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On some fixed point theorems on expansion mappings

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Abstract. In this note we give a unified approach to some fixed point theorems on expansion mappings.

1. Some recent papers [4,5,6,7,9] are concerned with fixed point theorems of expansion mappings on complete metric spaces.

Then purpose of this paper is to unify all these results giving a unified approach to fixed point theorems on expansion mappings. We will prove some common fixed point theorems for two expansion type mappings and as corollaries of our theorems we will obtain some results from [4,5,6,7,9].

2. In order to generalize the Banach fixed point theorem many authors [2,3,8] have introduced several generalized contractions. Many fixed point theorems on expansion mappings are duals of some results on contractive mappings [4,5].

We remind that in [2] Delbosco considered the set $\mathcal{G}$ of all continuous functions $g : [0,\infty)^3 \to [0,\infty)$ satisfying the conditions:

(i) $g(1,1,1) = h < 1$;

(ii) if $u, v \in [0,\infty)$ are such that $u \le g(v,v,u)$ or $u \le g(u,v,v)$ or $u \le g(v,u,v)$ then $u \le hv$;

in order to give a unified approach for contractive mappings:

Theorem A [2]**.** *Let S and T be two mappings of a complete metric space (X,d) into itself satisfying the inequality*

$$d(Sx,Ty) \le g(d(x,y), d(x,Sx), d(y,Ty))$$

for all $x, y \in X$, where $g \in \mathcal{G}$. Then S and T have a unique common fixed point.

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We introduce now two classes of functions:

Definition 1. Let ξ be the class of all continuous functions $g : [0,\infty)^3 \to [0,\infty)$ with the property that there exists $h > 1$ such that if $u, v \in (0,\infty)$ satisfy

$$u \geq g(v,u,v) \quad \text{or} \quad u \geq g(v,v,u)$$

then $u \geq hv$.

Definition 2. Let ξ' be the subclass of ξ consisting of the functions $g \in \xi$ satisfying the additional condition: if $x \in [0,\infty)$ satisfies

$$\min\{g(x,x,0), g(x,0,x)\} = 0$$

then $x = 0$.

We see that ξ' is a proper subclass of ξ since the function $g : [0,\infty)^3 \to [0,\infty)$, $g(x_1,x_2,x_3) = 2 \min\{x_1,x_2,x_3\}$ belongs to the class ξ but does not belong to the class ξ'. However, the class ξ' is not empty (we can see this observing that the function $g : [0,\infty)^3 \to [0,\infty)$, $g(x_1,x_2,x_3) = \frac{1}{2}(x_1 + x_2 + x_3)$ belongs to ξ').

The consideration of the class ξ' was suggested by the following observation: consider the dual set $\mathcal{G}'$ of Delbosco's set $\mathcal{G}$. The set $\mathcal{G}'$ is that of all continuous functions $g : [0,\infty)^3 \to [0,\infty)$ satisfying the conditions:

(i)′ $g(1,1,1) = h > 1$;

(ii)′ if $u, v \in [0,\infty)$ are such that $u \geq g(v,v,u)$ or $u \geq g(u,v,v)$ or $u \geq g(v,u,v)$ then $u \geq hv$. We observe that $\mathcal{G}' \subset \xi'$.

3. In this section we will prove the main results of the note:

Theorem 1. *Let (X,d) be a complete metric space and let $T, S : X \to X$ be surjective continuous mappings. If there exists a $g \in \xi'$ such that*

$$d(Sx,Ty) \geq g(d(x,y), d(x,Sx), d(y,Ty))$$

for all $x, y \in X$ with $x \neq y$, then S and T have a common fixed point.

Proof. Let $x_0 \in X$. We define the sequence $\{x_n\}$ as follows:

$$x_0 = Sx_1,\ x_1 = Tx_2, \ldots, x_{2n} = Sx_{2n+1},\ x_{2n+1} = Tx_{2n+2},\ \ldots$$

(this construction is possible because S and T are surjective mappings).

Suppose that for some $n \geq 0$ we have $x_{2n} = x_{2n+1}$. For $x_{2n+1} \neq x_{2n+2}$ we would obtain from the hypothesis that

$$0 = d(x_{2n}, x_{2n+1}) = d(Sx_{2n+1}, Tx_{2n+2}) \geq$$
$$\geq g(d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, Sx_{2n+1}), , d(x_{2n+2}, Tx_{2n+2})) =$$
$$= g(d(x_{2n+1}, x_{2n+2}), 0, d(x_{2n+1}, x_{2n+2})),$$

thus $d(x_{2n+1}, x_{2n+2}) = 0$ since $g \in \xi'$. This is a contradiction. In a similar way we prove that if there is a $k \geq 0$ with $x_{2k+1} = x_{2k+2}$, then $x_{2k+1} = x_{2k+2} = x_{2k+3}$. Thus we have that if the sequence $\{x_n\}$ has two consecutive terms equal, then S and T have a common fixed point.

Suppose now that $x_{2n} \neq x_{2n+1}$ and $x_{2n+1} \neq x_{2n+2}$ for all $n \geq 0$. We obtain by the hypothesis that

$$\begin{aligned} d(x_{2n}, x_{2n+1}) = d(Sx_{2n+1}, Tx_{2n+2}) \geq \\ \geq g(d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, Sx_{2n+1}), d(x_{2n+2}, Tx_{2n+2})) = \\ = g(d(x_{2n+1} + x_{2n+2}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})), \end{aligned}$$

thus $d(x_{2n}, x_{2n+1}) \geq hd(x_{2n+1}, x_{2n+2})$.

Similarly, we prove that $d(x_{2n+1}, x_{2n+2}) \geq h\ d(x_{2n+2}, x_{2n+3})$.

Since $h > 1$ we deduce that $\{x_n\}$ is a Cauchy sequence in X. Thus (X being a complete metric space) we have that there exists $\lim\limits_{n\to\infty} x_n = x \in X$. From the relations $x_{2n} = Sx_{2n+1}$, $x_{2n+1} = Tx_{2n+2}$ we obtain that $x = Sx = Tx$ since S and T are continuous. This completes the proof of Theorem 1.

Remark 1. If instead of $g \in \xi'$ we assume $g \in \xi$ in Theorem 1 we obtain that S or T has a fixed point.

Taking now $g \in \xi$ of a particular form, we obtain some recent results on fixed points of expansion mappings:

– if $g(x_1, x_2, x_3) = \sqrt{k \min\{x_1x_2, x_1x_3, x_2x_3\}}$ where $k > 1$, we obtain for $S = T$ a result of Popa [7];

– if $g(x_1, x_2, x_3) = \sqrt{k \min\{x_2^2, x_3^2, x_1x_2, x_1x_3\}}$ where $k > 1$, we obtain for $S = T$ a fixed point theorem of Popa [7].

Theorem 2. *Let (X, d) be a complete metric space and let $T, S : X \to X$ be surjective mappings. If there exists a $g \in \xi'$ such that*

$$d(Sx, Ty) \geq g(d(x, y), d(x, Sx), d(y, Ty))$$

for all $x, y \in X$ with $x \neq y$, then S and T have a common fixed point.

Proof. As in the proof of Theorem 1, we show that the sequence $\{x_n\}$ defined recurrently by

$$x_0 = Sx_1,\ x_1 = Tx_2,\ \dots,\ x_{2n} = Sx_{2n+1},\ x_{2n+1} = Tx_{2n+2},\ \dots$$

is a Cauchy sequence in X. Let $x = \lim\limits_{n\to\infty} x_n$.

Since S is onto, there exists a point $y \in X$ such that $Sy = x$. If $y = x_{2n+2}$ for infinitely many n, we get $y = x$ since $\lim\limits_{n\to\infty} x_n = x$, thus

$Sx = x$. If $y \neq x_{2n+2}$ for infinitely many n, we get

$$d(x, x_{2n+1}) = d(Sy, Tx_{2n+2}) \geq$$
$$\geq g(d(y, x_{2n+2}), d(y, Sy), d(x_{2n+2}, Tx_{2n+2})) =$$
$$= g(d(y, x_{2n+2}), d(y, x), d(x_{2n+2}, x_{2n+1})).$$

Letting $n \to \infty$, we obtain

$$0 \geq g(d(x, y), d(x, y), 0),$$

thus $x = y$ since $g \in \xi'$. It follows that $Sx = x$. Similarly we can prove that $Tx = x$. This completes the proof of the theorem.

Taking $g \in \xi'$ of a particular form, we obtain several fixed point theorems:

– if $g(x_1, x_2, x_3) = hx_1$ where $h > 1$, we obtain for $S = T$ a theorem of Gao, Iseki, Li, Wang [4] and a theorem of Gillespie and Williams [5];

– if $g(x_1, x_2, x_3) = ax_1 + bx_2 + cx_3$ where a, b, c are non-negative reals with $a+b+c > 1$, $b < 1$ and $c < 1$, we obtain a theorem of Taniguchi [9];

– if $g(x_1, x_2, x_3) = \sqrt{ax_1^2 + bx_2^2 + cx_3^2}$ where a, b, c are non-negative reals with $b < 1$, $c < 1$ and $a + b + c > 1$, we obtain for $S = T$ a result of Popa [7];

– if $g(x_1, x_2, x_3) = k\dfrac{x_2^2 + x_3^2 + x_2x_3}{x_2 + x_3}$ for $x_2+x_3 > 0$ and $g(x_1, 0, 0) = 0$ where $k \in \left(\frac{2}{3}, 1\right)$, we obtain for $S = T$ a result of Popa [7].

Remark 2. The fixed point may be not unique in Theorem 1 and Theorem 2. This can be seen letting $Sx = Tx = x$ and considering $g \in \xi'$

$$g(x_1, x_2, x_3) = \frac{1}{2}x_1 + x_2 + x_3.$$

4. It is of interest to investigate the consistency with respect to our recent result [1]. It is easy to see that if $T : X \to X$ is a bijection on a complete metric space (X, d) which is an α-contraction then $T^{-1} : X \to X$ is a $\frac{1}{\alpha}$-expansion.

The natural question arises whether a bijection satisfying the contractive conditions from [1] has the property that T^{-1} satisfies the expansive conditions of the present paper. That this is not the case can be seen from the following

Example. Let us consider

$$T : [0, \infty) \to [0, \infty), \quad Tx = \ln(1 + x).$$

Since

$$|\ln(1+x)-\ln(1+y)| \le \ln(1+|x-y|), \quad x,y\in[0,\infty),$$

we can apply the result of [1].

We will show that there is no $g\in\xi$ such that T^{-1} satisfies the conditions of Theorem 1 for this $g\in\xi$.

Suppose that there is a $g\in\xi$ such that

$$|e^x-e^y| \ge g(|x-y|, e^x-x-1, e^y-y-1)$$

for all $x,y\in[0,\infty)$, $x\neq y$. Taking $y=\ln(1+x)$ we would get that

$$e^x-x-1 \ge g(x-\ln(1+x), e^x-x-1, x-\ln(1+x))$$

and by the properties of $g\in\xi$ we obtain the existence of a $h>1$ such that

$$e^x-x-1 \ge h(x-\ln(1+x)), \quad x\in(0,\infty),$$

which is in contradiction with

$$\lim_{x\to 0}\frac{e^x-x-1}{x-\ln(1+x)} = 1.$$

We proved so that there exists no $g\in\xi$ such that T^{-1} satisfies the conditions of Theorem 1 for this $g\in\xi$.

Remark 3. It is easy to see that the result of [1] differs from the present results by observing that in [1] the continuity condition is essential whereas in Theorem 2 we do not require continuity conditions.

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