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On weakly SS-quasinormal minimal subgroups of finite groups

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Abstract. A subgroup H of a group G is said to be weakly SS-quasinormal if there exists a subgroup B of G such that HB is normal in G and for any prime p with $(p, |H|) = 1$, H permutes with every Sylow p-subgroup of B and $\mathrm{Syl}_p(B) \subseteq \mathrm{Syl}_p(G)$. In this article, we study the influence of weakly SS-quasinormal minimal subgroups of a finite group. Our results generalize the recent results obtained about the classification of a group by considering the SS-quasinormality of some subgroups.

1. Introduction

Throughout this article, only finite groups will be considered. The unexplained notation is standard and follows that in [7]. Two subgroups H and K of a group G are said to permute if $HK = KH$. A subgroup H of a group G is said to be S-*quasinormal* in G if H permutes with every Sylow subgroup of G. This embedding property was studied by KEGEL in [8] and was extended to the SS-quasinormality in [9], [10]. Recall that a subgroup H of G is said to be SS-*quasinormal* in G if there is a supplement B of H to G such that H permutable with every Sylow subgroup of B. In this article we consider a new permutability property in finite groups: the weakly SS-quasinormality.

Definition 1.1. Let H be a subgroup of a group G. We say that H is weakly SS-quasinormal if there exists a subgroup B of G such that HB is normal in G

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and for any prime p with $(p, |H|) = 1$, H permutes with every Sylow p-subgroup of B and $\mathrm{Syl}_p(B) \subseteq \mathrm{Syl}_p(G)$.

This embedding property is very close to the SS-quasinormality. The relationship between S-quasinormal subgroups and SS-quasinormal subgroups has been investigated in [9], [10]. For instance:

Proposition 1.1 ([9], Lemma 2.2)**.** *Let P be a p-subgroup of G. Then P is S-quasinormal if and only if P is SS-quasinormal and P is contained in $O_p(G)$.*

A significant role will be played by the following result, due to Kegel [8].

Proposition 1.2. *Let H be a subgroup of G. Then H is subnormal if H is S-quasinormal.*

It is clear that every SS-quasinormal subgroup are weakly SS-quasinormal. However the following example shows, in general, that a weakly SS-quasinormal subgroup need not be SS-quasinormal. This means that the set of weakly SS-quasinormal subgroups is bigger than that of SS-quasinormal subgroups. In what follows, $G = [A]B$ means B is a complement to the normal subgroup A in G.

Example 1.1. Let $G = [A]B$, where $A = \langle a, b \mid a^3 = b^3 = 1, b^a = b\rangle$, $B = \langle c, d \mid c^2 = d^2 = 1, d^c = d\rangle$ and $a^c = a, (cb)^2 = (ad)^2 = (bd)^2 = 1$. Then, $L = \langle bd\rangle$ is weakly SS-quasinormal but not SS-quasinormal.

In fact, because A is the only Sylow 3-subgroup of G, it is clear that L is weakly SS-quasinormal. However L is not SS-quasinormal. If not, let M be a supplement of L to G, then M is a subgroup with index 2, so either c or cd lies in M. By definition, L permutes with either $\langle c\rangle$ or $\langle cda\rangle$, a contradiction.

In order to develop the weakly SS-quasinormality, we give some introductions and statement of results.

A class $\mathcal{F}$ of groups is called a formation if $\mathcal{F}$ contains all homomorphic images of a group in $\mathcal{F}$, and if G/N_1 and G/N_2 are in $\mathcal{F}$, then $G/(N_1 \cap N_2)$ is in $\mathcal{F}$ for normal subgroups N_1, N_2 of G. A formation $\mathcal{F}$ is said to be saturated if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$. For a formation $\mathcal{F}$, each group G has a smallest normal subgroup N such that $G/N \in \mathcal{F}$. This uniquely determined normal subgroup of G is called the $\mathcal{F}$-residual subgroup of G and is denoted by $G^{\mathcal{F}}$. In this article, $\mathcal{U}$, $\mathcal{N}_p$ will denote the class of all supersolvable groups and the class of all p-nilpotent groups, respectively. As well-known results, $\mathcal{U}$, $\mathcal{N}_p$ are saturated formations.

A number of authors have studied the structure of a group G under the assumption that some subgroups of G are well located in G. For example, Shaalan [13] proved that, for a proper normal subgroup H of G, if $G/H \in \mathcal{U}$ and every subgroup of H of prime order is S-quasinormal in G, suppose further that one of the following conditions holds: (i) $2 \dagger |H|$, (ii) $2 \mid |H|$ and the Sylow 2-subgroups of H are abelian, (iii) $2 \mid |H|$ and every cyclic subgroup of H of order 4 is S-quasinormal in G, then $G \in \mathcal{U}$. Ballester-Bolinches and Pedraza-Aguilera [4] proved that if $\mathcal{F}$ is a saturated formation containing $\mathcal{U}$ and G is a group with normal subgroup H such that $G/H \in \mathcal{F}$, assume further that a Sylow 2-subgroup of G is abelian or all cyclic subgroups with order 4 of H are S-quasinormal in G and all minimal subgroups of H are permutable in G, then $G \in \mathcal{F}$. It is natural to limit the hypotheses of minimal subgroups to a smaller subgroup, say, the Fitting subgroup $F(H)$, of H, and to remove the abelian assumption of the Sylow 2-subgroup of G by replacing the hypothesis with the Q_8-free hypothesis by using the weakly SS-quasinormality.

2. Preliminaries

In this section, we collect some auxiliary results that are needed in the sequel. The first result is well known (cf. [14], VI, Aufgaben 16).

Lemma 2.1. *Suppose that G is a minimal non-supersolvable group (every proper subgroup of G is supersolvable but G is not supersolvable). Then G has the following properties:*

(i) *$G = [P]K$, where P is a normal Sylow p-subgroup of G and K is supersolvable* Hall-*subgroup of G;*

(ii) *P is either elementary abelian or superspecial with $Z(P) = P' = \Phi(P)$;*

(iii) *If $p > 2$, then the exponent of P is p. If $p = 2$, then the exponent of P is 2 or 4.*

Referring to Li ([10], Lemma 2.1), we have the following result.

Lemma 2.2. *Suppose that H is weakly SS-quasinormal in G, H a subgroup of G, and N a normal subgroup of G. We have:*

(i) *If H is a subgroup of K, then H is weakly SS-quasinormal in K;*

(ii) *If H is a p-subgroup and p a prime, then HN/N is weakly SS-quasinormal in G/N;*

(iii) *If N is a subgroup of K and K/N is weakly SS-quasinormal in G/N, then K is weakly SS-quasinormal in G.*

Lemma 2.3. *Let P be a p-subgroup of G. Then P is S-quasinormal in G if and only if P is weakly SS-quasinormal in G and P is contained in $O_p(G)$.*

Proof. First suppose that P is S-quasinormal in G. Then by Proposition 1.2, we get that P is subnormal in G. Thus, $P \leq O_p(G)$ by a result of Schmid ([12], p. 287).

Conversely, suppose $P \leq O_p(G)$ and P is weakly SS-quasinormal in G. Then PB is normal in G for some subgroup B of G, and for any prime $q \neq p$, $\mathrm{Syl}_q(B) \subseteq \mathrm{Syl}_q(G)$. This implies that $O^p(PB) = O^p(G)$. Let Q be any Sylow q-subgroup of B. By definition, we have $PQ = QP$. Moreover, $P = P(O_p(G) \cap Q) = O_p(G) \cap PQ$ is normal in PQ, and hence $O^p(PB) \leq N_G(P)$. It yields that $O^p(G) \leq N_G(P)$. By Lemma A of [12], P is S-quasinormal in G. $\square$

From Lemma 2.3 and Proposition 1.1, for any p-subgroup P of G, if $P \leq O_p(G)$, then P is SS-qusinormal if and only if P is weakly SS-quasinormal.

Lemma 2.4 ([6]). *Let A and B are supersolvable normal subgroups of G such that $|G : A|$ and $|G : B|$ are co-prime. Then G is supersolvable.*

Let Q_8 denote the quaternion group of order 8. A group G is called Q_8-free if no quotient group of any subgroup of G is isomorphic to Q_8. In what follows, if G is a p-group, $\Omega_1(G)$ will denote the subgroup of G generated by its elements of order p.

Lemma 2.5 ([5], Lemma 2.15). *If σ is an automorphism of odd order of Q_8-free 2-group G, and σ acts trivially on $\Omega_1(G)$, then $\sigma = 1$.*

3. Main results

Theorem 3.1. *Let p be a prime dividing the order of G and H a normal p-subgroup of G such that $G/H \in \mathcal{U}$. If every subgroup of H of order p is weakly SS-quasinormal in G, and if, in addition, every cyclic subgroup of H of order 4 is weakly SS-quasinormal in G or H is Q_8-free, then $G \in \mathcal{U}$.*

Proof. Assume the theorem is false and let G be a counterexample of minimal order.

It is obvious that the hypotheses of the theorem are inherited for subgroups of G. The minimal choice of G yields that $G \notin \mathcal{U}$ but every proper subgroup of G lies in $\mathcal{U}$. By Lemma 2.1, $G = [P]K$, where P is the normal Sylow p-subgroup of G and K is a supersolvable Hall-subgroup of G for some $p \in \pi(G)$.

If $(p, |P|) = 1$, then that $G = G/P \cap H$ is isomorphic to a subgroup of $G/P \times G/H$. This means that $G \in \mathcal{U}$, a contradiction. Hence, $H \leq P$. If $H \leq \Phi(P)$, then $G/\Phi(P) \in \mathcal{U}$. Noting that $\Phi(P) \leq \Phi(G)$, therefore, $G/\Phi(G) \in \mathcal{U}$. It follows that $G \in \mathcal{U}$, a contradiction. Hence, $H \nleq \Phi(P)$. Since $P/\Phi(P)$ is a normal minimal subgroup of $G/\Phi(P)$, then $H = P$. By hypotheses, every subgroup of H of order p is weakly SS-quasinormal in G, then is S-quasinormal in G by Lemma 2.3.

Case 1. If $p > 2$, then $\exp(P) = p$ by Lemma 2.1. Thus $G \in \mathcal{U}$ by a theorem of Asaad and Csorgo in [2], a contradiction.

Case 2. If $p = 2$, for any subgroup $\langle x\rangle$ of H of order 2, $\langle x\rangle$ is weakly SS-quasinormal by hypotheses and so S-quasinormal in G. So let q be odd prime and Q any Sylow q-subgroup of G, then $\langle x\rangle Q$ is a subgroup of G. In fact, $\langle x\rangle Q$ is a proper subgroup of G, and hence lies in $\mathcal{U}$. So $\langle x\rangle Q$ is nilpotent and Q centralizes $\langle x\rangle$. It follows that K acts trivially on $\Omega_1(P)$ by conjugation.

Case 2.1. Suppose H is Q_8-free. It is immediate from Lemma 2.5, K acts trivially on P, and consequently $G = P \times K$. Thus, we get that $(|G : P|, |G : K|) = 1$. Then Lemma 2.4 implies that $G \in \mathcal{U}$, a contradiction.

Case 2.2. Suppose every cyclic subgroup of H of order 4 is weakly SS-quasinormal in G. Let x be any generated element of P. Since $\exp(P) = 2$ or 4 by Lemma 2.1, x is then of order 2 or 4. If $\langle x\rangle$ is normal in G, then $1 \neq \langle x\rangle\Phi(P)/\Phi(P)$ is a normal minimal subgroup of $G/\Phi(P)$ and contained in $P/\Phi(P)$. That is $\langle x\rangle\Phi(P) = P$, and hence $\langle x\rangle = P$, a contradiction. Thus, all it remains to consider $N_G(\langle x\rangle)$ is a proper subgroup of G. Since every cyclic subgroup of H of order 2 or 4 is weakly SS-quasinormal in G by hypotheses and so $H \leq O_p(G)$. It follows that each $\langle x\rangle$ is S-quasinormal in G by Lemma 2.3. Hence $O^p(G) \leq N_G(\langle x\rangle) < G$. This means $P \nleq O^p(G)$, that is $P \cap O^p(G) \leq \Phi(P)$. On the other hand, $G/P \cap O^p(G) \in \mathcal{U}$. Trivially, $G/\Phi(P) \in \mathcal{U}$ and hence $G \in \mathcal{U}$, a contradiction. $\square$

Theorem 3.2. *Let p be a prime dividing the order of G with $(p-1, |G|) = 1$. If every minimal subgroup of G of order p is weakly SS-quasinormal in G, and if, in addition, every cyclic subgroup of G of order 4 is weakly SS-quasinormal in G or Sylow p-subgroup of G is Q_8-free, then $G \in \mathcal{N}_p$.*

Proof. Assume the theorem is false and let G be a counterexample of minimal order. It is obvious that the hypotheses of the theorem are inherited for subgroups of G. Our minimal choice yields that $G \notin \mathcal{N}_p$ but every proper subgroup of G lies in $\mathcal{N}_p$. A well-known result ([11], Theorem 9.1.9) implies that, $G = [P]Q$, where P is normal in G and Q is cyclic and is not normal in G for some $p \in \pi(G)$.

Case 1. $p > 2$. Let x be any generated element of P. Since $\exp(P) = p$, we have that x is of order p. If $\langle x\rangle$ is normal in G, then $1 \neq \langle x\rangle\Phi(P)/\Phi(P)$ is a normal minimal subgroup of $G/\Phi(P)$ and contained in $P/\Phi(P)$. Hence $\langle x\rangle\Phi(P) = P$ and so $\langle x\rangle = P$. Noting that $|\mathrm{Aut}(P)| = p-1$, we have that q divides $p-1$, a contradiction. So $N_G(\langle x\rangle)$ must be a proper subgroup of G. On the other hand, every subgroup of P of order p is weakly SS-quasinormal in G by hypotheses, then is S-quasinormal in G by Lemma 2.3. Thus $\langle x\rangle Q$ is a proper subgroup of G, then is nilpotent. Hence, Q acts trivially on P by conjugation, a contradiction.

Case 2. $p = 2$. Let $\langle x\rangle$ be any subgroup of P of order 2. Without loss of generality suppose that Q is any Sylow q-subgroup of G. Then, from Lemma 2.3, $\langle x\rangle$ must be S-quasinormal in G. So $\langle x\rangle$ permutes with Q. Then it must be true that $\langle x\rangle Q$ is a proper subgroup of G and hence a direct product of $\langle x\rangle$ and Q. It follows that Q acts trivially on $\Omega_1(P)$ by conjugation.

Case 2.1. If P is Q_8-free, then Lemma 2.5 implies that Q acts trivially on P. Therefore, $G = P \times Q$, a contradiction.

Case 2.2. If every cyclic subgroup of G of order 4 is weakly SS-quasinormal in G. Let x be any generated element of P. Since $\exp(P) = 2$ or 4 by Theorem 9.1.9 of [11], we have that x is of order 2 or 4. Since $P \leq O_p(G)$, by Lemma 2.3, each $\langle x\rangle$ is S-quasinormal in G. This means that $\langle x\rangle$ permutes with Q. It is easy to show that $\langle x\rangle Q$ is a proper subgroup of G, and is a direct product of $\langle x\rangle$ and Q. Hence, Q acts trivially on P by conjugation, a final contradiction. □

Theorem 3.3. *Let p be a prime dividing the order of G with $(p-1, |G|) = 1$. If every minimal subgroup of G of order $q \neq p$ is weakly SS-quasinormal in G, and if, in addition, every cyclic subgroup of G of order 4 is weakly SS-quasinormal in G or Sylow p-subgroup of G is Q_8-free, then G possesses Sylow tower.*

Proof. Theorem 3.2 implies $G \in \mathcal{N}_p$. Let K be a normal p-complement to G. Therefore, $G = PK$, where P is a Sylow p-subgroup of G. By induction on $|G|$, K possesses Sylow tower. Thus, G possesses Sylow tower. □

Theorem 3.4. *Let H be a normal subgroup of G such that $G/H \in \mathcal{U}$. If every minimal subgroup of H is weakly SS-quasinormal in G, and if, in addition, every cyclic subgroup of H of order 4 is weakly SS-quasinormal in G or H is Q_8-free, then $G \in \mathcal{U}$.*

Proof. Suppose H is a p-group. Then $G \in \mathcal{U}$ by Theorem 3.1. Thus $|H|$ is divisible by at least two distinct primes. Let p be the largest prime dividing $|H|$ and P a Sylow p-subgroup of H, then from Theorem 3.3, we immediately get that $G \in \mathcal{U}$. Thus P is normal in H, then is normal in G.

Now, let X/P be any prime order subgroup of H/P. Then $\langle h\rangle P = X$ for some prime order subgroup $\langle h\rangle$ of H. Since Lemma 2.2 implies that every prime order subgroup of H/P is weakly SS-quasinormal in G/P. Moreover, $(G/P)/(H/P) \cong G/H \in \mathcal{U}$. By induction on $|G|$, we have that $G/P \in \mathcal{U}$. It follows from Theorem 3.1, that $G \in \mathcal{U}$. □

As an immediate consequence of Theorem 3.4, we have:

Corollary 3.5. *Let p be a prime dividing the order of G with $(p-1, |G|) = 1$. If every minimal subgroup of G is weakly SS-quasinormal in G, then $G \in \mathcal{U}$ if and only if $G \in \mathcal{N}_p$.*

Proof. First suppose that $G \in \mathcal{U}$. It is clear that $G \in \mathcal{N}_p$. Conversely, suppose $G \in \mathcal{N}_p$ and hence G has a normal p-complement K. Then $G = PK$, where P is a Sylow p-subgroup of G. It follows that $G/K \cong P$. Then from Theorem 3.4, we immediately get that $G \in \mathcal{U}$. □

Ito Theorem has a generalization as follows.

Corollary 3.6. *Let p be a prime dividing the order of G with $(p-1, |G|) = 1$. If every minimal subgroup of G' is weakly SS-quasinormal in G, then $G' \in \mathcal{N}$.*

Proof. Theorem 3.4 immediately implies that $G \in \mathcal{U}$ if we let $H = G'$. It is clear that $G' \in \mathcal{N}$. This completes the proof. □

Corollary 3.7. *Let p be a prime dividing the order of G with $(p-1, |G|) = 1$ and q the largest prime dividing the order of G. If every minimal subgroup of G of order $q \neq p$ is weakly SS-quasinormal in G and every cyclic subgroup of G of order 4 is weakly SS-quasinormal in G or Sylow p-subgroup of G is Q_8-free, then $G/G_q \in \mathcal{U}$.*

Proof. Theorem 3.3 implies that G possesses Sylow tower. Therefore, $G = PK$, where K is a normal p-complement to G. It follows from Theorem 3.3, that K possesses Sylow tower. Then $K = RL$, where L is a normal r-complement to K, r is the smallest prime dividing the order of K and R Sylow r-subgroup of K. This means that $K/L \cong R$, and hence $G/P \cong K \in \mathcal{U}$ by Theorem 3.4. This completes the proof. □

The next question addresses wether Theorem 3.4 could be applied to the group formation $\mathcal{F}$, and what are the additional conditions needed in order to stay in the class. The following Theorem generalizes some results in [4], [13].

Theorem 3.8. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Then $G \in \mathcal{F}$ if and only if G has a normal subgroup H such that $G/H \in \mathcal{F}$, every minimal* *subgroup of H is weakly SS-quasinormal in G, and if, in addition, every cyclic subgroup of H of order 4 is weakly SS-quasinormal in G or H is Q_8-free.*

Proof. We need to prove only that the sufficiency is true. Assume the theorem is false and let G be a counterexample of minimal order. Theorem 3.4 implies that $H \in \mathcal{U}$. Let p be the largest prime dividing $|H|$ and P a Sylow p-subgroup of H. Clearly, P is normal in H, then is normal in G.

Let X/P be any cyclic subgroup of H/P of order prime or 4 and $\langle h\rangle$ some cyclic subgroup of H of order prime or 4. Since $(G/P)/(H/P) \cong G/H \in \mathcal{F}$, we have $\langle h\rangle P = X$. By Lemma 2.2, X/P is weakly SS-quasinormal in G/P. The minimal choice of G implies that $G/P \in \mathcal{F}$.

By hypotheses, every minimal subgroup and every cyclic subgroup of P of order 4 is weakly SS-quasinormal in G or H is Q_8-free. Then every minimal subgroup of P is S-quasinormal in G by Lemma 2.3, and every cyclic subgroup of P of order 4 is S-quasinormal in G or H is Q_8-free.

First suppose every cyclic subgroup of P of order 4 is S-quasinormal in G. Then by a known Theorem of [2], we get that $G \in \mathcal{F}$, a contradiction.

Now, suppose $p = 2$ and $P = H$ is Q_8-free. Since $G \notin \mathcal{F}$ but $G/P \in \mathcal{F}$, then $1 < G^{\mathcal{F}} \leq P$. By Theorem 3.5 of [3], $G = MF'(G)$, where M is a maximal subgroup of G, $F'(G) = \mathrm{Soc}(G \mod \Phi(G))$, and G modulo the G-core M_G of M does not lie in $\mathcal{F}$. Moreover, $G^{\mathcal{F}}$ is a p-group. Thus $G = MG^{\mathcal{F}} = MF(G)$ and consequently $(M, M\cap P)$ satisfies the conditions of the Theorem. The minimality of G implies $M \in \mathcal{F}$.

By Lemma 2 of [1], we get that $\exp(G^{\mathcal{F}}) = p$ or 4. If $G^{\mathcal{F}}$ is abelian, then $G^{\mathcal{F}}$ is a normal minimal subgroup of G, and so $G^{\mathcal{F}} \not\leq \Phi(G)$. It follows that, $G = M^*G^{\mathcal{F}}$, where M^* is a maximal subgroup of G and $M^* \cap G^{\mathcal{F}} = 1$. Since every minimal subgroup of P is S-quasinormal in G, we have that $\langle x\rangle Q$ is a subgroup of G for each element x of $G^{\mathcal{F}}$ and each Sylow q-subgroup Q of G of order odd. This means that $G^{\mathcal{F}} = \langle x\rangle$ and hence $G^{\mathcal{F}}$ is of order p. Thus $G \in \mathcal{F}$, a contradiction. If $G^{\mathcal{F}}$ is nonabelian, then Lemma 2 of [1] implies that $(G^{\mathcal{F}})' = Z(G^{\mathcal{F}}) = \Phi(G^{\mathcal{F}})$ is an elementary abelian group. Let $X/(G^{\mathcal{F}})'$ be any prime order subgroup of $G^{\mathcal{F}}/(G^{\mathcal{F}})'$. Then there exists a subgroup A of $G^{\mathcal{F}}$ such that $A(G^{\mathcal{F}})' = X$. If $|A|$ is a prime, then, since $(G^{\mathcal{F}}/(G^{\mathcal{F}})') \cap (\Phi(G)/(G^{\mathcal{F}})') = 1$, we have that $X/(G^{\mathcal{F}})'$ is normal in $G/(G^{\mathcal{F}})'$. Now, the minimality of $G^{\mathcal{F}}/(G^{\mathcal{F}})'$ implies that $X = G^{\mathcal{F}}$. Hence X is a cyclic group of order prime. Thus $G \in \mathcal{F}$, a contradiction.

Now, all it remains to consider that each element of $G^{\mathcal{F}}$ is of order 4. Then, $\Omega_1(G^{\mathcal{F}}) = (G^{\mathcal{F}})' = Z(G^{\mathcal{F}}) = \Phi(G^{\mathcal{F}})$. For any minimal subgroup X of $G^{\mathcal{F}}$ and every Sylow q-subgroup Q of G with $q \neq 2$. Since every minimal subgroup of P is

S-quasinormal in G, then $\langle x\rangle Q$ is a subgroup of G. Hence $Q \le C_G(x)$. It follows that every $2'$-element of G acts trivially on $\Omega_1(G^{\mathcal{F}})$ by conjugation, then acts trivially on $G^{\mathcal{F}}$ by Lemma 2.5. Because $G/(G^{\mathcal{F}})'$ is a chief factor of G, it follows that $G/(G^{\mathcal{F}})'$ is of order prime. Since $G/(G^{\mathcal{F}})'$ is G-isomorphic to $\mathrm{Soc}(G/M_G)$ and so $G/M_G \in \mathcal{F}$, a contradiction. □

The following result generalizes Theorem 3.5 in [9].

Theorem 3.9. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Then $G \in \mathcal{F}$ if and only if G has a normal subgroup H such that $G/H \in \mathcal{F}$, every minimal subgroup of $F(H)$ is weakly SS-quasinormal in G, and if, in addition, every cyclic subgroup of H of order 4 is weakly SS-quasinormal in G or H is Q_8-free.*

Proof. We need to prove only that the sufficiency is true. Assume the theorem is false and let G be a counterexample of minimal order.

If $F(H)$ is of order odd, then Lemma 2.2 and the main Theorem of [2] imply $G \in \mathcal{F}$, a contradiction. So $F(H)$ is of order even. Because any Sylow 2-subgroup P of $F(H)$ is normal in G. Let $L/P = F(H/P)$. Since $(G/P)/(H/P) \cong G/H \in \mathcal{F}$ and L/P is nilpotent, if $1 < P_1/P \in \mathrm{Syl}_2(L/P)$, then $P_1 \le F(H)$. By Sylow′s Theorem, $P_1 \le P$ and so $P_1 = P$, a contradiction. This means that L/P is of order odd. From Theorem 3.1, we get that $L \in \mathcal{U}$. Next we show that every cyclic subgroup X/P of L/P of order prime or 4 is weakly SS-quasinormal in G/P. The reason is that there exists a Sylow q-subgroup Q of L such that $X/P \le QP/P \in \mathrm{Syl}_q(L/P)$. Moreover, $L \in \mathcal{U}$ implies that $PQ \in \mathcal{U}$ and so Q is normal in PQ. Since L/P is nilpotent and so PQ is normal in G. It follows that Q is normal in G. Since $X = (X \cap Q)P$, we have that $X \cap Q$ is a cyclic subgroup of Q of order prime or 4 and is weakly SS-quasinormal in G by hypotheses. From Lemma 2.2, we get that X/P is weakly SS-quasinormal in G/P. It follows from the proof above that $(G/P, H/P)$ satisfies the conditions of the Theorem. The minimality of G implies $G/P \in \mathcal{F}$. Thus, $G \in \mathcal{F}$ by Theorem 3.8, a final contradiction. □

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