_\xi-2\mu_2\widetilde{B}^i_\gamma\widetilde{x}^\gamma\widetilde{x}^\xi+2\widetilde{A}^i_\gamma\widetilde{x}^\gamma\widetilde{x}^\xi+2T^i\widetilde{x}^\xi-|\widetilde{x}|^2A^\xi_i,\\
Y_4&:=-Q^\xi_{ik}x^k-(1-\mu_1|x|^2)B^\xi_i-2\mu_1B^\xi_kx^kx^i+2A^\xi_kx^kx^i+2\widetilde{T}^\xi x^i-|x|^2\widetilde{A}^i_\xi,
\end{aligned}
$$

$$
\widetilde{Q}^i_{\xi\gamma}:=\frac{\partial\widetilde{q}^\xi_\gamma}{\partial x^i}(0),\quad Q^\xi_{ik}:=\frac{\partial q^i_k}{\partial\widetilde{x}^\xi}(0),\quad\widetilde{B}^i_\xi:=\frac{\partial\widetilde{b}^\xi}{\partial x^i}(0),\quad B^\xi_i:=\frac{\partial b^i}{\partial\widetilde{x}^\xi}(0).
$$

Plugging (50) into (49) gives

$$
\begin{aligned}
&-\left\{(1+\mu_2|\widetilde{x}|^2)^2\frac{\partial q^i_k}{\partial\widetilde{x}^\xi}x^k+(1+\mu_1|x|^2)^2\frac{\partial\widetilde{q}^\xi_\gamma}{\partial x^i}\widetilde{x}^\gamma\right\}\\
&=\big\{8(\mu_1+\mu_2)\kappa+4(2\mu_1+\mu_2)\langle\widetilde{T},\widetilde{x}\rangle+4(\mu_1+2\mu_2)\langle T,x\rangle\\
&\quad+4((\mu_1+\mu_2)\langle Ax,\widetilde{x}\rangle\big\}x^i\widetilde{x}^\xi-2(2\mu_1+\mu_2)|\widetilde{x}|^2\widetilde{T}^\xi x^i-2(\mu_1+2\mu_2)|x|^2T^i\widetilde{x}^\xi\\
&\quad-2(\mu_1+\mu_2)|\widetilde{x}|^2A^\xi_kx^kx^i-2(\mu_1+\mu_2)|x|^2\widetilde{A}^i_\gamma\widetilde{x}^\gamma\widetilde{x}^\xi+(1+\mu_2|\widetilde{x}|^2)|x|^2\widetilde{A}^i_\gamma\\
&\quad+(1+\mu_1|x|^2)|\widetilde{x}|^2A^\gamma_i-\widetilde{Q}^i_{\xi\gamma}\widetilde{x}^\gamma-Q^\xi_{ik}x^k-(1-\mu_2|\widetilde{x}|^2)\widetilde{B}^i_\xi\\
&\quad-(1-\mu_1|x|^2)B^\xi_i-2\mu_2\widetilde{B}^i_\gamma\widetilde{x}^\gamma\widetilde{x}^\xi-2\mu_1B^\xi_kx^kx^i.
\end{aligned}\tag{51}
$$

Contracting (51) with x^i and $\widetilde{x}^\xi$ yields

$$
\begin{aligned}
&\big\{(\mu_1+\mu_2)\langle Ax,\widetilde{x}\rangle+2(2\mu_1+\mu_2)\langle\widetilde{T},\widetilde{x}\rangle+2(\mu_1+2\mu_2)\langle T,x\rangle+8(\mu_1+\mu_2)\kappa\big\}|x|^2|\widetilde{x}|^2\\
&\quad+(|x|^2+|\widetilde{x}|^2)\langle Ax,\widetilde{x}\rangle+\big(\mu_2|\widetilde{x}|^2\widetilde{B}^i_\xi+\mu_1|x|^2B^\xi_i\big)\widetilde{x}^\xi x^i-2\mu_2|\widetilde{x}|^2\widetilde{B}^i_\gamma\widetilde{x}^\gamma x^i\\
&\quad-2\mu_1|x|^2B^\xi_kx^k\widetilde{x}^\xi-\widetilde{Q}^i_{\xi\gamma}\widetilde{x}^\gamma\widetilde{x}^\xi x^i-Q^\xi_{ik}x^kx^i\widetilde{x}^\xi-(\widetilde{B}^i_\xi+B^\xi_i)x^i\widetilde{x}^\xi=0,
\end{aligned}\tag{52}
$$

where we have used the skew-symmetry of the matrices (q^i_k) and $(\widetilde{q}^\xi_\gamma)$. The coefficients of every order in (52) must be zero. As for the orders six, five, three and two, we have

$$(\mu_1+\mu_2)A=0, \qquad (2\mu_1+\mu_2)\xi=0, \qquad (\mu_1+2\mu_2)\eta=0, \tag{53}$$

$$\widetilde{Q}^i_{\xi\gamma}=0, \qquad Q^\xi_{ik}=0, \qquad \widetilde{B}^i_\xi+B^\xi_i=0. \tag{54}$$

The terms of order four in (52) can be simplified to

$$8(\mu_1+\mu_2)\kappa|x|^2|\widetilde{x}|^2+\big(\langle Ax-\mu_1Bx,\widetilde{x}\rangle\big)|x|^2+\big(\langle\widetilde{A}\widetilde{x}-\mu_2\widetilde{B}\widetilde{x},x\rangle\big)|\widetilde{x}|^2. \tag{55}$$

By (55) we have

$$(\mu_1+\mu_2)\kappa=0, \quad A-\mu_1B=0, \quad \widetilde{A}-\mu_2\widetilde{B}=0, \tag{56}$$

where the matrices $B:=(B^\xi_i)$ and $\widetilde{B}:=(\widetilde{B}^i_\xi)$. By (53), (54) and (56), we get (51) simplified to

$$\begin{aligned}-(1+\mu_2|\widetilde{x}|^2)^2\frac{\partial q^i_k}{\partial\widetilde{x}^\xi}x^k-(1+\mu_1|x|^2)^2\frac{\partial\widetilde{q}^\xi_\gamma}{\partial x^i}\widetilde{x}^\gamma&\\ =2\big\{(|x|^2+|\widetilde{x}|^2)A^\xi_i-A^\xi_kx^kx^i-A^\gamma_i\widetilde{x}^\gamma\widetilde{x}^\xi\big\}.&\end{aligned} \tag{57}$$

Differentiating (57) by x^k and then putting $x=0$, we have

$$\frac{\partial q^i_k}{\partial\widetilde{x}^\xi}=-\frac{1}{(1+\mu_2|\widetilde{x}|^2)^2}\frac{\partial^2\widetilde{q}^\xi_\gamma}{\partial x^i\partial x^k}(0)\widetilde{x}^\gamma. \tag{58}$$

Similarly, differentiating (57) by $\widetilde{x}^\gamma$ and then putting $\widetilde{x}=0$, we have

$$\frac{\partial\widetilde{q}^\xi_\gamma}{\partial x^i}=-\frac{1}{(1+\mu_1|x|^2)^2}\frac{\partial^2q^i_k}{\partial x^\xi\partial x^\gamma}(0)x^k. \tag{59}$$

By (58) and (59), we see that $\frac{\partial q^i_k}{\partial\widetilde{x}^\xi}$ are symmetric in the indices i, k, and $\frac{\partial\widetilde{q}^\xi_\gamma}{\partial x^i}$ are symmetric in the indices ξ, γ. Then by the skew symmetry of the matrices (q^i_k) and $(\widetilde{q}^\xi_\gamma)$, we see that

$$\frac{\partial q^i_k}{\partial\widetilde{x}^\xi}=0, \quad \frac{\partial\widetilde{q}^\xi_\gamma}{\partial x^i}=0, \tag{60}$$

which imply that the matrices $(q^i_k(\widetilde{x}))$ and $(\widetilde{q}^\xi_\gamma(x))$ are constant matrices. Again by (60) and (57) we get

$$A=\widetilde{A}=0. \tag{61}$$

Then by (54), (56) and (61) we conclude that $\mu_1 B = 0$, $\mu_2 \widetilde{B} = 0$; and $B = \widetilde{B} = 0$ if $(\mu_1)^2 + (\mu_2)^2 \neq 0$. Now by this fact, (54) and (61), we can solve the differential equations (50). We can obtain

$$b^i(\widetilde{x}) = \begin{cases} -T^i / \big[\mu_2(1+\mu_2|\widetilde{x}|^2)\big] + \theta^i, & \text{if } \ \mu_2 \neq 0, \\ |\widetilde{x}|^2 T^i - \widetilde{B}^i_\gamma \widetilde{x}^\gamma + \omega^i, & \text{if } \ \mu_2 = 0, \end{cases} \tag{62}$$

where (θ^i) and (ω^i) are constant vectors. Similarly, we have

$$\widetilde{b}^\xi(x) = \begin{cases} -\widetilde{T}^\xi / \big[\mu_1(1+\mu_1|x|^2)\big] + \widetilde{\theta}^\xi, & \text{if } \ \mu_1 \neq 0, \\ |x|^2 \widetilde{T}^\xi - B^\xi_k x^k + \widetilde{\omega}^\xi, & \text{if } \ \mu_1 = 0, \end{cases} \tag{63}$$

where $(\widetilde{\theta}^\xi)$ and $(\widetilde{\omega}^\xi)$ are constant vectors. By (53), (56) and (61) we have

$$(2\mu_1+\mu_2)\widetilde{T} = 0, \quad (\mu_1 + 2\mu_2)T = 0, \quad (\mu_1+\mu_2)\kappa = 0,$$
$$A = \widetilde{A} = 0, \quad \mu_1 B = 0, \mu_2 \widetilde{B} = 0.$$

Therefore, we have the following different cases:

(ia) $\mu_1 = \mu_2 = 0$, $A = \widetilde{A} = 0$.

(ib) $\mu_1 = -\mu_2 \neq 0$, $\widetilde{T} = T = 0$, $A = \widetilde{A} = 0$, $B = \widetilde{B} = 0$.

(ic) $\mu_1 = -\mu_2/2 \neq 0$, $\kappa = 0$, $T = 0$, $A = \widetilde{A} = 0$, $B = \widetilde{B} = 0$.

(id) $\mu_2 = -\mu_1/2 \neq 0$, $\kappa = 0$, $\widetilde{T} = 0$, $A = \widetilde{A} = 0$, $B = \widetilde{B} = 0$.

(ie) $\mu_1 \neq -\mu_2$, $\mu_1 \neq -\mu_2/2$, $\mu_2 \neq -\mu_1/2$, $\mu_1 \neq 0$, $\mu_2 \neq 0$, $\kappa = 0$, $\widetilde{T} = T = 0$, $A = \widetilde{A} = 0$, $B = \widetilde{B} = 0$.

(if) $\mu_1 = 0$, $\mu_2 \neq 0$, $\kappa = 0$, $\widetilde{T} = T = 0$, $A = \widetilde{A} = 0$, $B = \widetilde{B} = 0$.

(ig) $\mu_1 \neq 0$, $\mu_2 = 0$, $\kappa = 0$, $\widetilde{T} = T = 0$, $A = \widetilde{A} = 0$, $B = \widetilde{B} = 0$.

According to the above different cases, we get the corresponding S-curvature from (41); and by plugging (36), (42), (60), (62) and (63) into (7) and (8), we can obtain the expressions of W^i and W^ξ. □

5. Randers metrics of constant S-curvature

In this section, we construct some Randers metrics of constant S-curvature in every dimension $n \geq 3$, with the corresponding Riemann metrics h's from the navigation problem being not locally conformally flat.

First we show two results as follows:

Theorem 5.1. *Let $F = \alpha+\beta$ be a 3-dimensional Randers metric determined by the navigation data (h, W) on $\mathbf{R}^2 \times \mathbf{R}^1$ with*

$$h = \sqrt{e^{2\sigma(x^1,x^2)}\left[(y^1)^2 + (y^2)^2\right] + (y^3)^2},$$

where $W = W(x^1, x^2, x^3)$ is a vector field on $\mathbf{R}^2 \times \mathbf{R}^1$. Then F is of isotropic S-curvature if and only if the following hold ($i = 1, 2$):

$$\frac{\partial W^1}{\partial x^2} + \frac{\partial W^2}{\partial x^1} = 0, \qquad \frac{\partial W^1}{\partial x^1} = \frac{\partial W^2}{\partial x^2}, \tag{64}$$

$$e^{\sigma}\frac{\partial W^i}{\partial x^3} + \frac{\partial W^3}{\partial x^i} = 0, \qquad 2\frac{\partial W^1}{\partial x^1} + W^1\frac{\partial \sigma}{\partial x^1} + W^2\frac{\partial \sigma}{\partial x^2} = 2\frac{\partial W^3}{\partial x^3}, \tag{65}$$

In this case, the isotropic S-curvature $\mathbf{S} = 4c(x^1, x^2, x^3)F$ is determined by

$$c(x^1,x^2,x^3) = -\frac{1}{4}\left[2\frac{\partial W^1}{\partial x^1} + W^1\frac{\partial \sigma}{\partial x^1} + W^2\frac{\partial \sigma}{\partial x^2}\right] = -\frac{1}{2}\frac{\partial W^3}{\partial x^3}. \tag{66}$$

In Theorem 5.1, if $W^i = W^i(x^1, x^2)$ and $W^3 = W^3(x^3)$, then the first formula in (65) holds naturally and the S-curvature is constant given by (66). The proof of Theorem 5.1 is similar to that of Theorem 1.1(i).

Theorem 5.2. *Let $F = \alpha + \beta$ be an n-dimensional Randers metric determined by the navigation data (h, W) on a product manifold $M \times \widetilde{M}$ with $h = \sqrt{H^2 + \widetilde{H}^2}$, where (M, H) and $(\widetilde{M}, \widetilde{H})$ are two Riemann manifolds and $W = (W^i(x), W^{\xi}(\widetilde{x}))$ is a vector field on $M \times \widetilde{M}$. Let F_1 be a Randers metric determined by the navigation data $(H, \{W^i\})$ and F_2 be another Randers metric determined by the navigation data $(\widetilde{H}, \{W^{\xi}\})$. If F_1 and F_2 are of constant S-curvature with the same constant c, that is, $\mathbf{S}_1 = (m+1)cF_1$ and $\mathbf{S}_2 = (\widetilde{m}+1)cF_2$, then F is also of constant S-curvature $\mathbf{S} = (n+1)cF$.*

Theorem 5.2 is easily proved by (2). We omit the details. Now in the following, we construct examples of constant S-curvature in every dimension $n \geq 3$.

Let $F = \alpha + \beta$ be a 2-dimensional Randers metric determined by the navigation data (h, W), where $h_{ij} = e^{\sigma(x)}\delta_{ij}$ with

$$\sigma = A_{ij}x^ix^j + B_ix^i + C, \tag{67}$$

where the 2-order matrix (A_{ij}) is symmetric, B is a constant vector and C is a constant; and

$$W^1 = -a_1x^2 + b_1, \qquad W^2 = a_1x^1 + b_2, \tag{68}$$

where a_1, b_1, b_2 are constant. It has been proved in [12] that, if the coefficients in (67) and (68) satisfy

$$\begin{cases} a_1 = 0 \\ b_1A_{11} + b_2A_{12} = 0, \\ b_1A_{12} + b_2A_{22} = 0 \end{cases} \qquad \text{or} \qquad \begin{cases} A_{12} = 0 \\ A_{11} = A_{22} \\ B_2 = -\dfrac{2b_1}{a_1}A_{11} \\ B_1 = \dfrac{2b_2}{a_1}A_{11} \end{cases} \tag{69}$$

then F is of constant S-curvature $\mathbf{S} = 3cF$ given by

$$c = -\frac{1}{4}(b_1B_1 + b_2B_2). \tag{70}$$

If further in (69), $A_{11} + A_{22} \neq 0$, then h is not of constant sectional curvature.

Now Let $F = \alpha + \beta$ be a 3-dimensional Randers metric determined by the navigation data (h, W) on $\mathbf{R}^2 \times \mathbf{R}^1$, where h is defined by

$$h = \sqrt{e^{2\sigma}\left[(y^1)^2 + (y^2)^2\right] + (y^3)^2}$$

with $\sigma = \sigma(x^1, x^2)$ being given by (67), and for the vector field

$$W = \big(W^1(x^1, x^2), W^2(x^1, x^2), W^3(x^3)\big),$$

W^1 and W^2 are given by (68) satisfying (69) and $A_{11}+A_{22} \neq 0$, and $W^3 = -2cx^3$ with c being given by (70). Then by Theorem 5.1, F is of constant S-curvature $\mathbf{S} = 4cF$ with h being not locally conformally flat by Lemma 2.1(ii).

By using the above 2-dimensional Randers metrics of constant S-curvature and Theorem 5.2, we can construct a class of Randers metrics of constant S-curvature in every dimension $n \geq 4$, with the corresponding Riemann metrics h's being not locally conformally flat. For details, we suppose in Theorem 5.2 that H and $W^1(x^1, x^2), W^2(x^1, x^2)$ are given by (67), (68) satisfying (69) and $A_{11} + A_{22} \neq 0$, and let $\widetilde{H}$ and $W^\xi(\widetilde{x})$ be arbitrarily given such that F_2 is of constant S-curvature $\mathbf{S}_2 = (\widetilde{m}+1)cF_2$ with c given by (70), then F is of constant S-curvature $\mathbf{S} = (n+1)cF$ by Theorem 5.2, and the corresponding Riemann metric h is not locally conformally flat by Lemma 2.1(i).

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