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On a class of locally dually flat Finsler metrics of isotropic flag curvature

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Abstract. In this paper, we characterize a class of locally dually flat (α, β) metrics $F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$ defined by a Riemannian metric α and a non-zero 1-form β, where ϵ and k are non-zero constants. As an application, we prove that there is no locally dually flat metric in the form $F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$ $(\epsilon \neq 0,\ k \neq 0,\ \beta \neq 0)$ with isotropic S-curvature unless it is Minkowskian. Moreover, we prove that if $F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$ $(\epsilon \neq 0,\ k \neq 0,\ \beta \neq 0)$ is locally dually flat, then it is locally projectively flat if and only if it is of constant flag curvature, and there is no locally dually flat metrics in the form $F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$ $(\epsilon \neq 0,\ k \neq 0,\ \beta \neq 0)$ of isotropic flag curvature unless it is Minkowskian.

1. Introduction

Locally dually flat Finsler metrics are studied in information geometry and the notion of locally dually flat Finsler metrics is introduced in ([Sh1]). A Finsler metric $F = F(x, y)$ on an n-dimensional manifold M is called locally dually flat if at every point there is a coordinate system (x^i) in which the spray coefficients are in the following form

$$G^i = -\frac{1}{2}g^{ij}H_{y^j}, \tag{1.1}$$

where $H = H(x, y)$ is a local scalar function on the tangent bundle TM of M. Such a coordinate system is called an adapted coordinate system. In [Sh1], the

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author proved that a Finsler metric $F = F(x, y)$ on an open subset $\mathcal{U} \subset \mathbf{R}^n$ is dually flat if and only if it satisfies the following PDE

$$[F^2]_{x^k y^l} y^k - 2[F^2]_{x^l} = 0. \tag{1.2}$$

In this case, $H = -\frac{1}{6}[F^2]_{x^m} y^m$. Locally dually flat Finsler metrics are studied in Finsler information geometry in [Sh1]. Recently, the classification of locally dually flat Randers metrics with almost isotropic flag curvature is given in [CSZ].

It is known that a Riemannian metric $F = \sqrt{g_{ij}(x) y^i y^l}$ is locally dually flat if and only if in an adapted coordinate system,

$$g_{ij} = \frac{\partial^2 \psi}{\partial x^i \partial x^j}(x),$$

where $\psi = \psi(x)$ is a C^∞ function ([AN], [Sh1]). The first example of non-Riemannian dually flat metrics is the Funk metric given as follows (cf. [Sh1], [CSZ]):

$$F = \frac{\sqrt{(1 - |x|^2)|y|^2 + \langle x, y\rangle^2}}{1 - |x|^2} \pm \frac{\langle x, y\rangle}{1 - |x|^2}. \tag{1.3}$$

This metric is defined on the unit ball $\mathbf{B}^n \subset \mathbf{R}^n$ and is a Randers metric with constant flag curvature $K = -\frac{1}{4}$. This is only known example of locally dually flat metrics with non-zero constant flag curvature up to a normalization. These facts inspire us to consider a class of (α, β) metrics on M, which is expressed in the following form

$$F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}, \tag{1.4}$$

where $\alpha = \sqrt{a_{ij} y^i y^j}$ is a Riemannian metric, $\beta = b_i y^i$ is a non-zero 1-form with $b = \|\beta(x)\|_\alpha < b_0$ for and $x \in M$, ϵ, k are non-zero constants such that

$$\alpha^2 + \epsilon\alpha\beta + k\beta^2 > 0, \quad \alpha^2 + 2kb^2\alpha^2 - 3k\beta^2 > 0, \quad \left|\frac{\beta}{\alpha}\right| \leq b < b_0. \tag{1.5}$$

These metrics have been extensively studied (cf. [Sh1], [SY] and references therein). We firstly give an equivalent characterization (Theorem 3.1) of locally dually flat metrics (1.4) and give some applications. As one of applications of Theorem 3.1, we prove that if β is parallel with respect to α, then $F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$ ($\epsilon \neq 0$, $k \neq 0$, $\beta \neq 0$) is locally dually flat if and only if α is flat. In this case, F is Minkowskian.

The S-curvature S is an important non-Riemannian quantity in Finsler geometry ([CS], [Sh4], [ChS]). A Finsler metric F is said to be of isotropic S-curvature if $S = (n + 1)c(x)F$, where $c(x)$ is a scalar function on M. Another application of Theorem 3.1 shows there is no locally dually flat Finsler metric $F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$ $(\epsilon \neq 0,\ k \neq 0,\ \beta \neq 0)$ with isotropic S-curvature unless it is Minkowskian.

Let's recall another notion of locally projectively flat Finsler metrics. A Finsler metric $F = F(x, y)$ is called locally projectively flat if at every point there is a coordinate system (x^i) in which all geodesics are straight lines, or equivalently, the spray coefficients are in the following form

$$G^i = Py^i, \tag{1.6}$$

where $P = P(x, y)$ is a local scalar function. Locally projectively flat metrics have been studied extensively (see [Sh2], [Sh3], [LS], [SY], etc. and the references therein). In [CSZ], authors proved that every dually flat and projectively flat metric on an open subset $\mathcal{U}$ in $\mathbf{R}^n$ must be either a Minkowski metric or a Funk metric after a normalization. A natural question is when a dually flat metric on $\mathcal{U}$ is projectively flat. For the metric in the form (1.4), if it is locally dually flat, then it is projectively flat if and only if it is of constant flag curvature (Theorem 4.1).

The main purpose of this paper is to classify locally dually flat metrics in the form (1.4) with isotropic flag curvature. We prove that there exists no locally dually flat metric in the form (1.4) of isotropic flag curvature (especially constant flag curvature) unless it is Minkowskian (Theorem 4.2).

This paper is arranged as follows. Firstly we give an introduction of locally dually flat (α, β) metric in §2. In §3, we obtain an equivalent characterization for locally dually flat Finsler metric with the form (1.4) (see Theorem 3.1) and give some applications of Theorem 3.1. Finally, in §4, we prove that if the metric (1.4) is locally dually flat, then it is projectively flat if and only if it is of constant flag curvature (see Theorem 4.1). Moreover, we prove that the metric (1.4) is a locally dually flat metric of isotropic flag curvature if and only if $\epsilon^2 = 4k$, α is flat and β is parallel with respect to α. In this case, F is locally isometric to a Minkowski metric $\overline{F} = \frac{(|y| \pm \sqrt{k} b_i y^i)^2}{|y|}$, where $|\cdot|$ is Euclidean metric in $\mathbf{R}^n$ and $b_i (1 \leq i \leq n)$ are constants(see Theorem 4.2).

In the following, we will use Einstein sum convention.

2. (α, β)-metrics

Let M be an n-dimensional smooth manifold. We denote by TM the tangent bundle of M and by $(x, y) = (x^i, y^i)$ the local coordinates on the tangent bundle TM. A Finsler manifold (M, F) is a smooth manifold equipped with a function $F : TM \to [0, \infty)$, which has the following properties

(i) Regularity: F is smooth in $TM \setminus \{0\}$.

(ii) Positively homogeneity: $F(x, \lambda y) = \lambda F(x, y)$, for $\lambda > 0$.

(iii) Strong convexity: the Hessian matrix of F^2, $(g_{ij}(x, y)) := \frac{1}{2}\big(\frac{\partial^2 F^2(x,y)}{\partial y^i \partial y^j}\big)$, is positive definite on $TM \setminus \{0\}$. We call F and the tensor g_{ij} the Finsler metric and the fundamental tensor of M respectively.

In Finsler geometry, $(\alpha,\ \beta)$-metric is a class of important Finsler metric. By definition, an $(\alpha,\ \beta)$-metric is expressed as the following form,

$$F = \alpha\phi(s), \quad s := \frac{\beta}{\alpha}, \tag{2.1}$$

where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form. $\phi = \phi(s)$ is a C^∞ positive function on an open interval $(-b_0, b_0)$ satisfying

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \ |s| \le b < b_0, \tag{2.2}$$

where $b := \|\beta(x)\|_\alpha$. It is known that $F = \alpha\phi(s)$ is a Finsler metric if and only if $\|\beta(x)\|_\alpha < b_0$ for any $x \in M$ ([CS]). In particular, if $\phi(s) = 1 + s$, then (α, β)-metric is a Randers metric. If $\phi(s) = 1+\epsilon s+ks^2$, then (α, β)-metric is exactly the metric in the form (1.4). Let $G^i(x, y)$ and $G^i_\alpha(x, y)$ denote the spray coefficients of F and α, respectively. To express formulae for the spray coefficients G^i of F in terms of α and β, we need to introduce some notations. Let $b_{i;j}$ be a covariant derivative of b_i with respect to α. Denote

$$r_{ij} := \frac{1}{2}(b_{i;j} + b_{j;i}), \quad s_{ij} := \frac{1}{2}(b_{i;j} - b_{j;i}), \tag{2.3}$$

$$s^i{}_j := a^{ih}s_{hj}, \quad s_j := b_i s^i{}_j = s_{ij}b^i, \quad r_j = r_{ij}b^i, \tag{2.4}$$

$$r_0 := r_j y^j, \quad s_0 := s_j y^j, \quad r_{00} := r_{ij}y^i y^j. \tag{2.5}$$

Thus we have the following

Lemma 2.1 ([CS]). *The spray coefficients G^i are related to G^i_α by*

$$G^i = G^i_\alpha + \alpha Q s^i{}_0 + \Theta(-2\alpha Q s_0 + r_{00})\frac{y^i}{\alpha} + \Psi(-2\alpha Q s_0 + r_{00})b^i, \tag{2.6}$$

where

$$Q := \frac{\phi'}{\phi - s\phi'}, \tag{2.7}$$

$$\Theta := \frac{\phi'(\phi - s\phi')}{2\phi[(\phi - s\phi') + (b^2 - s^2)\phi'']} - s\Psi, \tag{2.8}$$

$$\Psi := \frac{\phi''}{2[(\phi - s\phi') + (b^2 - s^2)\phi'']}, \tag{2.9}$$

here $b^i := a^{ij}b_j$ and $b^2 := a^{ij}b_i b_j = b_j b^j$.

From (1.2), we can prove the following

Lemma 2.2. *An (α,β)-metric $F=\alpha\phi(s)$, where $s=\frac{\beta}{\alpha}$, is dually flat on an open subset $\mathcal{U}\subset\mathbf{R}^n$ if and only if*

$$2\alpha^2 a_{ml}G^m_\alpha + Q(3s_{l0}-r_{l0})\alpha^3 - \alpha^2\left(y_m\frac{\partial G^m_\alpha}{\partial y^l} + \alpha Q b_m\frac{\partial G^m_\alpha}{\partial y^l}\right) + Q\alpha(r_{00}+2b_mG^m_\alpha)y_l$$
$$+\left[2Q(y_mG^m_\alpha) + \frac{\phi'^2+\phi\phi''}{\phi(\phi-s\phi')}(\alpha r_{00}+2(b_m\alpha-sy_m)G^m_\alpha)\right](\alpha b_l - sy_l)=0, \quad (2.10)$$

where $r_{i0}:=r_{ij}y^j$, $s_{i0}:=s_{ij}y^j$ and $y_i:=a_{ij}y^j$.

Proof. By direct computation, F is dually flat on $\mathcal{U}$ if and only if

$$\alpha\phi^2(\alpha_{x^ky^l}y^k-2\alpha_{x^l})+\phi^2\alpha_{y_l}(\alpha_{x^k}y^k)+\alpha^2\phi\phi'(s_{x^ky^l}y^k-2s_{x_l})$$
$$+2\alpha\phi\phi'(\alpha_{y^l}s_{x^k}y^k+s_{y^l}\alpha_{x_k}y^k)+\alpha^2(\phi'^2+\phi\phi'')(s_{x^k}y^k)s_{y^l}=0. \quad (2.11)$$

On the other hand,

$$\alpha_{x^l}=\frac{1}{\alpha}\frac{\partial G^m_\alpha}{\partial y^l}y_m,\quad \alpha_{x^k}y^k=\frac{2}{\alpha}G^m_\alpha y_m,\quad \alpha_l=\frac{y_l}{\alpha}, \quad (2.12)$$

$$s_{x^l}=\frac{1}{\alpha}b_{m;l}y^m+\frac{1}{\alpha^2}(\alpha b_m-sy_m)\frac{\partial G^m_\alpha}{\partial y^l},\quad s_{y^l}=\frac{\alpha b_l-sy_l}{\alpha^2}, \quad (2.13)$$

$$s_{x^k}y^k=\frac{r_{00}}{\alpha}+\frac{2}{\alpha^2}(\alpha b_m-sy_m)G^m_\alpha, \quad (2.14)$$

$$\alpha_{x^ky^l}y^k-2\alpha_{x^l}=\frac{2}{\alpha^3}(a_{ml}\alpha^2-y_my_l)G^m_\alpha-\frac{1}{\alpha}\frac{\partial G^m_\alpha}{\partial y^l}y_m, \quad (2.15)$$

$$\begin{aligned} s_{x^ky^l}y^k-2s_{x^l} &= -\frac{r_{00}}{\alpha^3}y_l+\frac{2}{\alpha}s_{l0}-\frac{4y_l}{\alpha^4}(\alpha b_m-sy_m)G^m_\alpha \\ &\quad+\frac{2}{\alpha^2}\left(\frac{y_l}{\alpha}b_m-\frac{\alpha b_l-sy_l}{\alpha^2}y_m-sa_{ml}\right)G^m_\alpha \\ &\quad-\frac{1}{\alpha}b_{m;l}y^m-\frac{1}{\alpha^2}(\alpha b_m-sy_m)\frac{\partial G^m_\alpha}{\partial y^l}. \end{aligned} \quad (2.16)$$

Putting (2.12)–(2.16) into (2.11) and noting $b_{m;l}y^m=r_{0l}+s_{0l}$ yields

$$2\phi(\phi-s\phi')\alpha^2a_{ml}G^m_\alpha+\phi\phi'(3s_{l0}-r_{l0})\alpha^3$$
$$-\alpha^2\phi\left[(\phi-s\phi')y_m\frac{\partial G^m_\alpha}{\partial y^l}+\alpha\phi'b_m\frac{\partial G^m_\alpha}{\partial y^l}\right]+\phi\phi'\alpha(r_{00}+2b_mG^m_\alpha)y_l$$
$$+[2\phi\phi'y_mG^m_\alpha+(\phi'^2+\phi\phi'')(\alpha r_{00}+2(\alpha b_m-sy_m)G^m_\alpha)](\alpha b_l-sy_l)=0.$$

This completes the proof. □

3. Locally dually flat Finsler metrics $F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$

In the following, we consider a class of special (α, β)-metrics on a manifold M^n defined by the following form

$$F = \alpha\phi(s), \qquad \phi(s) = 1 + \epsilon s + ks^2, \tag{3.1}$$

that is,

$$F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}, \tag{3.2}$$

where ϵ, k are constants, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. From (2.2), we have

$$1 + \epsilon s + ks^2 > 0, \quad 1 + 2kb^2 - 3ks^2 > 0, \quad |s| \le b < b_0. \tag{3.3}$$

F is a Finsler metric if and only if β satisfies that $b = \|\beta(x)\|_\alpha < b_0$ for any $x \in M$. From (3.1) and Lemma 2.2, we can prove the following

Theorem 3.1. *Let $F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$ be a Finsler metric on a manifold M, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric, $\beta = b_i(x)y^i$ is a non-zero 1-form and ϵ, k are non-zero constants. Then F is locally dually flat if and only if in an adapted coordinate system, α and β satisfy*

$$r_{00} = \frac{2}{3}[\theta\beta - (\theta_l b^l)\alpha^2], \tag{3.4}$$

$$s_{l0} = \frac{1}{3}(\beta\theta_l - \theta b_l), \tag{3.5}$$

$$G^l_\alpha = \frac{1}{3}(\alpha^2\theta^l + 2\theta y^l), \tag{3.6}$$

where $\theta := \theta_i(x)y^i$ is a 1-form on M and $\theta^l := a^{lm}\theta_m$.

Proof. If (3.4)–(3.6) hold, the locally dually flatness of F follows from Lemma 2.2 directly. Conversely, since $\phi(s) = 1 + \epsilon s + ks^2$, equation (2.10) is reduced to the following equation:

$$2A\alpha^2 a_{ml}G^m + 2B\alpha(b_m G^m_\alpha)y_l + B\alpha^3(3s_{l0} - r_{l0}) - A\alpha^2 y_m\frac{\partial G^m_\alpha}{\partial y^l} - B\alpha^3 b_m\frac{\partial G^m_\alpha}{\partial y^l}$$
$$+B\alpha r_{00}y_l + \{2By_m G^m_\alpha + C[\alpha r_{00} + 2(b_m\alpha - sy_m)G^m_\alpha]\}(\alpha b_l - sy_l) = 0, \tag{3.7}$$

where

$$A(s) := \phi(\phi - s\phi') = 1 + \epsilon s - \epsilon ks^3 - k^2s^4,$$
$$B(s) := \phi\phi' = \epsilon + (\epsilon^2 + 2k)s + 3k\epsilon s^2 + 2k^2s^3,$$
$$C(s) := \phi'^2 + \phi\phi'' = \epsilon^2 + 2k + 6\epsilon ks + 6k^2s^2.$$

Multiplying (3.7) by α^4 and rewriting this equation as a polynomial in α, noting

$\epsilon \neq 0$, then the sum of odd power and even power of α are zero respectively. Dividing both sides of the former by α, one get

$$\begin{aligned}&\left(3s_{l0} - r_{l0} - b_m\frac{\partial G^m_\alpha}{\partial y^l}\right)\alpha^6 + \bigg[(6k\beta b_l + y_l)(r_{00} + 2b_m G^m_\alpha)\\&\quad + 3k\beta^2\left(3s_{l0} - r_{l0} - b_m\frac{\partial G^m_\alpha}{\partial y^l}\right) + 2(y_m G^m_\alpha)b_l + 2\beta a_{ml}G^m_\alpha - \beta y_m\frac{\partial G^m_\alpha}{\partial y^l}\bigg]\alpha^4\\&\quad + \bigg[-3k\beta^2(r_{00} + 2b_m G^m_\alpha)y_l - 2\beta(3k\beta b_l + y_l)(y_m G^m_\alpha) - 2k\beta^3 a_{ml}G^m_\alpha\\&\quad + k\beta^3 y_m\frac{\partial G^m_\alpha}{\partial y^l}\bigg]\alpha^2 + 6k\beta^3 y_m G^m_\alpha y_l = 0, \qquad (3.8)\end{aligned}$$

$$\begin{aligned}&\bigg[(\epsilon^2 + 2k)(r_{00} + 2b_m G^m_\alpha)b_l + (\epsilon^2 + 2k)\beta\left(3s_{l0} - r_{l0} - b_m\frac{\partial G^m_\alpha}{\partial y^l}\right) + 2a_{ml}G^m_\alpha\\&\quad - y_m\frac{\partial G^m_\alpha}{\partial y^l}\bigg]\alpha^6 + \bigg[6k^2\beta^2(r_{00} + 2b_m G^m_\alpha)b_l + 2k^2\beta^3\left(3s_{l0} - r_{l0} - b_m\frac{\partial G^m_\alpha}{\partial y^l}\right)\bigg]\alpha^4\\&\quad + \bigg[-4k^2\beta^3(r_{00} + 2b_m G^m_\alpha)y_l - 8k^2\beta^3(y_m G^m_\alpha)b_l - 2k^2\beta^4 a_{ml}G^m_\alpha\\&\quad + k^2\beta^4 y_m\frac{\partial G^m_\alpha}{\partial y^l}\bigg]\alpha^2 + 8k^2\beta^4(y_m G^m_\alpha)y_l = 0. \qquad (3.9)\end{aligned}$$

Contracting (3.8) and (3.9) with b^l yield

$$\begin{aligned}&\left(3s_0 - r_0 - \frac{\partial(b_m G^m_\alpha)}{\partial y^l}b_l\right)\alpha^6 + \bigg[\beta(12kb^2 + 5)b_m G^m_\alpha + \beta(6kb^2 + 1)r_{00}\\&\quad + 3k\beta^2\left(3s_0 - r_0 - \frac{\partial(b_m G^m_\alpha)}{\partial y^l}b_l\right) + 2b^2 y_m G^m_\alpha - \beta\frac{\partial(y_m G^m_\alpha)}{\partial y^l}b^l\bigg]\alpha^4\\&\quad + \bigg[-3k\beta^3 r_{00} - 9k\beta^3 b_m G^m_\alpha - 2\beta^2(3kb^2 + 1)y_m G^m_\alpha + k\beta^3\frac{\partial(y_m G^m_\alpha)}{\partial y^l}b^l\bigg]\alpha^2\\&\quad + 6k\beta^4 y_m G^m_\alpha = 0, \qquad (3.10)\end{aligned}$$

$$\begin{aligned}&\bigg[(2\epsilon^2 b^2 + 4kb^2 + 3)b_m G^m_\alpha + (\epsilon^2 + 2k)b^2 r_{00}\\&\quad + (\epsilon^2 + 2k)\beta\left(3s_0 - r_0 - \frac{\partial(b_m G^m_\alpha)}{\partial y^l}b^l\right) - \frac{\partial(y_m G^m_\alpha)}{\partial y^l}b^l\bigg]\alpha^6\\&\quad + \bigg[6k^2\beta^2 b^2(r_{00} + 2b_m G^m_\alpha) + 2k^2\beta^3\left(3s_0 - r_0 - \frac{\partial(b_m G^m_\alpha)}{\partial y^l}b^l\right)\bigg]\alpha^4\end{aligned}$$

$$+\left[-4k^2\beta^4 r_{00}-11k^2\beta^4 b_m G^m_\alpha-8k^2\beta^3 b^2 y_m G^m_\alpha+k^2\beta^4\frac{\partial(y_m G^m_\alpha)}{\partial y^l}b^l\right]\alpha^2$$
$$+8k^2\beta^5 y_m G^m_\alpha=0. \tag{3.11}$$

(3.10) $\times\, 4k\beta$ – (3.11) $\times\, 3$ and dividing by α^2 on both sides yields

$$\begin{aligned}(\alpha^2-k\beta^2)(3\alpha^2-k\beta^2)\frac{\partial(y_m G^m_\alpha)}{\partial y^l}b^l+\alpha^2\beta[(3\epsilon^2+2k)\alpha^2-6k^2\beta^2]\frac{\partial(b_m G^m_\alpha)}{\partial y^l}b^l\\ =[(3\epsilon^2+2k)\beta(3s_0-r_0)+3(2\epsilon^2b^2+4kb^2+3)b_m G^m_\alpha+3(\epsilon^2+2k)b^2r_{00}]\alpha^4\\ -[4k\beta^2(3kb^2+5)b_m G^m_\alpha+8kb^2\beta y_m G^m_\alpha+2k\beta^2(3kb^2+2)r_{00}\\ +6k^2\beta^3(3s_0-r_0)]\alpha^2+[3k^2\beta^4 b_m G^m_\alpha+8k\beta^3 y_m G^m_\alpha].\end{aligned} \tag{3.12}$$

From (3.10),we get

$$\begin{aligned}\beta\alpha^2(\alpha^2-k\beta^2)\frac{\partial(y_m G^m_\alpha)}{\partial y^l}b^l+\alpha^4(\alpha^2+3k\beta^2)\frac{\partial(b_m G^m_\alpha)}{\partial y^l}b^l=(3s_0-r_0)\alpha^6\\ +[(12kb^2+5)\beta b_m G^m_\alpha+2b^2y_m G^m_\alpha+(6kb^2+1)\beta r_{00}+3k\beta^2(3s_0-r_0)]\alpha^4\\ [-3k\beta^3r_{00}-9k\beta^3b_m G^m_\alpha-2(3kb^2+1)\beta^2y_m G^m_\alpha]\alpha^2+6k\beta^4y_m G^m_\alpha.\end{aligned} \tag{3.13}$$

Since F is non-Riemannian, $\alpha^2+3k\beta^2\neq 0$ and $(3\epsilon^2+2k)\alpha^2-6k^2\beta^2\neq 0$, (3.12) $\times\,\alpha^2(\alpha^2+3k\beta^2)$ – (3.13) $\times\,\beta[(3\epsilon^2+2k)\alpha^2-6k^2\beta^2]$ yields

$$\begin{aligned}\alpha^2(\alpha^2-k\beta^2)[(\alpha^2+k\beta^2)^2-\epsilon^2\alpha^2\beta^2]\left[\frac{\partial(y_m G^m_\alpha)}{\partial y^l}b^l-3b_m G^m_\alpha\right]\\ =D(b^2\alpha^2-\beta^2)[\alpha^2r_{00}+2\alpha^2b_m G^m_\alpha-2\beta y_m G^m_\alpha],\end{aligned} \tag{3.14}$$

where $D:=(\epsilon^2+2k)\alpha^4-3k\epsilon^2\alpha^2\beta^2+6k^3\beta^4$. Noting that

$$D=(\epsilon^2+2k)[(\alpha^2+k\beta^2)^2-\epsilon^2\alpha^2\beta^2]+(\epsilon^2-4k)[(\epsilon^2+k)\alpha^2\beta^2-k\beta^4]. \tag{3.15}$$

Case I: $\epsilon^2=4k$.

In this case, $D=6k[(\alpha^2+k\beta^2)^2-4k\alpha^2\beta^2]=6k(\alpha^2-k\beta^2)^2\neq 0$ and (3.14) is reduced to the following

$$\begin{aligned}\alpha^2(\alpha^2-k\beta^2)\left[\frac{\partial(y_m G^m_\alpha)}{\partial y^l}b^l-3b_m G^m_\alpha\right]\\ =6k(b^2\alpha^2-\beta^2)[\alpha^2r_{00}+2\alpha^2b_m G^m_\alpha-2\beta y_m G^m_\alpha].\end{aligned} \tag{3.16}$$

(1) If b^2 is not identically equal to $\frac{1}{k}$, then $(b^2\alpha^2-\beta^2)$, $(\alpha^2-k\beta^2)$ and α^2 are all irreducible polynomials of (y^i) and one of them is not divisible by another one. Thus, there is a function $\sigma=\sigma(x)$ on M such that

$$\frac{\partial(y_m G^m_\alpha)}{\partial y^l}b^l-3b_m G^m_\alpha=\sigma(b^2\alpha^2-\beta^2), \tag{3.17}$$

$$\alpha^2 r_{00}+2\alpha^2 b_m G^m_\alpha-2\beta y_m G^m_\alpha=\frac{\sigma}{6k}\alpha^2(\alpha^2-k\beta^2). \tag{3.18}$$

From (3.18), we have

$$2\beta y_m G^m_\alpha=\left[r_{00}+2b_m G^m_\alpha-\frac{\sigma}{6k}(\alpha^2-k\beta^2)\right]\alpha^2. \tag{3.19}$$

Since α^2 does not contain the factor β, there exist a 1-form $\theta:=\theta_i y^i$ on M such that

$$y_m G^m_\alpha=\theta\alpha^2, \tag{3.20}$$

$$b_m G^m_\alpha=\theta\beta-\frac{1}{2}r_{00}+\frac{\sigma}{12k}(\alpha^2-k\beta^2). \tag{3.21}$$

From (3.17), (3.20) and (3.21), we obtain

$$r_{00}=\frac{2}{3}\theta\beta-\frac{5}{6}\sigma\beta^2+\frac{2}{3}(\sigma b^2+\frac{\sigma}{4k}-\theta_l b^l)\alpha^2, \tag{3.22}$$

$$\frac{\partial(y_m G^m_\alpha)}{\partial y^l}=\theta_l\alpha^2+2\theta y_l, \tag{3.23}$$

$$\frac{\partial(b_m G^m_\alpha)}{\partial y^l}=\theta_l\beta+\theta b_l-r_{l0}+\frac{\sigma}{6k}(y_l-k\beta b_l). \tag{3.24}$$

Using (3.20)–(3.24), (3.8)–(3.9) become

$$\begin{aligned}
&3\beta(\alpha^2-k\beta^2)a_{ml}G^m_\alpha+3\alpha^2(\alpha^2+3k\beta^2)s_{l0}\\
&\quad+\beta\left(2k\theta\beta^2-2\theta\alpha^2-\frac{7}{6}\sigma\alpha^2\beta+\frac{1}{2}k\sigma\beta^3\right)y_l-2\alpha^2\beta(\alpha^2+k\beta^2)\theta_l\\
&\quad+\frac{1}{2}\alpha^2\left(2\theta\alpha^2+6k\theta\beta^2-k\sigma\beta^3+\frac{7}{3}\sigma\alpha^2\beta\right)b_l=0,
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
&3(\alpha^2+k\beta^2)(\alpha^2-k\beta^2)a_{ml}G^m_\alpha+6k\alpha^2\beta(3\alpha^2+k\beta^2)s_{l0}\\
&\quad+\left(2k^2\theta\beta^4-\sigma\alpha^4\beta+\frac{2}{3}k^2\sigma\beta^5-2\theta\alpha^4-k\sigma\alpha^2\beta^3\right)y_l\\
&\quad-\alpha^2(\alpha^4+6k\alpha^2\beta^2+k^2\beta^4)\theta_l\\
&\quad+\alpha^2\left(2k^2\theta\beta^3+6k\theta\alpha^2\beta+\sigma\alpha^4-\frac{2}{3}k^2\sigma\beta^4+k\sigma\alpha^2\beta^2\right)b_l=0.
\end{aligned} \tag{3.26}$$

Solving equations (3.25)–(3.26), we get

$$a_{ml}G^m_\alpha = \frac{1}{3}(2\theta + \sigma\beta)y_l + \frac{1}{3}(\theta_l - \sigma b_l)\alpha^2, \tag{3.27}$$

$$s_{l0} = \frac{1}{3}(\beta\theta_l - \theta b_l) + \frac{1}{18\alpha^2}\sigma\beta^2 y_l - \frac{1}{18}\sigma\beta b_l. \tag{3.28}$$

(3.28) implies

$$s_{lk} = \frac{1}{3}(b_k\theta_l - \theta_k b_l) + \frac{\sigma}{18\alpha^4}(-\alpha^4 b_k b_l + 2\alpha^2\beta b_k y_l - 2\beta^2 y_k y_l + a_{lk}\alpha^2\beta^2). \tag{3.29}$$

Since s_{lk} is anti-symmetric with respect to l and k, we have

$$\sigma[\alpha^4 b_k b_l - \alpha^2\beta(b_k y_l + b_l y_k) + 2\beta^2 y_k y_l - a_{lk}\alpha^2\beta^2] = 0. \tag{3.30}$$

Contracting (3.30) with b^k yields

$$\sigma(b^2\alpha^2 - 2\beta^2)(\alpha^2 b_l - \beta y_l) = 0. \tag{3.31}$$

(3.31) implies

$$\sigma\alpha^2 b_l = \sigma\beta y_l, \tag{3.32}$$

because of $(b^2\alpha^2 - 2\beta^2) \neq 0$. From (3.32), we have $\sigma b^2\alpha^2 = \sigma\beta^2$, which implies $\sigma = 0$, because of α not including the factor β. Thus (3.22), (3.27) and (3.28) imply (3.4),(3.5) and (3.6).

(2) If b^2 is equal to $\frac{1}{k}$ everywhere, then (3.16) is reduced to the following

$$\alpha^2\left[\frac{\partial(y_m G^m_\alpha)}{\partial y^l}b^l - 3b_m G^m_\alpha\right] = 6k[\alpha^2 r_{00} + 2\alpha^2 b_m G^m_\alpha - 2\beta y_m G^m_\alpha]. \tag{3.33}$$

From (3.33), $y_m G^m_\alpha$ must be divisible by α^2. Consequently, there is a 1-from $\theta = \theta_i y^i$ on M, such that

$$y_m G^m_\alpha = \theta\alpha^2. \tag{3.34}$$

Plugging (3.34) into (3.33) yields

$$b_m G^m_\alpha = \frac{1}{15}(\theta_0\alpha^2 + 14\theta\beta - 6r_{00}), \tag{3.35}$$

where $\theta_0 = \theta_i b^i$. Using (3.34)–(3.35), (3.8)–(3.9) are reduced to

$$3\beta(\alpha^2 - k\beta^2)a_{ml}G^m_\alpha + 3\alpha^2(\alpha^2 + 3k\beta^2)s_{l0} - \frac{1}{5}\alpha^2(\alpha^2 + 3k\beta^2)r_{l0}$$

$$+\frac{1}{15}[3(\alpha^2-3k\beta^2)r_{00}-4\beta(8\alpha^2-9k\beta^2)\theta-12k\alpha^2\beta^2\theta_0]y_l$$
$$-\frac{1}{15}\alpha^2\beta(29\alpha^2+27k\beta^2)\theta_l+\frac{2}{15}[9k\alpha^2\beta r_{00}+2\alpha^2(4\alpha^2+9k\beta^2)\theta$$
$$+6k\alpha^4\beta\theta_0]b_l=0, \tag{3.36}$$

$$3(\alpha^2+k\beta^2)(\alpha^2-k\beta^2)a_{ml}G^m_\alpha+6k\alpha^2\beta(3\alpha^2+k\beta^2)s_{l0}$$
$$-\frac{2}{5}k\alpha^2\beta(3\alpha^2+k\beta^2)r_{l0}-\frac{2}{15}[6k^2\beta^3r_{00}+(15\alpha^4-19k^2\beta^4)\theta$$
$$+6k\alpha^2\beta(\alpha^2+k\beta^2)\theta_0]y_l-\frac{1}{15}\alpha^2(15\alpha^4+84k\alpha^2\beta^2+13k^2\beta^4)\theta_l$$
$$+\frac{2}{5}\left[3k\alpha^2(\alpha^2+k\beta^2)r_{00}+\frac{2}{3}k\alpha^2\beta(21\alpha^2+5k\beta^2)\theta\right.$$
$$\left.+2k\alpha^4(\alpha^2+k\beta^2)\theta_0\right]b_l=0. \tag{3.37}$$

Solving (3.36) and (3.37), we get

$$a_{ml}G^m_\alpha=\frac{\alpha^2}{3}\theta_l+\frac{2}{15(\alpha^2-k\beta^2)}[(3k\beta r_{00}+5\alpha^2\theta-7k\beta^2\theta+2k\alpha^2\beta\theta_0)y_l$$
$$-k\alpha^2(3r_{00}+2\alpha^2\theta_0+2\beta\theta)b_l]; \tag{3.38}$$

$$s_{l0}=\frac{1}{15}r_{l0}-\frac{16}{45}\theta b_l+\frac{14}{45}\beta\theta_l+\frac{2\beta\theta-3r_{00}}{45\alpha^2}y_l. \tag{3.39}$$

From (3.39), we have

$$s_{lk}=\frac{1}{45\alpha^4}\big\{\alpha^4(3r_{lk}+14b_k\theta_l-16\theta_kb_l)+\alpha^2[(2\beta\theta-3r_{00})a_{lk}-6r_{k0}y_l$$
$$+2\theta b_ky_l+2\beta\theta_ky_l]-2(2\beta\theta-3r_{00})y_ky_l\big\}. \tag{3.40}$$

Using $s_{lk}=-s_{kl}$, we obtain

$$\alpha^4(3r_{lk}-\theta_kb_l-\theta_lb_k)+\alpha^2[(2\beta\theta-3r_{00})a_{lk}-3(r_{k0}y_l+r_{l0}y_k)$$
$$+\theta(b_ky_l+b_ly_k)+\beta(\theta_ky_l+\theta_ly_k)]-2(2\beta\theta-3r_{00})y_ky_l=0. \tag{3.41}$$

Since the first and second term in (3.41) include the factor α^2 respectively, there is a function $\sigma(x)$ on M such that

$$r_{00}=\frac{2}{3}(\theta\beta-\sigma\alpha^2). \tag{3.42}$$

Plugging (3.42) into (3.39), we get

$$s_{l0} = \frac{1}{3}(\beta\theta_l - \theta b_l). \tag{3.43}$$

By assumption that $b^2 = \frac{1}{k}$, we have

$$(b_j)_{x^k} b^j + b_j (b^j)_{x^k} = 0. \tag{3.44}$$

Noting that $(b_j)_{x^k} = b_{j;k} + \frac{\partial^2 G^i_\alpha}{\partial y^j \partial y^k} b_i$ and $(b^j)_{x^k} = b^j{}_{;k} - \frac{\partial^2 G^j_\alpha}{\partial y^i \partial y^k} b^i$. Thus, (3.44) is equivalent to

$$b_{j;k} b^j = 0 \Longleftrightarrow (r_{jk} + s_{jk}) b^j = 0. \tag{3.45}$$

From (3.42) and (3.43), we obtain $\frac{4}{3}(\theta_0 b_k - \sigma b_k) = 0$, which implies $\sigma = \theta_0$. Thus, (3.4)–(3.6) follow from $\sigma = \theta_0$, (3.42)–(3.43) and (3.38).

Case II: $\epsilon^2 \neq 4k$.

From the definition of D, we have

$$D = (\alpha^2 - k\beta^2)[(\epsilon^2 + 2k)\alpha^2 - 2k(\epsilon^2 - k)\beta^2] - 2k^2(\epsilon^2 - 4k)\beta^4. \tag{3.46}$$

Thus D is not divisible by $(\alpha^2 - k\beta^2)$. D is also not divisible by $[(\alpha^2 + k\beta^2)^2 - \epsilon^2\alpha^2\beta^2]$ from (3.15) and α^2. On the other hand, if b^2 is not identity equal to $\frac{1}{k}$, then $(b^2\alpha^2 - \beta^2)$ can not divisible by α^2, $(\alpha^2 - k\beta^2)$ and $[(\alpha^2 + k\beta^2)^2 - \epsilon^2\alpha^2\beta^2]$. Thus from (3.14), we have

$$\frac{\partial(y_m G^m_\alpha)}{\partial y^l} b^l - 3 b_m G^m_\alpha = 0, \tag{3.47}$$

$$\alpha^2 r_{00} + 2\alpha^2 b_m G^m_\alpha - 2\beta y_m G^m_\alpha = 0. \tag{3.48}$$

If $b^2 = \frac{1}{k}$ everywhere, then by the same discussion as above, we still obtain (3.47) and (3.48) from (3.14). Similarly, it follows from (3.47) and (3.48) that there exist a 1-form $\tau := \tau_i y^i$ such that

$$y_m G^m_\alpha = \tau\alpha^2, \tag{3.49}$$

$$b_m G^m_\alpha = \frac{1}{3}[2\tau\beta + (\tau_l b^l)\alpha^2], \tag{3.50}$$

$$r_{00} = \frac{2}{3}[\tau\beta - (\tau_l b^l)\alpha^2]. \tag{3.51}$$

Similar to case I, we have

$$a_{ml} G^m_\alpha = \frac{1}{3}(\alpha^2\tau_l + 2\tau y_l), \tag{3.52}$$

$$s_{l0} = \frac{1}{3}(\beta\tau_l - \tau b_l). \tag{3.53}$$

This completes the proof of theorem. □

Corollary 3.1. *Let $F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$ be a Finsler metric on M as Theorem 3.1. If β is parallel with respect to α, then F is locally dually flat if and only if α is flat. In this case, F is locally isometric to a Minkowski metric $\widetilde{F}(y) = |y| + \epsilon b_i y^i + k\frac{(b_i y^i)^2}{|y|}$ with zero flag curvature, where $|\cdot|$ is the Euclidean metric on R^n and $b_i (1 \le i \le n)$ are constants.*

Proof. It is trivial for the proof of sufficient condition of locally dually flat metric F from Lemma 2.2. Conversely, assume β is parallel with respect to α and F is dually flat. Then $b_{i;j} = 0$. Thus $s_{l0} = r_{l0} = 0$. By Theorem 3.1, we have

$$\beta\theta^l = \theta b^l = (b_i\theta^i)y^l, \tag{3.54}$$

which implies $\beta\theta = (b_i\theta^i)\alpha^2$ and

$$G^i = G^i_\alpha = \frac{1}{3}\alpha^2\theta^i + \frac{2}{3}\theta y^i = \frac{1}{3\beta}\alpha^2(b_i\theta^i)y^i + \frac{2}{3}\theta y^i = \theta y^i. \tag{3.55}$$

Hence F is both projectively flat metric and dually flat metric. By Proposition 2.6 in [CSZ], F is of constant flag curvature λ. On the other hand, the flag curvature of F is given by

$$K = \lambda = \frac{\theta^2 - \theta_{x^k}y^k}{F^2}. \tag{3.56}$$

Thus (3.56) is equivalent to

$$[\lambda\alpha^4 + (\epsilon\lambda\beta^2 + 2k\lambda\beta^2 - \theta^2 + \theta_{x^k}y^k)\alpha^2 + k^2\lambda\beta^4] + 2\epsilon\lambda(\alpha^2 + k\beta^2)\alpha\beta = 0. \tag{3.57}$$

We must have $2\epsilon\lambda(\alpha^2 + k\beta^2)\beta = 0$. Noting $\epsilon \neq 0$ and $\beta \neq 0$, So $\lambda = 0$. Thus, it follows $\theta_{x^k}y^k = \theta^2$ from (3.56).

Since F is a projectively flat metric with zero flag curvature, α is also projectively flat and of constant sectional curvature μ by Beltrami theorem. We can set

$$\alpha = \frac{\sqrt{(1 + \mu|x|^2)|y|^2 - \mu\langle x, y\rangle^2}}{1 + \mu|x|^2}, \tag{3.58}$$

where $\langle\, ,\rangle$ is the standard Euclidean inner product on $\mathbf{R}^n$ and $|\cdot|$ is a norm with respect to $\langle\, ,\rangle$. By direct computation, we have

$$G^i_\alpha = -\frac{\mu\langle x, y\rangle}{1 + \mu|x|^2}y^i. \tag{3.59}$$

From (3.55), we get

$$\theta = -\frac{\mu\langle x, y\rangle}{1 + \mu|x|^2}. \tag{3.60}$$

Using $\theta_{x^k}y^k = \theta^2$, we have $\mu = 0$ which implies $\alpha = |y|^2$ is flat and b_i is constant because of $b_{i;j} = 0$. □

Before we give another corollary, we recall the following theorem.

Theorem 3.2 ([ChS]). *Let $F = \alpha\phi(s), s = \frac{\beta}{\alpha}$ be an (α,β) metric on M. Suppose that $\phi \neq k_1\sqrt{1+k_2s^2} + k_3s$ for any constants $k_1 > 0, k_2$ and k_3. Then F is of isotropic S-curvature, $S = (n+1)c(x)F$, if and only if one of the following holds*

(1) *β satisfies $r_j + s_j = 0$ and ϕ satisfies $\Phi = 0$, where $r_j := r_{jk}b^k$, $s_j := s_{kj}b^k$ and Φ is defined by*

$$\Phi = -(Q - sQ')(n\Delta + 1 + sQ) - (b^2 - s^2)(1 + sQ)Q'', \tag{3.61}$$

here $\Delta = 1 + sQ + (b^2 - s^2)Q'$. In this case, $S = 0$.

(2) *β satisfies $r_{ij} = \mu(b^2a_{ij} - b_ib_j), s_j = 0$, where $\mu = \mu(x)$ is a scalar function, and ϕ satisfies*

$$\Phi = -2(n+1)a\frac{\phi\Delta^2}{b^2 - s^2}, \tag{3.62}$$

where a is a constant. In this case, $S = (n+1)cF$ with $c = a\mu$.

(3) *β satisfies $r_{ij} = s_j = 0$. In this case, $S = 0$, regardless of the choice of a particular ϕ.*

For the metric $F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$, where ϵ, k are non-zero constants and β is a non-zero 1-form, that is, $\phi = 1 + \epsilon s + ks^2$, by direct computation, we obtain that $\Phi = \frac{\overline{\Phi}}{(1-ks^2)^4}$, where $\overline{\Phi}$ is a polynomial in s and b of degree 7 and 2 respectively, and the coefficient of s^7 in $\overline{\Phi}$ is $-12nk^4$. Thus $\Phi = 0$ is impossible because of $k \neq 0$. On the other hand, we compute $\phi\Delta^2$ and $\phi\Delta^2 = \frac{\overline{\Delta}}{(1-ks^2)^4}$. where $\overline{\Delta}$ is also a polynomial in s and b of degree 10 and 4 respectively, and the coefficient of s^{10} in $\overline{\Delta}$ is $9k^5$. Thus, it is impossible that (3.62) holds. Hence by Theorem 3.2, we have that $F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$ $(k \neq 0)$ is a Finsler metric with isotropic S-curvature if and only if β satisfies $r_{ij} = s_j = 0$. In this case, $S = 0$. From this and Theorem 3.1, we know that F is locally dually flat with isotropic S-curvature if and only if $\theta = 0$, which implies $r_{ij} = s_{ij} = 0$ and $G^i_\alpha = 0$. So $b_{i;j} = 0$, that is, β is parallel with respect to α and α is flat. Hence, we obtain

Corollary 3.2. *Let $F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$ be a Finsler metric on M as Theorem 3.1. Then it is locally dually flat with isotropic S-curvature if and only if α is flat and β is parallel with respect to α. In this case, F is locally isometric to a Minkowski metric $\widetilde{F}(y) = |y| + \epsilon b_iy^i + k\frac{(b_iy^i)^2}{|y|}$ with zero S-curvature.*

4. Locally dually flat metrics $F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$ of isotropic flag curvature

In this section, we will classify the metrics $F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$ ($\epsilon \neq 0, k \neq 0$, $\beta \neq 0$), which are locally dually flat metrics of isotropic flag curvature(esp. constant flag curvature). Firstly, from Lemma 2.1, the spray coefficients G^i of F are given

$$G^i = G^i_\alpha + \alpha Q s^i{}_0 + \Theta(-2\alpha Q s_0 + r_{00})\frac{y^i}{\alpha} + \Psi(-2\alpha Q s_0 + r_{00})b^i, \tag{4.1}$$

with

$$Q = \frac{\epsilon + 2ks}{1 - ks^2}, \tag{4.2}$$

$$\Theta = \frac{\epsilon - 3k\epsilon s^2 - 4k^2 s^3}{2(1 + 2kb^2 - 3ks^2)(1 + \epsilon s + ks^2)}, \tag{4.3}$$

$$\Psi = \frac{k}{1 + 2kb^2 - 3ks^2}. \tag{4.4}$$

If $F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$ is locally dually flat, then the spray coefficients of F can be written as the following form by Theorem 3.1,

$$G^i = Py^i + L\theta^i + Tb^i, \tag{4.5}$$

where

$$P := \frac{2}{3}\{[1 + (s + b^2 Q)\Theta]\theta - (1 + sQ)\Theta\theta_0\alpha\}, \tag{4.6}$$

$$L := \frac{\alpha^2}{3}(1 + sQ), \tag{4.7}$$

$$T := \frac{\alpha}{3}[(2(s + b^2 Q)\Psi - Q]\theta - 2\Psi(1 + sQ)\theta_0\alpha], \tag{4.8}$$

and $\theta_0 := \theta_i b^i$. P is positively y-homogeneous of degree one and L, T are positively y-homogeneous of degree two respectively.

For any Finsler metric F and $y \in T_xM \setminus \{0\}$, the Riemann curvature $R_y := R^i{}_k(y)\frac{\partial}{\partial x^i} \otimes dx^k : T_xM \to T_xM$ is defined as a linear map with the property $R_y(y) = 0$ and $g_y(R_y(u), v) = g_y(u, R_y(v))$ for $u, v \in T_xM$ (cf. [CS]), where

$$R^i{}_k(y) := 2\frac{\partial G^i}{\partial x^k} - y^j\frac{\partial^2 G^i}{\partial x^j \partial x^k} + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{4.9}$$

For a flag $\Pi = \mathrm{span}\{y, u\} \subset T_x(M)$ with flagpole y, the flag curvature $K = K(\Pi, y)$ is defined by

$$K(\Pi, y) := \frac{g_y(u, R_y(u))}{g_y(y, y)g_y(u, u) - g_y(y, u)^2},$$

where $g_y = g_{ij}(x,y)dx^i \otimes dx^j$. It is the analogue of the sectional curvature in Riemannian geometry. We say that a Finsler metric F is of scalar flag curvature, if for any $y \in T_x(M) \setminus \{0\}$, the flag curvature $K = K(x,y)$ is independent of Π containing $y \in T_xM$. If $K = K(x)$ depends on $x \in M$ only, then F is said to be of isotropic flag curvature. F is said to be of constant flag curvature if K=constant. A basic fact ([CS], [Sh4]) is that a Finsler metric F is of isotropic flag curvature $K = K(x)$ if and only if

$$R^i{}_k = KF^2\left(\delta^i_k - \frac{y^i}{F}F_{y^k}\right). \tag{4.10}$$

Since Ricci curvature is defined as the trace of the Riemannian curvature, that is, Ric $:= R^m{}_m$, thus if F is of isotropic flag curvature $K = K(x)$, then we have

$$\mathrm{Ric} = (n-1)KF^2. \tag{4.11}$$

Lemma 4.1. *Let $F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$ be a dually flat Finsler metric on an open subset $\mathcal{U} \subset \mathbf{R}^n (n \geq 2)$ of isotropic flag curvature $\lambda = \lambda(x)$, where α is a Riemannian metric, β is a non-zero 1-form and ϵ, k are non-zero contants. Then*

(1) $\epsilon^2 = 4k$;

(2) $\theta = 0$;

(3) *F must be of constant flag curvature λ and $\lambda = 0$.*

Proof. By assumption, $F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$ is a dually flat metric of isotropic flag curvature. we get from (4.5) and (4.9)

$$R^i{}_k(y) = \Xi(y)\delta^i_k + \tau_k(y)y^i + \mu_k(y)\theta^i + \nu_k(y)b^i + \chi^i{}_k(y), \tag{4.12}$$

and

$$\mathrm{Ric} = (n-1)\Xi(y) + \mu_i(y)\theta^i + \nu_i(y)b^i + \chi^i{}_i(y), \tag{4.13}$$

where

$$\Xi(y) := P^2 - P_{x^j}y^j + 2LP_{y^j}\theta^j + 2TP_{y^j}b^j, \tag{4.14}$$

$$\tau_k(y) := 3(P_{x^k} - PP_{y^k} - L_{y^k}P_{y^j}\theta^j - T_{y^k}P_{y^j}b^j) + \Xi_{y^k}, \tag{4.15}$$

$$\begin{aligned}\mu_k(y) :=\ & 2L_{x^k} - L_{x^jy^k}y^j + 2LL_{y^jy^k}\theta^j + 2TL_{y^jy^k}b^j - L_{y^j}L_{y^k}\theta^j \\ & - L_{y^j}T_{y^k}b^j,\end{aligned} \tag{4.16}$$

$$\begin{aligned}\nu_k(y) :=\ & 2T_{x^k} - T_{x^jy^k}y^j + 2LT_{y^jy^k}\theta^j + 2TT_{y^jy^k}b^j - T_{y^j}L_{y^k}\theta^j \\ & - T_{y^j}T_{y^k}b^j,\end{aligned} \tag{4.17}$$

$$\chi^i{}_k(y) := 2L(\theta^i)_{x^k} + 2T(b^i)_{x^k} - L_{y^k}(\theta^i)_{x^j}y^j - T_{y^k}(b^i)_{x^j}y^j. \tag{4.18}$$

Hence by (4.11), we have

$$(n-1)\lambda F^2 = (n-1)\Xi(y) + \mu_i(y)\theta^i + \nu_i(y)b^i + \chi^i{}_i(y). \tag{4.19}$$

In order to formulate the equation (4.19) in α and θ, we need to compute all terms on the right hand of (4.19). Firstly, from Theorem 3.1 and (2.11)–(2.13), we get

$$\alpha_{x^k}y^k = 2\theta\alpha, \quad \alpha_{x^k}\theta^k = \frac{2}{3\alpha}(|\theta|^2\alpha^2 + 2\theta^2), \quad \alpha_{x^k}b^k = \frac{2}{3}(\theta_0\alpha + 2s\theta), \tag{4.20}$$

$$\alpha_{y^k}y^k = \alpha, \quad \alpha_{y^k}\theta^k = \frac{\theta}{\alpha}, \quad \alpha_{y^k}b^k = s, \tag{4.21}$$

$$s_{y^k} = \frac{\alpha b_k - sy_k}{\alpha^2}, \quad s_{x^k} = \frac{4(\alpha b_k - sy_k)}{3\alpha^2}\theta = \frac{4\theta}{3}s_{y^k}, \tag{4.22}$$

$$s_{x^k}y^k = s_{y^k}y^k = 0, \quad s_{y^k}\theta^k = \frac{\theta_0\alpha - s\theta}{\alpha^2}, \quad s_{y^k}b^k = \frac{b^2 - s^2}{\alpha}. \tag{4.23}$$

Let

$$p := \frac{2}{3}(1 + \overline{A}\Theta), \qquad q := -\frac{2}{3}\overline{B}\Theta;$$
$$l := 2\overline{B} - s\overline{B}_s, \qquad h := \overline{B}_s - s\overline{B}_{ss};$$
$$u := \frac{1}{3}(2\overline{A}\Psi - Q), \qquad v := -\frac{2}{3}\overline{B}\Psi,$$

where $\overline{A} := s + b^2Q$ and $\overline{B} := 1 + sQ$. Then

$$P = p\theta + q\theta_0\alpha, \quad L = \frac{1}{3}\overline{B}\alpha^2, \quad T = u\theta\alpha + v\theta_0\alpha^2. \tag{4.24}$$

Using (4.20)–(4.24), by direct computation, we get

$$P_{x^j}y^j = (p_b\theta + q_b\theta_0\alpha)b_{x^j}y^j + p\theta_{x^j}y^j + q\alpha(\theta_0)_{x^j}y^j + 2q\theta_0\theta\alpha, \tag{4.25}$$

$$P_{y^j}\theta^j = p|\theta|^2 + (p_s - sq_s + q)\frac{\theta_0\theta}{\alpha} - sp_s\frac{\theta^2}{\alpha^2} + q_s(\theta_0)^2, \tag{4.26}$$

$$P_{y^j}b^j = (p + sq + q_st)\theta_0 + p_st\frac{\theta}{\alpha}, \quad L_{x^j}\theta^j = \frac{4}{9}(\overline{B}|\theta|^2\alpha^2 + \overline{B}_s\theta_0\theta\alpha + l\theta^2), \tag{4.27}$$

$$L_{y^j}\theta^j = \frac{1}{3}(l\theta + \overline{B}_s\theta_0\alpha), \quad L_{y^j}b^j = \frac{1}{3}(sl + \overline{B}_sb^2)\alpha, \tag{4.28}$$

$$(L_{y^i}\theta^i)_{y^j}\theta^j = \frac{1}{3}\left(l|\theta|^2 + 2h\frac{\theta_0\theta}{\alpha} - sh\frac{\theta^2}{\alpha^2} + \overline{B}_{ss}\theta_0^2\right), \tag{4.29}$$

$$(L_{y^i}b^i)_{y^j}\theta^j = \frac{1}{3}\left[ht\frac{\theta}{\alpha} + (2\overline{B} + \overline{B}_{ss}t)\theta_0\right], \tag{4.30}$$

$$(L_{y^i}\theta^i)_{x^j}y^j = \frac{1}{3}[l\theta_{x^j}y^j + \overline{B}_s\alpha(\theta_0)_{x^j}y^j + 2\overline{B}_s\theta_0\theta\alpha], \tag{4.31}$$

$$T_{y^j}\theta^j = u|\theta|^2\alpha + (u - su_s)\frac{\theta^2}{\alpha} + (2v + u_s - sv_s)\theta_0\theta + v_s\theta_0^2\alpha, \tag{4.32}$$

$$T_{y^j}b^j = (u + 2sv + v_st)\theta_0\alpha + (su + u_st)\theta, \tag{4.33}$$

$$\begin{aligned}(T_{y^i}\theta^i)_{y^j}b^j &= (us + u_st)|\theta|^2 + [2(u - su_s) + (u_{ss} - sv_{ss} + v_s)t]\frac{\theta_0\theta}{\alpha}\\ &\quad - s(u - su_s + u_{ss}t)\frac{\theta^2}{\alpha^2} + (2v + u_s + v_{ss}t)\theta_0^2,\end{aligned} \tag{4.34}$$

$$(T_{y^i}b^i)_{y^j}b^j = [2us + 2b^2v + (2u_s + sv_s + v_{ss}t)t]\theta_0 + (u - su_s + u_{ss}t)t\frac{\theta}{\alpha}, \tag{4.35}$$

$$\begin{aligned}T_{x^j}b^j &= \frac{2}{3}[(u + 4sv + 2v_st)\theta_0\theta\alpha + 2(su + u_st)\theta^2 + 2v\theta_0^2\alpha^2]\\ &\quad u\alpha\theta_{x^j}b^j + v\alpha^2(\theta_0)_{x^j}b^j + (u_b\theta + v_b\theta_0\alpha)\alpha b_{x^j}b^j,\end{aligned} \tag{4.36}$$

$$\begin{aligned}(T_{y^i}b^i)_{x^j}y^j &= 2(u + 2sv + v_st)\theta_0\theta\alpha + (u + 2sv + v_st)\alpha(\theta_0)_{x^j}y^j\\ &\quad (u_b + 2bv_s + 2sv_b + v_{sb}t)\theta_0\alpha b_{x^j}y^j + (su_b + 2bu_s + u_{sb}t)\theta b_{x^j}y^j\\ &\quad + (su + u_st)\theta_{x^j}y^j,\end{aligned} \tag{4.37}$$

where $t = b^2 - s^2$ and $(\cdot)_s$, $(\cdot)_b$ or $(\cdot)_{sb}$ are the first or second differential with respect to s, b. Putting (4.24)–(4.37) into (4.19) yields

$$\begin{aligned}(n-1)\lambda F^2 &= c_1|\theta|^2\alpha^2 + c_2\theta^2 + c_3\theta_0\theta\alpha + c_4\theta_0^2\alpha^2 + c_5(b_{x^i}y^i)\theta_0\alpha + c_6((\theta_0)_{x^i}y^i)\alpha\\ &\quad + c_7(\theta_{x^i}b^i)\alpha + c_8(b_{x^i}y^i)\theta + c_9\alpha^2 + c_{10}\theta\alpha + c_{11}\theta_{x^i}y^i,\end{aligned} \tag{4.38}$$

where $c_i (1 \le i \le 11)$ are defined as follows:

$$c_1 := \frac{2}{3}\left[(n-1)p\overline{B} + \frac{4}{3}(l\overline{B} + \overline{B}) + \overline{B}(su + u_st) - u(ls + \overline{B}_sb^2)\right]; \tag{4.39}$$

$$c_i := c_{i1} + c_{i2} (2 \le i \le 4); \quad c_{21} := \frac{n-1}{3}(3p^2 - 2sp_s\overline{B} + 6up_st); \tag{4.40}$$

$$\begin{aligned}c_{22} &:= -\frac{1}{9}(2sh\overline{B} - 6uht + l^2 - 8l) + \frac{1}{3}\big[-2su\overline{B} + 2s^2u_s\overline{B} - 3u^2s^2\\ &\quad - 2(6suu_s - 4u_s - 3u^2 + su_{ss}\overline{B})t\big]\\ &\quad + (2uu_{ss} - u_s^2)t^2 - \frac{2}{3}(ls + \overline{B}_sb^2)(u - su_s);\end{aligned} \tag{4.41}$$

$$c_{31} := \frac{2(n-1)}{3}\big[3pq - 3q + \overline{B}(p_s - sq_s + q) + 3u(p + sq + q_s t) + 3vp_s t\big]; \quad (4.42)$$

$$c_{32} := \frac{2}{9}\big[2h\overline{B} + 12u\overline{B} + 3(u\overline{B}_{ss} + vh)t - l\overline{B}_s + \overline{B}_s\big] + \frac{1}{3}\big[-2u + 6u^2 s + 4sv - 4su_s\overline{B}\,\big] + \frac{2}{3}\big[v_s\overline{B} + v_s + 9uv + 3uu_s - 9svu_s + \overline{B}u_{ss} - s\overline{B}v_{ss}\big]t + 2(uv_{ss} + vu_{ss} - u_s v_s)t^2 - \frac{2}{3}(sl + \overline{B}_s b^2)(u_s - sv_s + 2v); \quad (4.43)$$

$$c_{41} := \frac{n-1}{3}\big[3q^2 + 2\overline{B}q_s + 6v(p + sq + q_s t)\big]; \quad (4.44)$$

$$c_{42} := \frac{1}{9}(2\overline{B}\overline{B}_{ss} - \overline{B}_s^2 + 6v\overline{B}_{ss}t) + \frac{1}{3}(8v + 8v\overline{B} + 2\overline{B}u_s - 3u^2) + 2\left(2vu_s + 2v^2 - svv_s - uv_s + \frac{1}{3}\overline{B}v_{ss}\right)t - (v_s^2 - 2vv_{ss})t^2 - \frac{2}{3}v_s(sl + \overline{B}_s b^2); \quad (4.45)$$

$$c_5 := -(n-1)q_b - u_b - 2bv_s - 2sv_b - v_{sb}t; \quad (4.46)$$

$$c_6 := -(n-1)q - \frac{1}{3}\overline{B}_s - 2sv - u - v_s t; \quad (4.47)$$

$$c_7 := 2u; \quad c_8 := -(n-1)p_b - su_b - 2bu_s - u_{sb}t; \quad (4.48)$$

$$c_9 := \frac{2}{3}\overline{B}(\theta^i)_{x^i} + 2v(\theta_0)_{x^i}b^i + 2\theta_0 v(b^i)_{x^i} + 2v_b\theta_0(b_{x^i}b^i); \quad (4.49)$$

$$c_{10} := 2[u_b b_{x^i}b^i + u(b^i)_{x^i}]; \quad c_{11} := -(n-1)p - \frac{1}{3}l - su - u_s t, \quad (4.50)$$

By a long but direct computation, the equation (4.38) can be reduced into the following

$$(n-1)\lambda F^2 = \frac{\overline{c}_1}{9A_1^3A_2^2}|\theta|^2\alpha^2 + \left[\frac{\overline{c}_{21}}{9A_1^2\phi(s)^2A_2^3} + \frac{\overline{c}_{22}}{9A_1^3A_2^4}\right]\theta^2 + \left[\frac{\overline{c}_{31}}{9A_1^2\phi(s)A_2^3} + \frac{\overline{c}_{32}}{9A_1^3A_2^4}\right]\theta_0\theta\alpha + \left[\frac{\overline{c}_{41}}{9A_1^3A_2^2} + \frac{\overline{c}_{42}}{9A_1^3A_2^4}\right]\theta_0^2\alpha^2 + \frac{\overline{c}_5}{A_1A_2^3}(b_{x^i}y^i)\theta_0\alpha + \frac{\overline{c}_6}{A_1A_2^2}((\theta_0)_{x^i}y^i)\alpha + \frac{\overline{c}_7}{A_1A_2}(\theta_{x^i}b^i)\alpha + \frac{\overline{c}_8}{A_1\phi(s)A_2^3}(b_{x^i}y^i)\theta + \frac{\overline{c}_9}{A_1A_2^2}\alpha^2 + \frac{\overline{c}_{10}}{A_1A_2^2}\theta\alpha + \frac{\overline{c}_{11}}{A_1\phi(s)A_2^2}\theta_{x^i}y^i, \quad (4.51)$$

here $A_1 := 1 - ks^2$, $A_2 := 1 + 2kb^2 - 3ks^2$, $\overline{c}_i(i = 1, \text{ or } 5 \le i \le 11)$, $\overline{c}_{i1}(2 \le i \le 4)$ and $\overline{c}_{i2}(2 \le i \le 4)$ are polynomials in s and b respectively, in particular, $\overline{c}_{21} =$

$9A_1^2\phi(s)^2A_2^3c_{21}$ and $\bar{c}_{i2} = 9A_1^3A_2^4c_{i2}$ $(2 \le i \le 4)$. From (4.51) and $\lambda = \lambda(x)$, we know that $\bar{c}_{21}$ must be divisible by $\phi(s)$. On the other hand, let

$$f_1(b) := \frac{1}{k^6}\big[2k^6b^6 + (8k - 7\epsilon^2)k^4b^4 + 2(5k^2 - 13k\epsilon^2 + 4\epsilon^4)k^2b^2 + 4k^3 - 21k^2\epsilon^2 + 16k\epsilon^4 - 3\epsilon^6\big], \tag{4.52}$$

and

$$f_2(b) := \frac{\epsilon}{k^6}\big[2k^6b^6 + (15k - 7\epsilon^2)k^4b^4 + 2(14k^2 - 17k\epsilon^2 + 4\epsilon^4)k^2b^2 + (15k^3 - 34k^2\epsilon^2 + 19k\epsilon^4 - 3\epsilon^6)\big]. \tag{4.53}$$

We compute $\bar{c}_{21}$ directly and get

$$\bar{c}_{21} \equiv -3(n-1)(\epsilon^2 - 4k)^3 f(s,b) \mod \phi(s), \tag{4.54}$$

where $f(s,b) := f_1(b) + f_2(b)s$ is a polynomial in s and b of degree 1 and 6 respectively. By assumption, $n \ge 2$. Thus, either $\epsilon^2 = 4k$ or $f(s,b) = 0$. But $f(s,b) = 0$ implies $f_1(b) = f_2(b) = 0$, which is impossible unless $\epsilon = k = 0$ because of the arbitrary of s and b. The assumption of lemma implies $\epsilon^2 = 4k$. (1) holds.

In the case when $\epsilon^2 = 4k$, the coefficients $\bar{c}_{22}$, $\bar{c}_{32}$ and $\bar{c}_{42}$ in (4.51) satisfy $\bar{c}_{22} = \phi(s)\bar{\bar{c}}_{22}$, $\bar{c}_{32} = \phi(s)^2\bar{\bar{c}}_{32}$ and $\bar{c}_{42} = \phi(s)^2\bar{\bar{c}}_{42}$, where $\bar{\bar{c}}_{22}$, $\bar{\bar{c}}_{32}$ and $\bar{\bar{c}}_{42}$ are polynomials in s and b. Moreover, from (4.51) and $\lambda = \lambda(x)$, we know that $\bar{c}_{22}\theta^2 + \bar{c}_{32}\theta_0\theta\alpha + \bar{c}_{42}\theta_0^2\alpha^2$ must be divisible by A_2, i.e.,

$$\bar{\bar{c}}_{22}\theta^2 + \phi(s)\bar{\bar{c}}_{32}\theta_0\theta\alpha + \phi(s)\bar{\bar{c}}_{42}\theta_0^2\alpha^2 \equiv 0 \mod A_2. \tag{4.55}$$

On the other hand, using $\epsilon^2 = 4k$, we compute by maple program

$$\bar{\bar{c}}_{22}\theta^2 + \phi(s)\bar{\bar{c}}_{32}\theta_0\theta\alpha + \phi(s)\bar{\bar{c}}_{42}\theta_0^2\alpha^2 \equiv g(s,b) \mod A_2, \tag{4.56}$$

where $g(s,b) := [(g_1(b) + g_2(b)s]\theta^2 + [g_3(b) + g_4(b)s]\theta_0\theta\alpha + [g_5(b) + g_6(b)s]\theta_0^2\alpha^2$ and

$$g_1(b) := -\frac{1}{9}\left(\frac{1}{12}\epsilon^{12}b^{12} - \frac{7}{8}\epsilon^{10}b^{10} + \frac{11}{4}\epsilon^8b^8 - \frac{5}{3}\epsilon^6b^6 - 4\epsilon^4b^4 + 16\epsilon^2b^2 - \frac{128}{3}\right); \tag{4.57}$$

$$g_2(b) := -\frac{\epsilon}{9}\left(\frac{1}{4}\epsilon^{10}b^{10} - \frac{11}{4}\epsilon^8b^8 + \frac{17}{2}\epsilon^6b^6 + 2\epsilon^4b^4 - 40\epsilon^2b^2 + 32\right); \tag{4.58}$$

$$g_3(b) := \frac{\epsilon^3b^2}{3}\left(\frac{1}{8}\epsilon^8b^8 - \frac{5}{4}\epsilon^6b^6 + 3\epsilon^4b^4 + 4\epsilon^2b^2 - 16\right); \tag{4.59}$$

$$g_4(b) := \frac{\epsilon^2}{9}\left(\frac{1}{8}\epsilon^{10}b^{10} - \epsilon^8 b^8 + \frac{1}{2}\epsilon^6 b^6 + 10\epsilon^4 b^4 - 8\epsilon^2 b^2 - 32\right); \tag{4.60}$$

$$g_5(b) := -\frac{\epsilon^2}{9}\left(\frac{1}{32}\epsilon^{10}b^{10} - \frac{1}{16}\epsilon^8 b^8 - \frac{7}{4}\epsilon^6 b^6 + 7\epsilon^4 b^4 + 4\epsilon^2 b^2 + 32\right); \tag{4.61}$$

$$g_6(b) := -\frac{\epsilon^3}{3}\left(\frac{1}{16}\epsilon^8 b^8 - \frac{5}{8}\epsilon^6 b^6 + \frac{3}{2}\epsilon^4 b^4 + 2\epsilon^2 b^2 - 8\right). \tag{4.62}$$

From (4.55) and (4.56), we get $g(s,b) = 0$, which implies θ is divisible by α. Noting that α is irrational and θ is a 1-form. Hence $\theta \equiv 0$. Thus from (4.51), we obtain $\lambda = 0$. This completes the proof of lemma. □

By Theorem 4.6 in [Zh], the metric $F = \frac{(\alpha \pm \sqrt{k}\beta)^2}{\alpha}$ with constant flag curvature must be projectively flat. Combining Lemma 4.1 and Proposition 2.6 in [CSZ], we have

Theorem 4.1. *Let $F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$ be a locally dually flat metric on $M^n (n \geq 2)$, where α is a Riemannian metric, β is a non-zero 1-form and ϵ, k are non-zero constants. Then it is locally projectively flat if and only if it is of constant flag curvature.*

Theorem 4.2. *Let $F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$ be a Finsler metric on $M^n (n \geq 2)$, where α is a Riemannian metric, β is a non-zero 1-form and ϵ, k are non-zero constants. Then F is a locally dually flat Finsler metric of isotropic flag curvature if and only if $\epsilon^2 = 4k$, α is flat and β is parallel with respect to α. In this case, F is locally isometric to $\overline{F} = \frac{(|y| + \sqrt{k}b_i y^i)^2}{|y|}$, which is a Minkowski metric with zero flag curvature, where $|\cdot|$ is the Euclidean metric on $\mathbf{R}^n$ and $b_i (1 \leq i \leq n)$ are non-zero constants.*

Proof. It is obvious that the Minkowski metric $F = \frac{(|y| + \sqrt{k}b_i y^i)^2}{|y|}$ is dually flat metric with constant curvature from Lemma 2.2. Conversely, assume that F is a locally dually flat Finsler metric of isotropic flag curvature. From Lemma 4.1, we have $\epsilon^2 = 4k$, $\theta = 0$ and F is of zero flag curvature. By Theorem 3.1, we conclude that $r_{jk} = s_{jk} = G^i_\alpha = 0$. We have completed the proof of theorem. □

Remark 4.1. The another proof of necessity of Theorem 4.2 may follow from the conclusion (1), (3) in Lemma 4.1, Theorem 4.6 in [Zh] and Theorem 1.2 in [SY] directly.

Remark 4.2. Theorem 4.2 shows that there exists no locally dually flat metric in the form $F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$ $(\epsilon \neq 0,\ k \neq 0,\ \beta \neq 0)$ of isotropic flag curvature unless it is Minkowskian.

Remark 4.3. Theorem 4.2 implies that locally dually flat metrics $F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$ $(\epsilon \neq 0,\ k \neq 0,\ \beta \neq 0)$ of isotropic flag curvature K are equivalent to locally dually flat metrics $F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$ $(\epsilon \neq 0,\ k \neq 0,\ \beta \neq 0)$ with constant flag curvature K. In both cases, we have $K = 0$.

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References

[AN] S.-I. Amari and H. Nagaoka, Methods of Information Geometry, Vol. 191, AMS Translation of Math. Monographs, *Oxford University Press*, 2000.

[ChS] X. Cheng and Z. Shen, A class of Finsler metrics with isotropic S-curvature, *Israel J. Math.* **169** (2009), 317–340.

[CS] S. S. Chern and Z. Shen, Riemannian-Finsler Geometry, *World Scientific Publisher, Singapore*, 2005.

[CSZ] X. Cheng, Z. Shen and Y. Zhou, On locally dually flat Randers metrics, 2009 (*in print*).

[LS] B. Li and Z. Shen, On a class of projectively flat Finsler metrics with constant flag curvature, *Inter. J. Math.* **18**, no. 7 (2007), 1–12.

[Sh1] Z. Shen, Riemann-Finsler Geometry with application to information geometry, *Chin. Ann. Math.* **27B**, no. 1 (2006), 73–94.

[Sh2] Z. Shen, Projectively flat Finsler metrics of constant flag curvature, *Trans. Amer. Math. Soc.* **355**, no. 4 (2003), 1713–1728.

[Sh3] Z. Shen, On projectively flat (α, β) metrics, *Canadian Math. Bull.* **52**, no. 1 (2009), 132–144.

[Sh4] Z. Shen, Differential Geometry of Spray and Finsler Spaces, *Kluwer Academic Publishers*, 2001.

[SY] Z. Shen and G. C. Yildirim, On a class of projectively flat metrics with constant flag curvature, *Canadian. J. Math.* **60**, no. 2 (2008), 443–456.

[Zh] L. Zhou, Some results on a class of (α, β) metrics with constant flag curvature, *Differential Geom. Appl.* **28**, no. 2 (2010), 170–193.

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