*Publ. Math. Debrecen*
**78/1** (2011), 89–102
DOI: 10.5486/PMD.2011.4566

# Asymptotic stability of differential equations with several delays

By CHUHUA JIN (Guangdong) and JIAOWAN LUO (Guangdong)

**Abstract.** The linear scalar differential equation with several delays

$$x'(t) = -\sum_{i=1}^{N} b_i(t)x(t-\tau_i(t))$$

is investigated, where $b_i(t) \in C(R^+, R)$ and $\tau_i(t) \in C(R^+, R^+)$ for $i = 1, 2, \dots, N$. Using fixed point theory, some new conditions for asymptotic stability of the zero solution are established. For $N = 1$, our theory improves the results in the earlier publications. For $N = 2$, two examples, which the results in the literature can not be applied to, are given to show the feasibility and effectiveness of our result.

## 1. Introduction

In this paper we consider the linear differential equation with several delays of the form

$$x'(t) = -\sum_{i=1}^{N} b_i(t)x(t-\tau_i(t)), \tag{1.1}$$

where $b_i \in C(R^+, R)$ and $\tau_i \in C(R^+, R^+)$ with $t - \tau_i(t) \to \infty$ as $t \to \infty$ for $i = 1, 2, \dots, N$, $R^+ = [0, +\infty)$. When $N = 1$ and $N = 2, \tau_1 = 0$, equation (1.1) reduces to

$$x'(t) = -b(t)x(t-\tau(t)) \tag{1.2}$$

*Mathematics Subject Classification:* 34K20, 34K40.
*Key words and phrases:* fixed points, stability, delay differential equations, several delays.
This work was partially supported by NSC (Grant No. 10871052).

and

$$x'(t) = -b_1(t)x(t) - b_2(t)x(t - \tau(t)), \tag{1.3}$$

respectively.

Equation (1.2), as well as more general cases, has been studied by many authors. Yorke [14] showed the following well-known result: If there are two positive numbers $\beta$ and $q$ such that

$$0 < b(t) \leq \beta, \quad \tau(t) \leq q, \quad \text{and} \quad \beta q < \frac{3}{2}, \tag{1.4}$$

then the trivial solution of (1.2) is uniformly stable. Yoneyama [12] generalized condition (1.4) to

$$\inf_{t\geq 0} \int_t^{t+q} b(s)ds > 0, \quad \tau(t) \leq q, \quad \text{and} \quad \sup_{t\geq 0} \int_t^{t+q} b(s)ds < \frac{3}{2}. \tag{1.5}$$

Krisztin [8] extended the Yorke's theorem and obtained that if $b_i : R^+ \to R^+$ is continuous, $b_i(t) \leq \beta_i$, and $\tau_i : R^+ \to [0, q_i]$ is continuous for $i = 1, 2, \ldots, N$, then the zero solution of (1.1) is uniformly stable if $\Sigma_{i=1}^N \beta_i q_i \leq 1$, and is uniformly asymptotically stable if $\Sigma_{i=1}^N \beta_i q_i < 1$.

Under conditions that $b(t) \geq 0$ and $\tau(t)$ is bounded, Yoneyama [13] also gave a generalization of the result of Yorke [14] and showed that if $\lambda = \sup_{t\geq 0} \int_{t-\tau(t)}^t b(s)ds < \frac{3}{2}$ and $\mu = \inf_{t\geq 0} \int_{t-\tau(t)}^t b(s)ds > 0$, then the zero solution of (1.2) is uniformly asymptotically stable, Yoneyama [11] showed that in case $\int_{t-\tau(t)}^t b(s)ds \to 0$ as $t \to \infty$, the zero solution of $x'(t) = -b(t)f(x(t - \tau(t)))$ is uniformly stable, and Hara *et al.* [6] showed that if $\sup_{t\geq 0} \int_{t-\tau(t)}^t b(s)ds < 1$ and $\int_0^t b(s)ds = \infty$, then the zero solution of (1.2) is uniformly stable and asymptotically stable. Furthermore, Muroya [9] told us that $\inf_{t\geq 0} \int_{t-\tau(t)}^t b(s)ds > 0$ is usually not an essential condition for the zero solution of (1.2) to be asymptotically stable.

All the above mentioned papers usually require that the delays be bounded. However, in a paper by Graef *et al.* [5], boundedness and stability are obtained without asking that $\tau(t)$ be bounded. We state their theorem for equation (1.2).

**Theorem A** ([5]). *Suppose that $b(t) \geq 0$ for $t \geq 0$, $t - \tau(t) \to \infty$ as $t \to \infty$ and*

$$\limsup_{t\to\infty} \int_{t-\tau(t)}^t b(s)ds = \mu < 1. \tag{1.6}$$

*Then the zero solution of* (1.2) *is asymptotically stable if and only if* $\int_0^\infty b(s)ds = \infty$.

Note that the sign condition $b(t) \geq 0$ is required by Theorem A. Burton [3] eliminated this condition for the case $\tau(t) = r$, a constant, by applying fixed point theory with an appropriate mapping function. It has been shown that many problems encountered in the study of stability by means of Lyapunov's direct method can be solved by using fixed point theory, see [2], [4].

**Theorem B** ([3]). *Suppose that $\tau(t) = r$ and there exists a constant $\alpha < 1$ such that*

$$\int_{t-r}^{t} |b(s+r)|ds + \int_{0}^{t} |b(s+r)|e^{-\int_s^t b(u+r)du} \int_{s-r}^{s} |b(u+r)|duds \leq \alpha \tag{1.7}$$

*for all $t \geq 0$ and $\int_0^\infty b(s)ds = \infty$. Then for every continuous initial function $\psi : [-r, 0] \to R$, the solution $x(t) = x(t, 0, \psi)$ of* (1.2) *is bounded and tends to zero as $t \to \infty$.*

Zhang [16] generalized Theorem B to (1.1) for unbounded $\tau_i(t)$'s and showed that $\int_0^\infty b(s)ds = \infty$ is a necessary condition for asymptotic stability.

**Theorem C** ([16]). *Suppose that $\tau_i$ is differentiable, the inverse function $g_i(t)$ of $t - \tau_i(t)$ exists, and there exists a constant $\alpha \in (0, 1)$ such that for $t \geq 0$,*

$$\liminf_{t\to\infty} \int_0^t Q(s)ds > -\infty, \tag{1.8}$$

$$\sum_{i=1}^{N} \Bigg[ \int_{t-\tau_i(t)}^{t} |b_i(g_i(s))|ds + \int_0^t e^{-\int_s^t Q(u)du} |Q(s)| \int_{s-\tau_i(s)}^{s} |b_i(g_i(v))|dvds \\ + \int_0^t e^{-\int_s^t Q(u)du} |b_i(s)|\,|\tau_i'(s)|ds \Bigg] \leq \alpha. \tag{1.9}$$

*Then the zero solution of* (1.1) *is asymptotically stable if and only if*

$$\int_0^t Q(s))ds \to \infty \quad \text{as} \quad t \to \infty, \tag{1.10}$$

*where $Q(t) := \sum_{i=1}^N b_i(g_i(t))$.*

Jin and Luo [7] generalized and improved Theorems B and C for (1.2). Raffoul [10] investigated equation (1.3) and obtained

**Theorem D** ([10]). *Suppose that there exists a constant $\alpha \in (0, 1)$ such that for $t \geq 0$*

$$\int_0^t b_1(s)ds \to \infty \quad \text{as} \quad t \to \infty, \tag{1.11}$$

*and*

$$\int_0^t e^{-\int_s^t b_1(u)du}|b_2(s)|ds \le \alpha. \tag{1.12}$$

*Then every solution* $x(t) = x(t, 0, \psi)$ *of* (1.3) *with small continuous initial function* $\psi(t)$, *is bounded and tends to zero as* $t \to \infty$.

Moreover, ZHANG [15] showed that if (1.12) holds, (1.11) is a necessary condition for asymptotic stability for (1.3).

For more 100 years, Lyapunov's direct method has been the main tool for investigating stability properties of ordinary, functional and partial differential equations. Yet, there is a large set of problems for which it has been ineffective. The purpose of this paper is to establish some new asymptotic stability conditions for (1.1) by fixed-point methods. These conditions do not require the boundedness of delays $\tau_i(t)$, nor do they ask for a fixed sign on the coefficient functions $b_i(t)$. Our theory improves the results in [5]–[7] for (1.2). Two examples are presented to show that our result is stronger than those in [3], [10], [15], [16].

## 2. Main results

Let $C(S_1, S_2)$ denote the set of all continuous functions $\phi : S_1 \to S_2$. Define $m_i(\theta) = \inf\{s - \tau_i(s) : s \ge \theta\}$, $\widetilde{m}(\theta) = \min\{m_i(\theta) : 1 \le i \le N\}$, and $\widetilde{C}(\theta) = C([\widetilde{m}(\theta), \theta], R)$ with the supremum norm $||\cdot||$.

For each $(t_0, \phi) \in R^+ \times \widetilde{C}(t_0)$, a solution of (1.1) through $(t_0, \phi)$ is a continuous function $x : [\widetilde{m}(t_0), t_0 + \alpha) \to R^n$ for some positive constant $\alpha > 0$ such that $x(t)$ satisfies (1.1) on $[t_0, t_0 + \alpha)$ and $x(s) = \phi(s)$ for $s \in [\widetilde{m}(t_0), t_0]$. We denote such a solution by $x(t) = x(t, t_0, \phi)$. For each $(t_0, \phi) \in R^+ \times \widetilde{C}(t_0)$, there exists a unique solution $x(t) = x(t, t_0, \phi)$ of (1.1) defined on $[t_0, +\infty)$. Stability definitions may be found in [1], for example.

**Theorem 2.1.** *Suppose that there exist a constant* $\alpha \in (0, 1)$ *and functions* $\tau_0 \in C(R^+, R^+)$ *with* $t - \tau_0(t) \to \infty$ *as* $t \to \infty$, $h \in C([m_0(0), \infty), R)$, *where* $m_0(\theta) = \inf\{s - \tau_0(s) : s \ge \theta\}$, *such that for* $t \ge 0$,

(i) $$\liminf_{t\to\infty} \int_0^t h(s)ds > -\infty,$$

(ii) $$\int_{t-\tau_0(t)}^t |h(s)|ds + \int_0^t e^{-\int_s^t h(u)du}|h(s)| \int_{s-\tau_0(s)}^s |h(v)|dvds$$

$$+\int_0^t e^{-\int_s^t h(u)du}\left|h(s-\tau_0(s))(1-\tau_0'(s))-\sum_{k=1}^N b_k(s)\right|ds$$
$$+\sum_{k=1}^N\int_0^t e^{-\int_s^t h(u)du}\left|\sum_{j=k}^N b_j(s)\right|\left|\int_{s-\tau_{k-1}(s)}^{s-\tau_k(s)}\sum_{i=1}^N|b_i(v)|dv\right|ds\le\alpha.$$

*where $b_i(t)$ is well-defined and continuous in $[m(0),\infty)$ ($m(\theta)=\min\{m_i(\theta):0\le i\le N\}$) for $i=1,2,\ldots,N$.*

*Then the zero solution of* (1.1) *is asymptotically stable if and only if*

(iii) $$\int_0^t h(s)ds\to\infty\quad as\quad t\to\infty.$$

Proof. First, suppose that (iii) holds. For each $t_0\ge0$, we set

$$K=\sup_{t\ge0}\{e^{-\int_0^t h(s)ds}\}. \tag{2.1}$$

and define $C(t_0)=C([m(t_0),t_0],R)$ with the supremum norm $\|\cdot\|$. Let $\phi\in C(t_0)$ be fixed and define

$$S=\{x\in C([m(t_0),\infty),R):x(t)\to0\text{ as }t\to\infty,\ x(s)=\phi(s)\text{ for }s\in[m(t_0),t_0]\}.$$

Then $S$ is a complete metric space with metric $\rho(x,y)=\sup_{t\ge t_0}\{|x(t)-y(t)|\}$.

Rewrite (1.1) in the following form:

$$\begin{aligned}x'(t)&=-\sum_{k=1}^N b_k(t)x(t-\tau_0(t))-\sum_{k=1}^N b_k(t)\int_{t-\tau_0(t)}^{t-\tau_1(t)}x'(s)ds\\&\quad-\sum_{k=2}^N b_k(t)\int_{t-\tau_1(t)}^{t-\tau_2(t)}x'(s)ds-\cdots-b_N(t)\int_{t-\tau_{N-1}(t)}^{t-\tau_N(t)}x'(s)ds\\&=-\sum_{k=1}^N b_k(t)x(t-\tau_0(t))-\sum_{k=1}^N\sum_{j=k}^N b_j(t)\int_{t-\tau_{k-1}(t)}^{t-\tau_k(t)}\sum_{i=1}^N b_i(s)x(s-\tau_i(s))ds.\end{aligned}$$

Multiplying both sides of (2.2) by $e^{\int_0^t h(s)ds}$ and integrating from $t_0$ to $t$, we obtain

$$\begin{aligned}x(t)&=\phi(t_0)e^{-\int_{t_0}^t h(s)ds}-\int_{t_0}^t e^{-\int_s^t h(u)du}\sum_{k=1}^N b_k(s)x(s-\tau_0(s))ds\\&\quad+\int_{t_0}^t e^{-\int_s^t h(u)du}h(s)x(s)ds-\sum_{k=1}^N\int_{t_0}^t e^{-\int_s^t h(u)du}\sum_{j=k}^N b_j(s)\end{aligned}$$

$$\times \int_{s-\tau_{k-1}(s)}^{s-\tau_k(s)} \sum_{i=1}^{N} b_i(v)x(v-\tau_i(v))dvds. \tag{2.2}$$

Then an integration by parts yields the following integral equation

$$\begin{aligned}
x(t) = &\left(\phi(t_0) - \int_{t_0-\tau_0(t_0)}^{t_0} h(s)\phi(s)ds\right)e^{-\int_{t_0}^{t} h(u)du} + \int_{t-\tau_0(t)}^{t} h(s)x(s)ds \\
&- \int_{t_0}^{t} e^{-\int_s^t h(u)du} h(s) \int_{s-\tau_0(s)}^{s} h(v)x(v)dvds \\
&+ \int_{t_0}^{t} e^{-\int_s^t h(u)du}\left\{h(s-\tau_0(s))(1-\tau_0'(s)) - \sum_{k=1}^{N} b_k(s)\right\}x(s-\tau_0(s))ds \\
&- \sum_{k=1}^{N}\int_{t_0}^{t} e^{-\int_s^t h(u)du}\sum_{j=k}^{N} b_j(s)\int_{s-\tau_{k-1}(s)}^{s-\tau_k(s)} \sum_{i=1}^{N} b_i(v)x(v-\tau_i(v))dvds.
\end{aligned} \tag{2.3}$$

Use (2.3) to define the operator $P : S \to S$ by $(Px)(t) = \phi(t)$ for $t \in [m(t_0), t_0]$ and

$$\begin{aligned}
(Px)(t) = &\left(\phi(t_0) - \int_{t_0-\tau_0(t_0)}^{t_0} h(s)\phi(s)ds\right)e^{-\int_{t_0}^{t} h(u)du} + \int_{t-\tau_0(t)}^{t} h(s)x(s)ds \\
&- \int_{t_0}^{t} e^{-\int_s^t h(u)du} h(s) \int_{s-\tau_0(s)}^{s} h(v)x(v)dvds \\
&+ \int_{t_0}^{t} e^{-\int_s^t h(u)du}\left\{h(s-\tau_0(s))(1-\tau_0'(s)) - \sum_{k=1}^{N} b_k(s)\right\}x(s-\tau_0(s))ds \\
&- \sum_{k=1}^{N}\int_{t_0}^{t} e^{-\int_s^t h(u)du}\sum_{j=k}^{N} b_j(s)\int_{s-\tau_{k-1}(s)}^{s-\tau_k(s)} \sum_{i=1}^{N} b_i(v)x(v-\tau_i(v))dvds
\end{aligned} \tag{2.4}$$

for $t \geq t_0$. It is clear that $(Px) \in C([m(t_0), \infty), R)$. Now consider the asymptotic behavior of each of the above terms as $t \to \infty$. The first term tends to 0 by (ii). Because $x(t) \to 0$ as $t \to \infty$, the same is true of the second term. Next we will prove that last term $I_5$ in (2.4) approaches to zero. Since $x(t) \to 0$ and $t - \tau_i(t) \to \infty$ as $t \to \infty$, for each $\varepsilon > 0$, there exist $T_1$, $T_2 > 0$ such that $s \geq T_1$ implies $s - \tau_i(s)) > T_2$ and $v \geq T_2$ implies $|x(v - \tau_i(v))| < \varepsilon$ for $i = 1, 2, \ldots, N$. Thus, for $t \geq T_1$,

$$|I_5| = \left|\sum_{k=1}^{N}\int_{t_0}^{t} e^{-\int_s^t h(u)du}\sum_{j=k}^{N} b_j(s)\int_{s-\tau_{k-1}(s)}^{s-\tau_k(s)} \sum_{i=1}^{N} b_i(v)x(v-\tau_i(v))dvds\right|$$

$$\leq \sum_{k=1}^{N}\int_{t_0}^{T_1} e^{-\int_s^t h(u)du}\Big|\sum_{j=k}^{N} b_j(s)\Big|\Big|\int_{s-\tau_{k-1}(s)}^{s-\tau_k(s)}\sum_{i=1}^{N}|b_i(v)|\,|x(v-\tau_i(v))|dv\Big|ds$$

$$+\sum_{k=1}^{N}\int_{T_1}^{t} e^{-\int_s^t h(u)du}\Big|\sum_{j=k}^{N} b_j(s)\Big|\Big|\int_{s-\tau_{k-1}(s)}^{s-\tau_k(s)}\sum_{i=1}^{N}|b_i(v)||x(v-\tau_i(v))|dv\Big|ds$$

$$\leq \sup_{\sigma\geq m(t_0)}|x(\sigma)|\sum_{k=1}^{N}\int_{t_0}^{T_1} e^{-\int_s^t h(u)du}\Big|\sum_{j=k}^{N} b_j(s)\Big|\Big|\int_{s-\tau_{k-1}(s)}^{s-\tau_k(s)}\sum_{i=1}^{N}|b_i(v)|dv\Big|ds$$

$$+\varepsilon\sum_{k=1}^{N}\int_{T_1}^{t} e^{-\int_s^t h(u)du}\Big|\sum_{j=k}^{N} b_j(s)\Big|\Big|\int_{s-\tau_{k-1}(s)}^{s-\tau_k(s)}\sum_{i=1}^{N}|b_i(v)|dv\Big|ds.$$

By (iii), there exists $T_3 > T_1$ such that $t \geq T_3$ implies

$$\sup_{\sigma\geq m(t_0)}|x(\sigma)|\sum_{k=1}^{N}\int_{t_0}^{T_1} e^{-\int_s^t h(u)du}\Big|\sum_{j=k}^{N} b_j(s)\Big|\Big|\int_{s-\tau_{k-1}(s)}^{s-\tau_k(s)}\sum_{i=1}^{N}|b_i(v)|dv\Big|ds$$

$$=\sup_{\sigma\geq m(t_0)}|x(\sigma)|e^{-\int_{T_1}^t h(u)du}\sum_{k=1}^{N}\int_{t_0}^{T_1} e^{-\int_s^{T_1} h(u)du}\Big|\sum_{j=k}^{N} b_j(s)\Big|$$

$$\times\Big|\int_{s-\tau_{k-1}(s)}^{s-\tau_k(s)}\sum_{i=1}^{N}|b_i(v)|dv\Big|ds < \varepsilon.$$

Apply (ii) to obtain $|I_5| \leq \varepsilon + \alpha\varepsilon < 2\varepsilon$. Thus, $I_5 \to 0$ as $t \to \infty$. Similarly, we can show that the rest of the terms in (2.4) approach zero as $t \to \infty$. This yields $(Px)(t) \to 0$ as $t \to \infty$, and hence $Px \in S$. Also, by (ii), $P$ is a contraction mapping with contraction constant $\alpha$. By the Contraction Mapping Principle, $P$ has a unique fixed point $x$ in $S$ which is a solution of (1.1) agreeing with the initial function $\phi(s)$ on $[m(t_0), t_0]$ and $x(t) = x(t, t_0, \phi) \to 0$ as $t \to \infty$.

In order to obtain the asymptotic stability, we need to prove that the zero solution of (1.1) is stable. Let $\varepsilon > 0$ be given and choose $\delta > 0 (\delta < \varepsilon)$ satisfying $2\delta K e^{\int_0^{t_0} h(u)du} + \alpha\varepsilon < \varepsilon$. If $x(t) = x(t, t_0, \phi)$ is a solution of (1.1) with $\|\phi\| < \delta$, then $x(t) = (Px)(t)$ defined in (2.4). We claim that $|x(t)| < \varepsilon$ for all $t \geq t_0$. Notice that $|x(s)| < \varepsilon$ on $[m(t_0), t_0]$. If there exists $t^* > t_0$ such that $|x(t^*)| = \varepsilon$ and $|x(s)| < \varepsilon$ for $m(t_0) \leq s < t^*$, then it follows from (2.4) that

$$|x(t^*)| = \|\phi\|\Big(1+\int_{t_0-\tau_0(t_0)}^{t_0}|h(s)|ds\Big)e^{-\int_{t_0}^{t^*} h(u)du} + \varepsilon\int_{t^*-\tau_0(t^*)}^{t^*}|h(s)|ds$$

$$+\varepsilon\int_{t_0}^{t^*} e^{-\int_s^{t^*} h(u)du}|h(s)|\int_{s-\tau_0(s)}^{s}|h(v)|dvds$$

$$
\begin{aligned}
&+\varepsilon\int_{t_0}^{t^*} e^{-\int_s^{t^*} h(u)du}\Big|h(s-\tau_0(s))(1-\tau_0'(s))-\sum_{k=1}^{N} b_k(s)\Big|ds\\
&+\varepsilon\sum_{k=1}^{N}\int_{t_0}^{t^*} e^{-\int_s^{t^*} h(u)du}\Big|\sum_{j=k}^{N} b_j(s)\Big|\Big|\int_{s-\tau_{k-1}(s)}^{s-\tau_k(s)}\sum_{i=1}^{N}|b_i(v)|dv\Big|ds\\
\le\ & 2\delta K e^{\int_0^{t_0} h(u)du}+\alpha\varepsilon<\varepsilon, \qquad (2.5)
\end{aligned}
$$

which contradicts the definition of $t^*$. Thus $|x(t)| < \varepsilon$ for all $t \ge t_0$, and the zero solution of (1.1) is stable. This shows that the zero solution of (1.1) is asymptotically stable if (iii) holds.

Conversely, suppose (iii) fails. Then by (i) there exists a sequence $\{t_n\}, t_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty}\int_0^{t_n} h(u)du = l$ for some $l \in R$. We may also choose a positive constant $J$ satisfying

$$
-J \le \int_0^{t_n} h(s)ds \le J
$$

for all $n \ge 1$. Let

$$
\begin{aligned}
\omega(s) = |h(s)|\int_{s-\tau_0(s)}^{s}|h(v)|dv + \Big|h(s-\tau_0(s))(1-\tau_0'(s)) - \sum_{k=1}^{N} b_k(s)\Big|\\
+\sum_{k=1}^{N}\Big|\sum_{j=k}^{N} b_j(s)\Big|\Big|\int_{s-\tau_{k-1}(s)}^{s-\tau_k(s)}\sum_{i=1}^{N}|b_i(v)|dv\Big|
\end{aligned}
$$

for all $s \ge 0$. By (ii), we have

$$
\int_0^{t_n} e^{-\int_s^{t_n} h(u)du}\omega(s)ds \le \alpha,
$$

which leads to

$$
\int_0^{t_n} e^{\int_0^s h(u)du}\omega(s)ds \le \alpha e^{\int_0^{t_n} h(u)du} \le e^J.
$$

The sequence $\{\int_0^{t_n} e^{\int_0^s h(u)du}\omega(s)ds\}$ is bounded, so there exists a convergent subsequence. For brevity in notation, we may assume

$$
\lim_{n\to\infty}\int_0^{t_n} e^{\int_0^s h(u)du}\omega(s)ds = \gamma
$$

for some $\gamma \in R^+$ and choose a positive integer $\bar{k}$ so large that

$$
\int_{t_{\bar{k}}}^{t_n} e^{\int_0^s h(u)du}\omega(s)ds < \delta_0/4K
$$

for all $n \geq \bar{k}$, where $\delta_0 > 0$ satisfies $2\delta_0 K e^J + \alpha < 1$.

By (i), $K$ in (2.1) is well defined. We now consider the solution $x(t) = x(t, t_{\bar{k}}, \phi)$ of (1.1) with $\phi(t_{\bar{k}}) = \delta_0$ and $|\phi(s)| \leq \delta_0$ for $s \leq t_{\bar{k}}$. An argument similar to that in (2.5) shows $|x(t)| \leq 1$ for $t \geq t_{\bar{k}}$. We may choose $\phi$ so that

$$\phi(t_{\bar{k}}) - \int_{t_{\bar{k}} - \tau_0(t_{\bar{k}})}^{t_{\bar{k}}} h(s)\phi(s)ds \geq \frac{1}{2}\delta_0.$$

It follows from (2.4) with $x(t) = (Px)(t)$ that for $n \geq t_{\bar{k}}$,

$$\begin{aligned}
\left| x(t_n) - \int_{t_n - \tau_0(t_n)}^{t_n} h(s)x(s)ds \right| &\geq \frac{1}{2}\delta_0 e^{-\int_{t_{\bar{k}}}^{t_n} h(u)du} - \int_{t_{\bar{k}}}^{t_n} e^{-\int_s^{t_n} h(u)du}\omega(s)ds \\
&= \frac{1}{2}\delta_0 e^{-\int_{t_{\bar{k}}}^{t_n} h(u)du} - e^{-\int_0^{t_n} h(u)du}\int_{t_{\bar{k}}}^{t_n} e^{\int_0^s h(u)du}\omega(s)ds \\
&= e^{-\int_{t_{\bar{k}}}^{t_n} h(u)du}\left(\frac{1}{2}\delta_0 - e^{-\int_0^{t_{\bar{k}}} h(u)du}\int_{t_{\bar{k}}}^{t_n} e^{\int_0^s h(u)du}\omega(s)ds\right) \\
&\geq e^{-\int_{t_{\bar{k}}}^{t_n} h(u)du}\left(\frac{1}{2}\delta_0 - K\int_{t_{\bar{k}}}^{t_n} e^{\int_0^s h(u)du}\omega(s)ds\right) \\
&\geq \frac{1}{4}\delta_0 e^{-\int_{t_{\bar{k}}}^{t_n} h(u)du} \geq \frac{1}{4}\delta_0 e^{-2J} > 0. \qquad (2.6)
\end{aligned}$$

On the other hand, if the zero solution of (1.1) is asymptotically stable, then $x(t) = x(t, t_{\bar{k}}, \phi) \to 0$ as $t \to \infty$. Since $t_n - \tau(t_n) \to \infty$ as $n \to \infty$ and (ii) holds, we have

$$x(t_n) - \int_{t_n - \tau_0(t_n)}^{t_n} h(s)x(s)ds \to 0 \quad \text{as} \quad n \to \infty$$

which contradicts (2.6). Hence condition (iii) is necessary for the asymptotic stability of the zero solution of (1.1). The proof is complete. □

**Corollary 2.2.** *Suppose that there exist a constant $\alpha \in (0,1)$ and functions $\tau_0 \in C(R^+, R^+)$ with $t - \tau_0(t) \to \infty$ as $t \to \infty$, $h \in C([m_0(0), \infty), R)$ such that for $t \geq 0$*

(i) $$\liminf_{t\to\infty} \int_0^t h(s)ds > -\infty,$$

(ii) $$\int_{t-\tau_0(t)}^t |h(s)|ds + \int_0^t e^{-\int_s^t h(u)du}|h(s)| \int_{s-\tau_0(s)}^s |h(v)|dvds$$

$$+\int_0^t e^{-\int_s^t h(u)du}\Big|h(s-\tau_0(s))(1-\tau_0'(s))-b(s)\Big|ds$$
$$+\int_0^t e^{-\int_s^t h(u)du}|b(s)|\Big|\int_{s-\tau_0(s)}^{s-\tau(s)}|b(v)|dv\Big|ds\le\alpha.$$

*Then the zero solution of* (1.2) *is asymptotically stable if and only if*

$$\int_0^t h(s)ds\to\infty \quad \textit{as} \quad t\to\infty.$$

*Remark 2.3.* It follows from the first part of the proof of Theorem 2.1 that the zero solution of (1.1) is stable under (i) and (ii). Moreover, Theorem 2.1 still holds if (ii) is satisfied for $t\ge t_\sigma$ for some $t_\sigma\ge R^+$.

*Remark 2.4.* When $b(t)\ge 0$, we choose $\tau_0(t)\equiv 0$ and $h(t)\equiv b(t)$, Corollary 2.2 reduces to Theorem A. Thus, the condition $b(t)\ge 0$ is eliminated and hence, Corollary 2.2 improves Theorem A.

*Remark 2.5.* We choose $\tau_0(t)\equiv\tau(t)$, Corollary 2.2 reduces to the result of [7] for (1.2).

*Remark 2.6.* For two functions $\tau_0$ and $h(t)$ being introduced, our result is stronger than those in [3,10,15,16], See Examples 3.1 and 3.2.

## 3. Examples

*Example 3.1.* Consider the scalar equation with two delays

$$x'(t)=-b_1(t)x(t-\tau_1(t))-b_2(t)x(t-\tau_2(t)), \tag{3.1}$$

where $\tau_1(t)=0.15t$, $\tau_2(t)=0.28t$, and $b_1(t)=\frac{1.55}{0.85t+1}$, $b_2(t)=\frac{0.45}{0.72t+1}$. Following the notation in Theorem C, we have $b_1(g_1(t))=\frac{1.55}{t+1}$, $b_2(g_2(t))=\frac{0.45}{t+1}$. Thus, as $t\to\infty$,

$$\int_{t-\tau_1(t)}^t |b_1(g_1(s))|ds=\int_{0.85t}^t\frac{1.55}{s+1}ds$$
$$=1.55\ln\frac{t+1}{0.85t+1}\longrightarrow -1.55\ln 0.85=0.2519,$$
$$\int_{t-\tau_2(t)}^t |b_2(g_2(s))|ds=\int_{0.72t}^t\frac{0.45}{s+1}ds$$

$$= 0.45 \ln \frac{t+1}{0.72t+1} \longrightarrow -0.45 \ln 0.72 = 0.1478,$$

$$\int_0^t e^{-\int_s^t Q(u)du} |Q(s)| \int_{s-\tau_1(s)}^s |b_1(g_1(v))| dv ds \longrightarrow 0.2519,$$

$$\int_0^t e^{-\int_s^t Q(u)du} |Q(s)| \int_{s-\tau_2(s)}^s |b_2(g_2(v))| dv ds \longrightarrow 0.1478,$$

$$\int_0^t e^{-\int_s^t Q(u)du} |b_1(s)|\, |\tau_1'(s)| ds$$

$$= \int_0^t e^{-\int_s^t \frac{2}{u+1} du} \frac{1.55 \times 0.15}{0.85s+1} ds \longrightarrow \frac{1.55 \times 0.15}{2 \times 0.85} = 0.1368.$$

$$\int_0^t e^{-\int_s^t Q(u)du} |b_2(s)|\, |\tau_2'(s)| ds$$

$$= \int_0^t e^{-\int_s^t \frac{2}{u+1} du} \frac{0.45 \times 0.28}{0.72s+1} ds \longrightarrow \frac{0.45 \times 0.28}{2 \times 0.72} = 0.0875.$$

Thus, we have

$$\lim_{t\to\infty} \sum_{j=1}^{2} \Bigg\{ \int_{t-\tau_j(t)}^t |b_j(g_j(s))| ds + \int_0^t e^{-\int_s^t Q(u)du} |Q(s)| \int_{s-\tau_j(s)}^s |b_j(g_j(v))| dv ds + \int_0^t e^{-\int_s^t Q(u)du} |b_j(s)|\, |\tau_j'(s)| ds \Bigg\} = 1.0237.$$

Thus there exists some $t_0 > 0$ such that $t \geq t_0$,

$$\sum_{j=1}^{2} \Bigg\{ \int_{t-\tau_j(t)}^t |b_j(g_j(s))| ds + \int_0^t e^{-\int_s^t Q(u)du} |Q(s)| \int_{s-\tau_j(s)}^s |b_j(g_j(v))| dv ds + \int_0^t e^{-\int_s^t Q(u)du} |b_j(s)|\, |\tau_j'(s)| ds \Bigg\} > 1.02.$$

This implies that condition (1.9) does not hold. Thus, Theorem C cannot be applied to equation (3.1).

However, choosing $\tau_0(t) = \tau_1(t) = 0.15t$ and $h(t) = \frac{2.3}{t+1}$, we have

$$\int_{t-\tau_0(t)}^t |h(s)| ds = \int_{0.85t}^t \frac{2.3}{s+1} ds = 2.3 \ln \frac{t+1}{0.85t+1} < 0.3738,$$

$$\int_0^t e^{-\int_s^t h(u)du} |h(s)| \int_{s-\tau_0(s)}^s |h(v)| dv ds < 0.3738,$$

$$
\begin{aligned}
&\int_0^t e^{-\int_s^t h(u)du}|h(s-\tau_0(s))(1-\tau_0'(s))-b_1(s)-b_2(s)|ds\\
&\quad=\int_0^t e^{-\int_s^t \frac{2.3}{u+1}du}\left(\frac{0.45}{0.72s+1}-\frac{0.405}{0.85s+1}\right)ds\\
&\quad=\int_0^t e^{-\int_s^t \frac{2.3}{u+1}du}\frac{0.0909s+0.045}{(0.72s+1)(0.85s+1)}ds\\
&\quad\le\frac{0.0909}{0.72}\int_0^t e^{-\int_s^t \frac{2.3}{u+1}du}\times\frac{1}{0.85s+1}ds\le\frac{0.0909}{2.3\times 0.72\times 0.85}<0.0646.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
\left|\int_{t-\tau_1(t)}^{t-\tau_2(t)}(|b_1(s)|+|b_2(s)|)ds\right|&=\int_{0.72s}^{0.85s}\left(\frac{1.55}{0.85s+1}+\frac{0.45}{0.72s+1}\right)ds\\
&\le\left(\frac{1.55}{0.85}+\frac{0.45}{0.72}\right)\ln\frac{0.85}{0.72}<0.4065,
\end{aligned}
$$

so

$$
\begin{aligned}
&\int_0^t e^{-\int_s^t h(u)du}|b_2(s)|\left|\int_{s-\tau_1(s)}^{s-\tau_2(s)}(|b_1(v)|+|b_2(v)|)dv\right|ds\\
&\qquad\le 0.4065\int_0^t e^{-\int_s^t \frac{2.3}{u+1}du}\frac{0.45}{0.72s+1}ds\le\frac{0.45\times 0.4065}{2.3\times 0.72}<0.1105.
\end{aligned}
$$

Let $\alpha := 0.3738+0.3738+0.0646+0.1105=0.9227<1$, then the zero solution of (3.1) is asymptotically stable by Theorem 2.1.

*Example 3.2.* Consider the following equation

$$
x'(t)=-b_1(t)x(t)-b_2(t)x(t-\tau(t)), \tag{3.2}
$$

where $\tau(t)=0.14t$, $b_1(t)=\frac{0.84}{0.87t+1}$, $b_2(t)=\frac{0.84}{0.86t+1}$. Obviously,

$$
\lim_{t\to\infty}\int_0^t e^{-\int_s^t b_1(u)du}|b_2(s)|ds=\frac{0.87}{0.86},
$$

hence there exists some $t_0>0$ such that $t\ge t_0$,

$$
\int_0^t e^{-\int_s^t b_1(u)du}|b_2(s)|ds>1.
$$

This implies that condition (1.12) does not hold. Thus, Theorem D cannot be applied to equation (3.2).

However, choosing $\tau_0(t) = 0.13t$ and $h(t) = \frac{1.92}{t+1}$, by calculation similar to Example 3.1, we have

$$\begin{aligned}\int_{t-\tau_0(t)}^{t} |h(s)|ds + \int_0^t e^{-\int_s^t h(u)du}|h(s)| \int_{s-\tau_0(s)}^{s} |h(v)|dvds \\ + \int_0^t e^{-\int_s^t h(u)du}\big|h(s-\tau_0(s))(1-\tau_0'(s)) - b_1(s) - b_2(s)\big|ds \\ + \int_0^t e^{-\int_s^t h(u)du}|b_1(s)+b_2(s)|\left|\int_{s-\tau_0(s)}^{s} \big(|b_1(v)|+|b_2(v)|\big)dv\right|ds \\ + \int_0^t e^{-\int_s^t h(u)du}|b_2(s)|\left|\int_s^{s-\tau(s)} \big(|b_1(v)|+|b_2(v)|\big)dv\right|ds < 0.9913,\end{aligned}$$

then the zero solution of (3.2) is asymptotically stable by Theorem 2.1.

Acknowledgement. The authors are grateful to the referees for their valuable suggestions and comments.

## References


[1] T. A. Burton, Stability and Periodic Solutions of Ordinary and Functional Differential Equations, *Academic Press*, *New York*, 1985.

[2] T. A. Burton and T. Furumochi, Krasnoselskii's fixed point theorem and stability, *Nonlinear Anal.* **49** (2002), 445–454.

[3] T. A. Burton, Stability by fixed point theory or Liapunov's theory: a comparison, *Fixed Point Theory* **4** (2003), 15–32.

[4] T. A. Burton, Stability by Fixed Point Theory for Functional Differential Equations, *Dover Publications, New York*, 2006.

[5] J. R. Graef, C. Qian and B. Zhang, Asymptotic behavior of solutions of differential equations with variable delays, *Proc. London Math. Soc.* **81** (2000), 72–92.

[6] T. Hara, T. Yoneyama and R. Miyazaki, Some refinements of Razumikhin's method and their applications, *Funkcial. Ekvac.* **35** (1992), 279–305.

[7] C. H. Jin and J. W. Luo, Fixed points and stability in neutral differential equations with variable delays, *Proc. Amer. Math. Soc.* **136** (2008), 809–818.

[8] T. Krisztin, On the stability properties for one dimensional functional equations, *Funkcial. Ekvac.* **34** (1991), 241–256.

[9] Y. Muroya, On Yoneyama's $\frac{3}{2}$ stability theorems for one-dimensional delay differential equations, *J. Math. Anal. Appl.* **247** (2000), 314–322.

[10] Y. N. Raffoul, Stability in neutral nonlinear differential equations with functional delays using fixed-point theory, *Math. Comput. Modelling* **40** (2004), 691–700.

[11] T. Yoneyama, On the stability for the delay-differential equation $x'(t) = -a(t)f(x(t-r(t)))$, *J. Math. Anal. Appl.* **120** (1986), 271–275.

[12] T. Yoneyama, On the $\frac{3}{2}$ stability theorem for one-dimensional functional delay-differential equations, *J. Math. Anal. Appl.* **125** (1987), 161–173.

[13] T. Yoneyama, The $\frac{3}{2}$ stability theorem for one-dimensional delay-differential equations with unbounded delay, *J. Math. Anal. Appl.* **165** (1992), 133–143.

[14] J. A. Yorke, Asymptotic stability for one-dimensional functional differential-delay equations, *J. Differential Equations* **7** (1970), 189–202.

[15] B. Zhang, Contraction mapping and stability in a delay-differential equation, *Dynam. Systems and Appl.* **4** (2004), 183–190.

[16] B. Zhang, Fixed points and stability in differential equations with variable delays, *Nonlinear Anal.* **63** (2005), e233–e242.

CHUHUA JIN
FACULTY OF APPLIED MATHEMATICS
GUANGDONG UNIVERSITY OF TECHNOLOGY
GUANGZHOU, GUANGDONG 510006
P.R. CHINA

*E-mail:* jinchuhua@tom.com

JIAOWAN LUO
SCHOOL OF MATHEMATICS
AND INFORMATION SCIENCE
GUANGZHOU UNIVERSITY
GUANGZHOU, GUANGDONG 510006
P.R. CHINA

*E-mail:* mathluo@yahoo.com

*(Received March 12, 2009; revised June 2, 2010)*